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Brief Report

Nonexistence of Finite-Time Stable Equilibria in a Class of Nonlinear Integral Equations

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Abstract: This brief report studies conditions to ensure the nonexistence of finite-time stable equilibria in a class of systems that are described by means of nonlinear integral equations, whose kernels are part of some Sonine kernel pairs. It is firstly demonstrated that, under certain criteria, a real-valued function that converges in finite-time to a constant value, different from the initial condition, and remains there afterwards, cannot have a Sonine derivative that also remains at zero after some finite time. Then, the concept of equilibrium is generalized to the case of equivalent equilibrium, and it is demonstrated that a nonlinear integral equation, whose kernel is part of some Sonine kernel pair, cannot possess equivalent finite-time stable equilibria. Finally, illustrative examples are presented.

Keywords: integral equations; finite-time stability; Sonine derivatives; fractional calculus

1. Introduction

Integer-order integrals and derivatives conform classical tools that allow us to describe local behavior, producing dynamic models that can be considered to approximate real-world dynamics to a good extent, when certain constraints are imposed. Nevertheless, fractional-order operators permit a better comprehension of non-local phenomena [1]; for this reason, fractional-order integrals and derivatives stand as formidable tools for the modeling and control of dynamical systems with more advanced characteristics.

While searching for the key of generalization, additional operators have been considered, such as variable-order [2,3] and distributed-order operators [4–9]. In addition, ingenious definitions have been proposed in [10–12], relying on non-singular kernels. Nonetheless, dynamical systems with non-singular kernel derivatives lead to a different class of integral equations, which is beyond the scope of the present study. Additional formulations of derivative operators can be found in [13–17].

In this report, the Sonine conditions are deemed to show the equivalence of the proposed models with well-suited integral equations [18]. Sonine derivatives have been previously studied in the inspiring works [19–23], and these operators generalize some previously defined non-local derivatives, permitting us to consider a larger class of dynamic responses.

The study of Sonine systems in relation to integral equations have interested the research community in recent years. The authors of [3,24] considered Sonine operators to propose and analyze well-suited variable-order derivative operators, leading to the study of a class of dynamical systems whose solutions are equivalent to some integral equations. In [25], some stability criteria were studied regarding a class of nonlinear integral equations, which are equivalent to suitable systems that are described in terms of Sonine derivatives, where the assumption of integer-order differentiable solutions is not considered. In [26], a proportional-integral controller was designed to compensate for a larger class of disturbances and un-modeled effects. Other outstanding applications in the same direction consider processes with anomalous diffusion [27].

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The contribution of this brief report is the formulation of some criteria to assure the nonexistence of finite-time stable equilibria, considering a class of integral equations, which are equivalent to some Sonine derivative based systems. The present study would be considered of paramount importance in the design of more general control systems, accounting for the intriguing properties of advanced physical systems and engineering processes.

The case of the nonexistence of finite-time stable equilibria has been previously investigated for the case of fractional-order systems [28,29]. Nevertheless, integer-order systems can still possess finite-time stable equilibria by virtue of their local behavior; an interesting application of finite-time stability of integer-order systems can be found in [30].

It is important to comment that the main contribution of this paper is different from that of [31], which studies the blow-up phenomenon, or finite-time escape, in a class of nonlinear integral equations with continuous solutions. Furthermore, the kernel in [31] is continuous and differentiable at all points of its domain, contrary to the case of the kernels that are considered in this paper.

The remainder of this document is organized as follows: the next section exposes preliminaries on Sonine integrals and derivatives, including an extension to the case of not necessarily integer-order differentiable functions. Section 3 presents the main results on the nonexistence of finite-time stable equilibria for generalized systems. Illustrative examples are studied in Section 4, and the main conclusions are presented in Section 5.

2. Preliminaries

2.1. Generalized Operators

The exposition of this section is given in detail in [20–23,25]. The following definition of a kernel pair is interesting for the study of generalized calculus:

Definition 1. *Let $\kappa(t)$ and $\lambda(t) \in \mathcal{L}^1_{loc}([0,\infty))$ be two non-negative functions that satisfy the so-called Sonine condition,*

$$\int_0^t \lambda(t-\tau)\kappa(\tau)d\tau = \int_0^t \lambda(\tau)\kappa(t-\tau)d\tau = 1. \tag{1}$$

It is said that the functions $\kappa(t)$ and $\lambda(t)$ conform a kernel pair.

In this report, it is assumed that both $\kappa(t)$ and $\lambda(t)$ have Laplace transforms, leading to

$$s\mathcal{K}(s)\Lambda(s) = 1, \tag{2}$$

for $F(s) = \int_0^\infty e^{-st} f(t)dt$ the Laplace transform of function $f(t)$.

The following properties are considered in [25]:

- $\lim_{t\to\infty} \kappa(t) = 0$ and $\lim_{t\to\infty} \lambda(t) = 0$.
- $\lim_{t\to 0} \kappa(t) = \infty$ and $\lim_{t\to 0} \lambda(t) = \infty$.
- $\lim_{t\to 0} \int_0^t \kappa(\tau)d\tau = 0$ and $\lim_{t\to 0} \int_0^t \lambda(\tau)d\tau = 0$.
- $\lim_{t\to\infty} \int_0^t \kappa(\tau)d\tau = \infty$ and $\lim_{t\to\infty} \int_0^t \lambda(\tau)d\tau = \infty$.

The following definitions constitute the generalized integral and derivative operators [20–22,25].

Definition 2. *Let $f \in \mathcal{L}^\infty_{loc}([0,\infty))$ and $\kappa \in \mathcal{L}^1_{loc}([0,\infty))$, with κ absolutely continuous on $[t_0, t]$ for arbitrary $t_0 > 0$. Then,*

$$\mathscr{I}^{\kappa(t)} f(t) = \int_0^t \kappa(t-\tau) f(\tau)d\tau \tag{3}$$

is the generalized, or Sonine, integral of function $f(t)$ associated to kernel $\kappa(t)$.

Definition 3. *Let $y(t)$ be a good enough continuous function, such that*

$$\lim_{z\to 0}\lambda(z)|y(t+z)-y(t)|=0.$$

Then, whenever $\int_0^t \frac{\partial\lambda(t-\tau)}{\partial\tau}[y(t)-y(\tau)]d\tau$ exists,

$$\mathscr{D}^{\lambda(t)}y(t)=\lambda(t)[y(t)-y(0)]+\int_0^t \frac{\partial\lambda(t-\tau)}{\partial\tau}[y(t)-y(\tau)]d\tau \tag{4}$$

is the generalized derivative of $y(t)$ with respect to $\lambda(t)$, and $y(t)$ is called λ-differentiable.

If function $y(t)$ is integer-order differentiable, one has that the derivative in (4) can be rewritten in a form that resembles the Caputo derivative [25].

Proposition 1. *Let $y(t)$ be integer-order and λ-differentiable. Then,*

$$\mathscr{D}^{\lambda(t)}y(t)=\mathscr{I}^{\lambda(t)}\dot{y}(t)=\int_0^t \lambda(t-\tau)\dot{y}(\tau)d\tau. \tag{5}$$

The following result is of great interest since it allows us to relate the solution of a generalized differential equation with a well-suited integral equation [25].

Theorem 1. *Consider $y(t)$ a Laplace transformable function, such that*

$$y(t)=y(0)+\int_0^t \kappa(t-\tau)\varphi(\tau)d\tau, \tag{6}$$

for some $\varphi\in\mathcal{L}^{\infty}_{loc}[0,\infty)$. Then,

$$\mathscr{I}^{\kappa(t)}\mathscr{D}^{\lambda(t)}y(t)=y(t)-y(0), \tag{7}$$

and $\varphi(t)=\mathscr{D}^{\lambda(t)}y(t)$ at least almost everywhere.

2.2. Generalized Systems

The sort of systems considered in this report are represented by means of nonlinear integral equations of the form

$$x(t)=x(0)+\int_0^t \kappa(t-\tau)f(\tau,x(\tau))d\tau, \tag{8}$$

where $x:[0,\infty)\to\mathbb{R}$ is the pseudo-state, $t\in[0,\infty)$ is the time, $f\in\mathcal{L}^{\infty}_{loc}([0,\infty))$ is the integrable flow function, and $\kappa(t)$ is part of the kernel pair $\{\kappa(t),\lambda(t)\}$.

System (8) can be rewritten as the generalized differential equation,

$$\mathscr{D}^{\lambda(t)}x(t)=f(t,x(t)). \tag{9}$$

The definition below, for the case of generalized systems, is inspired in [32].

Definition 4. *The point $x=x^*$ is an equilibrium of (9) if $\mathscr{D}^{\lambda(t)}x^*=f(t,x^*)$ for all $t\geq 0$.*

Remark 1. *It can be noted that $\mathscr{D}^{\lambda(t)}c=0$, whenever c is a constant. Then, $x=x^*$ is an equilibrium of (9) if and only if $f(t,x^*)=0$ for all $t\geq 0$. Additionally, considering $y(t)=x(t)-x^*$, Equation (9) becomes*

$$\mathscr{D}^{\lambda(t)}y(t)=\bar{f}(t,y(t)), \tag{10}$$

with $\bar{f}(t, y(t)) = f(t, y(t) + x^*)$*, which possess the equilibrium* $y^* = 0$.

The previous remark implies that one can consider $x^* = 0$ as the equilibrium of (9), otherwise, it is possible to perform a change of coordinates.

3. Main Results

A natural way to prove that an equilibrium cannot be finite-time stable relies on a contrapositive argument, which consists in demonstrating that, if the solutions of (9) converge to x^* in finite-time and stay there afterwards, then, x^* is not an equilibrium of (9). In other words, if $x(t) \to x^*$ after $T < \infty$, then $\mathscr{D}^{\lambda(t)} x(t) = f(t, x(t))$ cannot stay at zero after some finite time $T' \geq T$.

Theorem 2. *Let* $x(t)$ *be a continuous solution of system* (9)*, and suppose that* $\exists T > 0$*, a finite time, such that* $x(t) = 0 \ \forall t \geq T$*. If*

$$\int_0^T \frac{1}{\lambda(t)} \frac{\partial \lambda(t-\tau)}{\partial \tau} d\tau \to 0 \quad as \quad t \to \infty. \tag{11}$$

Then, $\mathscr{D}^{\lambda(t)} x(t)$ *cannot stay at zero after some finite moment.*

Proof. The generalized derivative of $x(t)$ is

$$\mathscr{D}^{\lambda(t)} x(t) = \lambda(t)[x(t) - x(0)] + \int_0^t \frac{\partial \lambda(t-\tau)}{\partial \tau} [x(t) - x(\tau)] d\tau. \tag{12}$$

Additionally, since $x(t) = 0 \ \forall t \geq T$, one gets for $t \geq T$ that

$$\mathscr{D}^{\lambda(t)} x(t) = -\lambda(t) x(0) - \int_0^T \frac{\partial \lambda(t-\tau)}{\partial \tau} x(\tau) d\tau. \tag{13}$$

Suppose there is $T' \geq T$, with $\mathscr{D}^{\lambda(t)} x(t) = 0$ whenever $t \geq T'$. Then,

$$\lambda(t) x(0) = -\int_0^T \frac{\partial \lambda(t-\tau)}{\partial \tau} x(\tau) d\tau, \tag{14}$$

for $t \geq T'$. Furthermore, it is possible to realize that

$$\begin{aligned} |x(0)| &= \frac{1}{\lambda(t)} \left| \int_0^T \frac{\partial \lambda(t-\tau)}{\partial \tau} x(\tau) d\tau \right| \\ &\leq \max_{\varsigma \in [0,T]} |x(\varsigma)| \int_0^T \frac{1}{\lambda(t)} \frac{\partial \lambda(t-\tau)}{\partial \tau} d\tau. \end{aligned} \tag{15}$$

Finally, condition (11) implies that $x(0) = 0$, and $x^* = 0$ is a finite-time stable equilibrium only considering the set of solutions with zero initial conditions. □

In accordance with [29], Theorem 2 also applies in the case of fractional-order systems. Nonetheless, condition (11) in Theorem 2 can be difficult to compute, and is impractical in more general application scenarios. A less stringent condition can be considered if one imposes that $f(t, x(t))$ is continuous in both t and x.

Theorem 3. *Let* $x(t)$ *be a continuous solution of system* (9)*, and suppose that* $\exists T > 0$*, a finite time, such that* $x(t) = 0 \ \forall t \geq T$*. If* $f(t, x(t))$ *is a continuous function on* t *and* x*. Then,* $x^* = 0$ *is not an equilibrium of system* (9)*.*

Proof. Considering the equivalence between (8) and (9), one has for $t \geq T$ that

$$0 = x(0) + \int_0^t \kappa(t-\tau) f(\tau, x(\tau)) d\tau, \tag{16}$$

since $x(t) = 0$ for $t \geq T$.

If one considers that $x^* = 0$ is an equilibrium of (9), one has that $f(\tau, 0) = 0 \ \ \forall \tau \geq 0$ and, consequently,

$$0 = x(0) + \int_0^T \kappa(t-\tau) f(\tau, x(\tau)) d\tau. \tag{17}$$

Relying on the continuity of $f(\cdot)$, the extreme value theorem allows us to express

$$\begin{aligned} |x(0)| &\leq \max_{\varsigma \in [0,T]} |f(\varsigma, x(\varsigma))| \int_0^T \kappa(t-\tau) d\tau \\ &= \max_{\varsigma \in [0,T]} |f(\varsigma, x(\varsigma))| \int_{t-T}^t \kappa(\tau) d\tau. \end{aligned} \tag{18}$$

Finally, remembering that $\kappa(t)$ is absolutely continuous in any closed interval $[t_0, t]$, with $t_0 > 0$, one has that $\kappa(t)$ is bounded and absolutely integrable in $[t-T, t]$ for $t > T$. Therefore,

$$|x(0)| \leq T \max_{\varsigma \in [0,T]} |f(\varsigma, x(\varsigma))| \max_{\varrho \in [t-T,t]} \kappa(\varrho), \tag{19}$$

which implies that $x(0) = 0$ since $k(t) \to 0$ as $t \to \infty$. $\square$

The above result is consistent with [28] for the case of fractional-order systems. Then, it is worth noting that any feedback of the form $u_{\alpha,\gamma}(t) = -\gamma |x(t)|^\alpha \text{sign}(x(t))$, with gain $\gamma > 0$ and exponent $\alpha \in (0,1)$, in $\mathscr{D}^{\lambda(t)} x(t) = u_{\alpha,\gamma}(t)$, is not able to enforce finite-time stable solutions. However, for the integer-order system $\dot{x} = -\gamma |x(t)|^\alpha \text{sign}(x(t))$, the origin $x = 0$ is a finite-time stable equilibrium. If $x \to 0$ in finite-time, $\gamma |x(t)|^\alpha \text{sign}(x(t)) \to 0$ is also in finite-time; but since $x^* = 0$ is an equilibrium, it cannot be finite-time stable, for the case of generalized systems (including those of fractional order).

The case of a continuous flow $f(\cdot)$ is interesting and covers a broad spectrum of potential applications. Nevertheless, discontinuous feedback is very interesting since it considers commuting devices for controller implementation. In the case where $f(\cdot)$ is discontinuous at $x^* = 0$, it is possible that $\mathscr{D}^{\lambda(t)} x(t)$ and $f(t, x(t))$ are not identical at $x = x^*$, but they have the same average or equivalent values. In this latter case, the definition below, which extends the concept of equilibrium, is of particular interest.

Definition 5. *Consider system (8), with $f \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}_+) \cap \mathcal{L}^\infty(\mathbb{R}_+)$. If*

$$\int_{\mathcal{I}} f(t, x^*) dt = 0,$$

for any open interval, $\mathcal{I} \subset [0, \infty)$. Then, the point $x = x^*$ is an equivalent equilibrium of (8).

Theorem 4. *Let $x(t)$ be a continuous solution of system (8), and suppose that $\exists T > 0$, a finite time, such that $x(t) = 0 \ \forall t \geq T$. Then, $x^* = 0$ is not an equivalent equilibrium of (8).*

Proof. We proceed by contradiction, that is, one supposes that $x^* = 0$ is a finite-time stable equivalent equilibrium of system (8).

From the fact that $x^* = 0$ is finite-time stable with convergence time $T > 0$, for $t \geq T$ one has that

$$\begin{aligned} 0 &= x(0) + \int_0^t \kappa(t-\tau) f(\tau, x(\tau)) d\tau \\ &= x(0) + \int_0^T \kappa(t-\tau) f(\tau, x(\tau)) d\tau + \int_T^t \kappa(t-\tau) f(\tau, 0) d\tau. \end{aligned} \tag{20}$$

It is possible to appreciate that the initial condition is bounded as

$$|x(0)| \leq \int_0^T \kappa(t-\tau)|f(\tau,x(\tau))|d\tau + \int_T^t \kappa(t-\tau)|f(\tau,0)|d\tau. \tag{21}$$

For the first integral in the right-hand side of (21) and $t > T$, one has that

$$\begin{aligned} \int_0^T \kappa(t-\tau)|f(\tau,x(\tau))|d\tau &\leq \|f(\varsigma,x(\varsigma))\|_{\mathcal{L}^\infty([0,T])} \int_0^T \kappa(t-\tau)d\tau \\ &= \|f(\varsigma,x(\varsigma))\|_{\mathcal{L}^\infty([0,T])} \int_{t-T}^t \kappa(\tau)d\tau \\ &\leq T\|f(\varsigma,x(\varsigma))\|_{\mathcal{L}^\infty([0,T])} \max_{\varrho\in[t-T,T]} \kappa(\varrho). \end{aligned} \tag{22}$$

Therefore, considering that $\kappa(\tau) \to 0$ as $\tau \to \infty$, one has that $\int_0^T \kappa(t-\tau)|f(\tau,x(\tau))|d\tau \to 0$.

For the second integral and $t > T$, one gets

$$\begin{aligned} \int_T^{t-\varepsilon} \kappa(t-\tau)|f(\tau,0)|d\tau &\leq \max_{\varsigma\in[T,t-\varepsilon]} \kappa(t-\varsigma) \int_T^{t-\varepsilon} |f(\tau,0)|d\tau \\ &\leq \kappa(\varepsilon) \int_T^{t-\varepsilon} |f(\tau,0)|d\tau \end{aligned} \tag{23}$$

for all $\varepsilon \in (0, t-T)$. Then, the fact that $x^* = 0$ is an equivalent equilibrium of (8) implies that $\int_T^{t-\varepsilon} |f(\tau,0)|d\tau = 0$ and, consequently,

$$\int_T^{t-\varepsilon} \kappa(t-\tau)f(\tau,0)d\tau = 0, \ \ for \ \ t-T > \varepsilon > 0. \tag{24}$$

Finally, fixing some $t > T$, and considering the continuity of the function

$$\mathfrak{F}(\varepsilon) = \int_T^{t-\varepsilon} \kappa(t-\tau)f(\tau,0)d\tau \tag{25}$$

at $\varepsilon = 0$, it results in

$$\int_T^t \kappa(t-\tau)f(\tau,0)d\tau = 0, \ \ \forall t \geq T > 0. \tag{26}$$

Therefore, $x(0) = 0$ and, thus, the only way $x^* = 0$ is a finite-time stable equivalent equilibrium of (8) is that $x(0) = x^*$. □

Theorem 4 includes Theorem 3 as a particular case; however, it is convenient to present both cases separately. As a conclusion of Theorem 4, the solution of an integral equation, whose kernel is a member of a Sonine kernel pair, cannot have finite-time stable equilibria with the assumption that its flow is a Lebesgue integrable and an essentially bounded function.

4. Examples

4.1. Fractional-Order Systems with General Analytic Kernels

In accordance with [33], let $[a,b] \subset \mathbb{R}$, $\hat{\alpha}, \hat{\beta} \in \mathbb{C}$ with $\Re(\hat{\alpha}) > 0$, $\Re(\hat{\beta}) > 0$, and $R \in \mathbb{R}_+$ such that $R > (b-a)^{\Re(\hat{\beta})}$. Consider a complex function that is analytic on the disc $\{z \in \mathbb{C} : |z| < D\}$, defined by the locally uniformly convergent power series

$$A\left(t^{\hat{\beta}}\right) = \sum_{k=0}^{\infty} a_k t^{\hat{\beta}k}, \tag{27}$$

where the coefficients a_k could depend on $\hat{\alpha}$ and $\hat{\beta}$. Let $\kappa(t)$ be a general analytic kernel defined by

$$\kappa(t) = t^{\hat{\alpha}-1} A\left(t^{\hat{\beta}}\right); \tag{28}$$

then, the fractional-order integral operator with a general analytic kernel is given by

$$\mathscr{I}^{\kappa(t)} f(t) = \int_0^t (t-\tau)^{\hat{\alpha}-1} A\left((t-\tau)^{\hat{\beta}}\right) f(\tau)\, d\tau = {}^{A}\mathcal{I}_{0+}^{\hat{\alpha},\hat{\beta}} f(t). \tag{29}$$

In this way, considering Definition 1, one has

$$\int_0^t (t-\tau)^{\hat{\alpha}-1} A\left((t-\tau)^{\hat{\beta}}\right) \lambda(\tau)\, d\tau = 1, \tag{30}$$

that is,

$$\mathscr{L}\left\{t^{\hat{\alpha}-1} A\left(t^{\hat{\beta}}\right) * \lambda(t)\right\} = \mathscr{L}\{1\}. \tag{31}$$

Recalling the uniform convergence of the series, it follows that

$$\Lambda(s) \sum_{k=0}^{\infty} a_k \frac{\Gamma\left(\hat{\beta}k + \hat{\alpha}\right)}{s^{\hat{\beta}k+\hat{\alpha}}} = \frac{1}{s}, \tag{32}$$

with $\Gamma(\cdot)$ the gamma function [34]. Defining

$$A_\Gamma\left(s^{-\hat{\beta}}\right) = \sum_{k=0}^{\infty} a_k \Gamma\left(\hat{\beta}k + \hat{\alpha}\right) \left(s^{-\hat{\beta}}\right)^k, \tag{33}$$

one has that

$$\Lambda(s) s^{1-\hat{\alpha}} A_\Gamma\left(s^{-\hat{\beta}}\right) = 1. \tag{34}$$

Finally, according to the Fernandez–Özarslan-Baleanu function [35], one gets

$$\lambda(t) = \mathscr{L}^{-1}\left\{\frac{s^{\hat{\alpha}-1}}{A_\Gamma\left(s^{-\hat{\beta}}\right)}\right\} = \mathscr{A}_0\left(t; \hat{\alpha}, \hat{\beta}, 1\right). \tag{35}$$

From the previous analysis, consider $\hat{\alpha} = \alpha$, $\hat{\beta} = 0$ and the series

$$\begin{aligned} A(1) &= \sum_{k=0}^{\infty} a_k = \frac{1}{\Gamma(\alpha)}, \\ A_\Gamma(1) &= \Gamma(\alpha) \sum_{k=0}^{\infty} a_k = \Gamma(\alpha) A(1) = 1. \end{aligned} \tag{36}$$

Then, from (28) and (35), the associated kernel pair is

$$\begin{aligned} \kappa(t) &= \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \\ \lambda(t) &= \mathscr{L}^{-1}\left\{\frac{s^{\alpha-1}}{A_\Gamma(1)}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s^{1-\alpha}}\right\} = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \end{aligned} \tag{37}$$

where $\alpha \in (0,1)$. By considering the pair (37), the Riemann–Liouville integral of order α is obtained in (29). In addition, since $\kappa(t) \to 0$ as $t \to \infty$, it is proved that fractional-order systems do not have finite-time stable (equivalent) equilibria for an integrable flow $f(\cdot)$. Furthermore, it is also possible to reach the same conclusion by noting that

$$\begin{aligned}\int_0^T \frac{1}{\lambda(t)}\frac{\partial\lambda(t-\tau)}{\partial\tau}d\tau &= \frac{\Gamma(1-\alpha)}{t^{-\alpha}}\left[\frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)}\right]_{\tau=0}^{\tau=T} \\ &= \frac{1}{t^{-\alpha}}\left[(t-T)^{-\alpha}-t^{-\alpha}\right] \\ &= \left[\left(1-\frac{T}{t}\right)^{-\alpha}-1\right] \to 0 \quad \text{as} \quad t\to\infty.\end{aligned} \tag{38}$$

This result has been previously reported in [28] for the case of continuous flows, and in [29] for more general cases that include discontinuous feedback; nonetheless, those contributions rely on a more restrictive class of integro-differential operators than those studied in this report.

A more general version of the above example can be obtained as follows. Let $\hat{\alpha} = \beta$, $\hat{\beta} = \alpha$, and

$$a_k = \frac{(-\omega)^k\Gamma(\gamma+k)}{\Gamma(\alpha k+\beta)\Gamma(\gamma)k!}, \qquad 0<\alpha\leq 1,; \tag{39}$$

then, from (27), the three-parameter Mittag–Leffler function is obtained as

$$A(t^\alpha) = \sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{(-\omega t^\alpha)^k}{k!} = E^{\gamma}_{\alpha,\beta}(-\omega t^\alpha), \tag{40}$$

with $(\gamma)_k$ as the Pochhammer symbol [36]. In addition, considering the generalized geometric series, one has that

$$A_\Gamma\left(s^{-\alpha}\right) = \sum_{k=0}^{\infty} a_k\Gamma(\alpha k+\beta)\left(s^{-\alpha}\right)^k = \sum_{k=0}^{\infty}\frac{(\gamma)_k}{k!}\left(-\omega s^{-\alpha}\right)^k = \frac{1}{(1+\omega s^{-\alpha})^\gamma}. \tag{41}$$

The above gives rise to the kernel pair,

$$\begin{aligned}\kappa(t) &= t^{\beta-1}E^{\gamma}_{\alpha,\beta}(-\omega t^\alpha), \\ \lambda(t) &= \mathscr{L}^{-1}\left\{\frac{s^{\beta-1}}{A_\Gamma(s^{-\alpha})}\right\} = \mathscr{L}^{-1}\left\{\frac{s^{\alpha(-\gamma)-(1-\beta)}}{(s^\alpha+\omega)^{-\gamma}}\right\} = t^{-\beta}E^{-\gamma}_{\alpha,1-\beta}(-\omega t^\alpha).\end{aligned} \tag{42}$$

Considering the pair (42), the Prabhakar integral and Prabhakar derivative, respectively, are obtained [37–39]. For $\beta \in (0,1)$ with $\alpha\gamma < \beta < 1+\alpha\gamma$, one has that

$$\lim_{t\to\infty}\kappa(t) = \lim_{s\to 0} s\mathcal{K}(s) = \lim_{s\to 0}\frac{s^{\alpha\gamma-\beta+1}}{(s^\alpha+\omega)^\gamma} = 0. \tag{43}$$

In addition, $\lim_{t\to\infty}\lambda(t) = 0$, as well as $\int_0^t \kappa(\tau)d\tau$, $\int_0^t \lambda(\tau)d\tau \to \infty$ as $t\to\infty$, showing that those systems, which are modeled through Prabhakar derivatives, do not have finite-time stable equilibria. For example, consider the function depicted in Figure 1 given by

$$x(t) = \begin{cases} 1-t, & t\in[0,1] \\ 0, & t\geq 1. \end{cases} \tag{44}$$

In order to calculate the derivative associated with function (44), from Definition 3 and the kernel pair (42), the Extended Prabhakar derivative is

$$\begin{aligned}\mathscr{D}^{\lambda(t)}_{t_0}x(t) &= (t-t_0)^{-\beta}E^{-\gamma}_{\alpha,1-\beta}(-\omega(t-t_0)^\alpha)[x(t)-x(t_0)] \\ &\quad - \int_{t_0}^{t}(t-\tau)^{-\beta-1}E^{-\gamma}_{\alpha,-\beta}(-\omega(t-\tau)^\alpha)\left[x(t)-x(\tau)\right]d\tau.\end{aligned} \tag{45}$$

Then, according to the function (44), with $t_0 = 0$, it follows that

$$\mathscr{D}^{\lambda(t)} x(t) = t^{-\beta} E^{-\gamma}_{\alpha,1-\beta}(-\omega t^{\alpha})[x(t) - 1] - x(t) \int_0^t (t-\tau)^{-\beta-1} E^{-\gamma}_{\alpha,-\beta}(-\omega(t-\tau)^{\alpha})\, d\tau + \int_0^1 (t-\tau)^{-\beta-1} E^{-\gamma}_{\alpha,-\beta}(-\omega(t-\tau)^{\alpha})\, d\tau - \int_0^1 (t-\tau)^{-\beta-1} E^{-\gamma}_{\alpha,-\beta}(-\omega(t-\tau)^{\alpha})\, \tau\, d\tau. \tag{46}$$

Evaluating the integrals in the above equation leads to

$$\int_0^t (t-\tau)^{-\beta-1} E^{-\gamma}_{\alpha,-\beta}(-\omega(t-\tau)^{\alpha})\, d\tau = t^{-\beta} E^{-\gamma}_{\alpha,-\beta+1}(-\omega t^{\alpha}) \tag{47}$$

$$\int_0^1 (t-\tau)^{-\beta-1} E^{-\gamma}_{\alpha,-\beta}(-\omega(t-\tau)^{\alpha})\, d\tau = t^{-\beta} E^{-\gamma}_{\alpha,-\beta+1}(-\omega t^{\alpha}) - (t-1)^{-\beta} E^{-\gamma}_{\alpha,-\beta+1}(-\omega(t-1)^{\alpha}) \tag{48}$$

$$\int_0^1 (t-\tau)^{-\beta-1} E^{-\gamma}_{\alpha,-\beta}(-\omega(t-\tau)^{\alpha})\, \tau\, d\tau = -(t-1)^{-\beta} E^{-\gamma}_{\alpha,-\beta+1}(-\omega(t-1)^{\alpha}) - (t-1)^{-\beta+1} E^{-\gamma}_{\alpha,-\beta+2}(-\omega(t-1)^{\alpha}) + t^{-\beta+1} E^{-\gamma}_{\alpha,-\beta+2}(-\omega t^{\alpha}). \tag{49}$$

Finally,

$$\mathscr{D}^{\lambda(t)} x(t) = (t-1)^{-\beta+1} E^{-\gamma}_{\alpha,-\beta+2}(-\omega(t-1)^{\alpha}) - t^{-\beta+1} E^{-\gamma}_{\alpha,-\beta+2}(-\omega t^{\alpha}). \tag{50}$$

Figures 2 and 3 show the behavior of function $\lambda(t)$ in Equation (42), associated with the extended Prabhakar derivative of function in (44), resulting in Equation (50). The results are obtained considering the parameters $\alpha = 0.65$, $\beta = 0.5$, $\gamma = -0.25$. Note that there is no finite-time of convergence in the derivative.

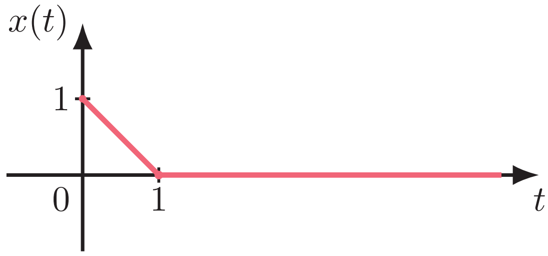

Figure 1. Function $x(t)$ vs. time t.

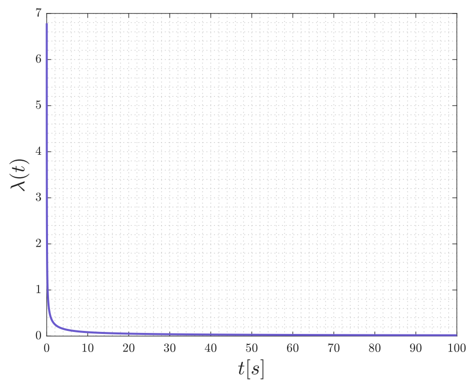

Figure 2. $\lambda(t)$ (Equation (42)) with $\alpha = 0.65$, $\beta = 0.5$, and $\gamma = -0.25$.

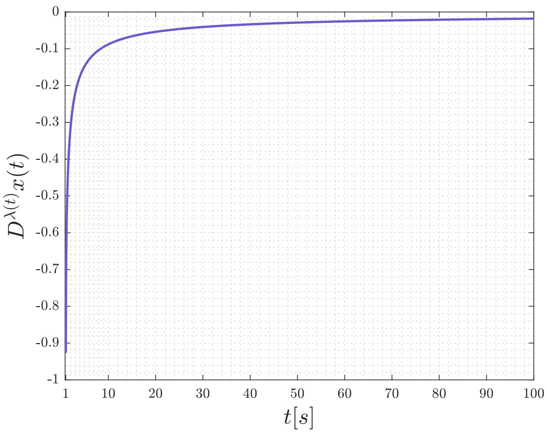

Figure 3. $\mathscr{D}^{\lambda(t)}x(t)$ (Equation (50)) with $\alpha = 0.65$, $\beta = 0.5$, and $\gamma = -0.25$.

4.2. Distributed-Order System

The following system, from [25], is related to a distributed-order system, where the kernel pair is

$$\begin{aligned} \kappa(t) &= \int_t^\infty \frac{e^{t-z}}{z} dz, \\ \lambda(t) &= \int_0^1 \frac{t^{-\alpha}}{\Gamma(1-\alpha)} d\alpha. \end{aligned} \tag{51}$$

It is possible to demonstrate that $s\mathcal{K}(s)\Lambda(s) = 1$ and $\kappa(t) \to 0$ as $t \to \infty$, implying the nonexistence of finite-time stable equilibria for a large class of well-behaved flows. In addition, it is possible to notice that $\lambda'(z)|_{z=t-\tau}$ goes faster to zero than $\lambda(t)$, as with $t \to \infty$, implying that the generalized derivative of a continuous function $x(t)$, with the non-zero Lebesgue measure and compact support, cannot remain at zero after some finite-time.

5. Conclusions

This report studied the finite-time stability concept for the case of a general class of nonlinear integral equations whose kernels belong to some Sonine kernel pairs. It was demonstrated that, under some conditions, the nonexistence of finite-time stable equilibria can be guaranteed. The present result provides a basis upon which to understand the dynamic properties of a large family of integro-differential operators, which include fractional- and distributed-order derivatives and integrals as particular cases, with potential applications in a family of variable-order operators, recently proposed in the literature. This result prevents the search for continuous and discontinuous controllers that guarantee the enforcement of finite-time stable equilibria, in a vast class of generalized systems, although finite-time convergence is still possible thorough the principle of dynamic extension.

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