fractal and fractional

Article

Fractional-Order Nonlinear Multi-Agent Systems: A Resilience-Based Approach to Consensus Analysis with Distributed and Input Delays

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Abstract: In this article, a resilient consensus analysis of fractional-order nonlinear leader and follower systems with input and distributed delays is assumed. To make controller design more practical, it is considered that the controller is not implemented as it is, and a disturbance term is incorporated into the controller part. A multi-agent system's topology ahead to a weighted graph which may be directed or undirected is used. The article examines a scenario of leader–follower consensus through the application of algebraic graph theory and the fractional-order Razumikhin method. Numerical simulations are also provided to show the effectiveness of the proposed design for the leader–follower consensus.

Keywords: fractional-order nonlinear system; Razumikhin approach; input delay; distributed delay; leader–following consensus

MSC: 34K37; 34B15

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1. Introduction

There are different fields where cooperative control has achieved rising importance for its vast application, for example satellite formation flying [1], -multi-vehicle cooperative control [2], etc. Multi-agent system consensus has attained great attention. Great investigation is made on leader–follower consensus [3–7] and nonleader–follower consensus [8–15]. Some researchers give importance to dynamical standards of integer order [16–18]. Therefore, fractional derivatives are convenient for describing many complicated phenomena and processes in comparing with classical derivatives of integer order [19]. By using graph theory and the Lyapunov function method, Yu et al. suppose a group of fractional-order leader–follower consensus [20]. Bai et al. investigated a multi-agent system's consensus by designing a useful controller [21]. By using the proposed lemma and by constructing a suitable Lyapunov function, Xu et al. analyse the replica of complicated systems of fractional-order [22,23].

In a real dynamical system, a delay in time is a common phenomenon which influences the behavioural system of dynamical standards, and due to it, the system can become ambiguous.

An investigation of the fractional-order-delayed system's consensus can be conducted by an analysis method of frequency-domain, such as in [24,25], considering the system's delay in input. Directed multi-agent systems with delays in nonuniform input and communication were studied by Shen et al. [26]. With a delay in heterogeneous input, an undirected multi-agent system is considered in [27]. In time domain analysis, the most suitable access was given by the theory of Lyapunov stability for finding adherence and a complicated dynamical system's consensus.

For dealing with the adherence of differential equations of fractional-order, the most suitable approach was presented by Liu et al. in inequality on Riemann–Liouville derivatives of quadratic function [28]. In many systems of fractional-order, certain stability criteria are obtained by using a proposed lemma [29,30]. By using Caputo sense, a little progress is made in systems of fractional-order through stability analysis. Since the fractional-order operational composition property does not hold, some problems may occur while studying a Caputo fractional-order delayed system's consensus. The main reason for using the Lyapunov direct method is convenience. Xu et al. investigated a fractional-order nonautonomous system's global asymptotical stability by selecting the convenient Lyapunov function [31]. It is still a challenging problem involving which Lyapunov function to select and how to show certain conditions for a fractional-order nonlinear delayed system consensus. Wang et al. compared a fractional-order delayed system's exponential consensus with heterogeneous impulsive controllers, where an undirected graph was drawn with the topology of coupling [32]. Zhu et al. studied systems of fractional-order with delay in input by finding states of error where the topology of coupling headed a directed graph[33]. Generally, the topology of coupling leads to a weighted graph (directed).

The objective of this paper is to analyze the resilient consensus of a nonlinear multi-agent system with distributed and input delays. To achieve this goal, the authors employ the fractional Razumikhin approach and algebraic graph theory to derive algebraic conditions for leader–follower consensus. This paper also includes examples to demonstrate the applicability of the presented cases for consensus checking.
The main contributions can be described as follows.

(1) The parameters of controllers and multi-agent systems are co-designed based on the model of nonlinear MASs. Compared with published results, the obtained fractional-order controller is resilient to uncertainties.
(2) The majority of the results mentioned in previous related references [24–28] deals with the assumptions that the nonlinear part $f(\tau, u(\tau)) = 0$ as $u(\tau) = 0$. However, the remainder of the nonlinear term $f(\tau, u(\tau))$ in the system dynamics model is not negligible and cannot be completely canceled. Since $f(\tau, u(\tau)) \neq 0$ and $u(\tau) = 0$ in many cases, it should be well addressed in the design of the controller. Furthermore, in the abovementioned references, it is assumed that the controllers derived by these techniques are precise, accurate and exactly implemented, but this is not always appropriate as it is difficult to have exact dynamics of the system. Therefore, in this paper, we consider both the effect of uncertainty in the controller and the nonvanishing nonlinearity in multi-agent dynamic systems to enhance the implementation of the controller.

2. Preliminaries

First, we offer some definitions and some important lemmas which will be subsequently used. Consider a directed weighted graph $A = (\mu, \epsilon)$, which contains a set of vertices $B = \{\varsigma_1, \varsigma_2, \varsigma_3, ..., \varsigma_M\}$ and a directed edge's set $\epsilon \subset \{\{\varsigma_j, \varsigma_k\} : \varsigma_j, \varsigma_k \in B\}$, where every directed edge ϵ_{jk} is an ordered pair of vertices $(\varsigma_j, \varsigma_k)$ which shows an edge which originates at vertices ς_k and ends at vertices ς_j. ς_j is called the tail, and ς_k is the head. $M_j = \{\varsigma_k | (\varsigma_k, \varsigma_j) : \varsigma_j, \varsigma_k \in \epsilon\}$ shows the neighbor's set of the vertices ς_j. Consider $C = (c_{jk})_{M\times M}$ as a weighted adjacency matrix, where $c_{jk} > 0$ for $(\varsigma_j, \varsigma_k) \in \epsilon$. Otherwise, $c_{jk} = 0$. Let A be a digraph.Then, its directed spanning tree is a subgraph of A, where by following the directed edge, the root vertices can approach every other vertex [34].

Let $D = (I_{jk})_{M\times M}$ be the Laplacian matrix of graph A

$$\begin{cases} I_{jk} = -c_{jk}, \ \ j \neq k \\ I_{jj} = \sum_{k=1,k\neq j}^{M} c_{jk} \ \ j = k. \end{cases}$$

It is obvious that $\sum_{k=1}^{M} I_{jk} = 0 \; for \; j = 1, 2, ..., M$.

Lemma 1 ([34]). *The directed graph A has a directed spanning tree if and only if the eigenvalue of the Laplacian matrix D is zero, and all other eigenvalue's real components are non-negative.*

Definition 1 ([19]). *Let $f(\tau)$ be a function. Then, the annotation of Caputo derivative with order α is*

$$ {}^{C}_{\tau_0}D^{\alpha}_{\tau}f(\tau) = \frac{1}{\Gamma(n-\alpha)}\int_{\tau_0}^{\tau}\frac{f^n(s)}{(\tau-s)^{\alpha-n+1}}ds. \tag{1}$$

$0 \le n-1 < \alpha \le n,\ n \in Z^+$.

Definition 2 ([35]). *Let $f : R^m \times R \to R^m$ be a function. Then, it is $QUAD(\Delta, e)$ if there exists a diagonal matrix $\Delta \in R^{m\times m}$ and a constant $e > 0$ satisfy*

$$(x-y)^T[f(\tau,x) - f(\tau,y)] - (x-y)^T\Delta(x-y) \le -e(x-y)^T(x-y),\ for\ any\ x,y \in R^m. \tag{2}$$

Lemma 2 ([36]). *Let E, F, G be three matrices. Then, the inequality*

$$\begin{pmatrix} E & F \\ E^T & G \end{pmatrix} < 0 \tag{3}$$

is equivalent to these inequalities

$$E < 0 \text{ and } G - F^T E^{-1} F < 0. \tag{4}$$

Lemma 3 ([37]). *Let $x(t) \in R^m$ be a continuously differentiable function, $H > 0$. The following relationship holds*

$$\frac{1}{2}{}^{C}_{\tau_0}D^{\alpha}_{\tau}(x^{\tau}(\tau)Hx(\tau) \le x^T(\tau)H{}^{C}_{\tau_0}D^{\alpha}_{\tau}x(\tau), \forall\alpha \in (0,1). \tag{5}$$

Consider $G = \{\theta|\theta : [-r_1, 0] \to R^m$ is continous$\}$ denotes the Banach space containing a supremum norm. Suppose a general fractional nonlinear equation with delay in time

$${}^{C}_{\tau_0}D^{\alpha}_{\tau}u(\tau) = f(\tau, u_\tau), \tau \ge \tau_0 \tag{6}$$

for $0 < \alpha \le 1$ and $u_\tau(\varpi) = u(\tau+\varpi)$, $\varpi \in [-r_1, 0]$, f maps $R\times$ (bounded sets of G) into bounded sets of R^m which satisfy $f(\tau, 0) = 0$.

Lemma 4 ([38]). *Suppose $\gamma_1, \gamma_2, \gamma_3 : R \to R$ are continuous increasing functions, $\gamma_1(s)$ and $\gamma_2(s)$ are positive if s is positive, and $\gamma_1(0) = \gamma_2(0) = 0$, $\dot{\gamma}_2 > 0$ if differential function $J : R \times R^m \to R$ and continuous increasing function $\eta(s)$ greater than s for $s > 0$ exists such that for $\theta \in G$ and $u \in R^m$*

$$\gamma_1(||u||) \le J(\tau, u) \le \gamma_2(||u||), \tag{7}$$

if

$$J(\tau + \phi, \theta(\phi)) \le \eta(J(\tau, \theta(0))).$$

$${}^{C}_{\tau_0}D^{\alpha}_{\tau}J(\tau, \theta(0)) \le -\gamma_3(||\theta(0)||), \tau \ge \tau_0. \tag{8}$$

Zero solution $u = 0$ of equation (6) is asymptotically stable for $\phi \in [-\varpi, 0]$. $u = 0$ is globally stable if $\gamma_1(s) \to \infty$ as $s \to \infty$.

Lemma 5 ([39]). *If $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of matrix $E \in R^{nxn}$ and $\mu_1, \mu_2, ..., \mu_m$, are eigenvalues of matrix $F \in R^{mxm}$, then $\lambda_j\mu_k(j = 1, 2, ..., n, k = 1, 2, ..., m)$ are the eigenvalues of $E \otimes F$.*

3. Leader-Following Consensus

Here, with distributed and input delay, resilient-based consensus analysis of a fractional-order nonlinear multi-agent system is discussed. On the basis of a fractional-order Razumikhin approach, many convenient conditions are shown.

Consider the jth agent is

$$ {}_{\tau_0}^{C}D_{\tau}^{\alpha}u_j(\tau) = Fu_j(\tau) + f(\tau, u_j(\tau)) + G\int_{-r_1}^{0} u_j(\tau+\phi)d\phi + \hat{x}_j(\tau) + \Delta x_j(\tau). \tag{9} $$

$j = 1, 2, ..., M$, where $u_j(\tau) = (u_{j1}(\tau), u_{j2}(\tau), ..., u_{jm}(\tau))^T$ and F is a constant matrix, and $\Delta x_j(\tau)$ is a disturbance in the controller and $\Delta x_j(\tau) = sint$, which is assumed to be bounded here. The leader satisfies

$$ {}_{\tau_0}^{C}D_{\tau}^{\alpha}u_0(\tau) = Fu_0(\tau) + f(\tau, u_0(\tau)) + G\int_{-r_1}^{0} u_0(\tau+\phi)d\phi. \tag{10} $$

The controller will be designed as follows:

$$ x_j(\tau) = K\sum_{k=1}^{M} c_{jk}(u_k(\tau-r_2) - u_j(\tau-r_2)) + Kc_{j0}(u_0(\tau-r_2) - u_j(\tau-r_2)) + \Delta x_j(\tau), j = 1, 2, ..., M \tag{11} $$

where r_2 is the input delay, r_1 is the distributed delay, and K is the constant matrix whose eigenvalues are positive. If a directed connection is present from $u_k(\tau)$ to $u_j(\tau)$, $j = 1, 2, ..., M$, $k = 0, 1, ..., M$. Then, $E = (c_{jk})_{M\times M}$, and $c_{jk} = 0$ otherwise.

Definition 3 ([6])**.** *Under the control law (11), leader–follower consensus of the multi-agent system (9) and (10) is attained if for any $j = 1, 2, ..., M$.*

$$ \lim_{\tau\to\infty} ||u_j(\tau) - u_0(\tau)|| = 0. \tag{12} $$

We need some lemmas and assumptions for obtaining results.

(H1). f is $QUAD(\Delta, e)$.

(H2). With the leader rooted, the multi-agent system's corresponding diagraph has a spanning tree.

Lemma 6 ([33])**.** *Consider $L = D + A_0$, $A_0 = diag(c_{10}, ..., c_{M0})$. (H2) holds if and only if all eigenvalues of matrix L have positive real parts.*

Lemma 7. *According to Lemma 5, if $(H2)$ is satisfied, then eigenvalues of matrix of $L\otimes K$ have non-negative real parts.*

Lemma 8. *If there are a scalar $\beta > 0$ and a scalar $\sigma > 0$ and a positive definite matrix $R > 0$ and if $(H1)$ and $(H2)$ hold, then under the control law (11) the leader and follower consensus of system (9) and (10) can be obtained.*

$$ I_M \otimes (F + F^T + 2\Delta - 2eI_m + \beta I_m + R + \frac{4\Delta x(\tau)}{\sigma}) + \frac{1}{\beta}(L^T\otimes K^T)(L\otimes K) < 0. \tag{13} $$

$$ I_M \otimes (\beta I_m - \frac{1}{r_1}R) + \frac{1}{\beta}(I_M\otimes G^T)(I_M\otimes G) < 0. \tag{14} $$

Proof. Putting Equation (11) in Equation (9)

$$ \begin{aligned} {}_{\tau_0}^{C}D_{\tau}^{\alpha}u_j(\tau) \;=\;& Fu_j(\tau) + f(\tau, u_j(\tau)) + G\int_{-r_1}^{0} u_j(\tau+\phi)d\phi + K\sum_{k=1}^{M} c_{jk}(u_k(\tau-r_2) - u_j(\tau-r_2)) \\ & + Kc_{j0}(u_0(\tau-r_2) - u_j(\tau-r_2)) + \Delta xj(\tau) + \Delta x_j(\tau)). \end{aligned} \tag{15} $$

Subtracting Equation (10) from Equation (15)

$$\begin{aligned}
{}_{\tau_0}^{C}D_{\tau}^{\alpha}u_j(\tau) - {}_{\tau_0}^{C}D_{\tau}^{\alpha}u_0(\tau) &= Fu_j(\tau) + f(\tau, u_j(\tau)) + G\int_{-r_1}^{0} u_j(\tau+\phi)d\phi + K\sum_{k=1}^{M} c_{jk}(u_k(\tau - r_2) \\
&\quad -u_j(\tau - r_2)) + Kc_{j0}(u_0(\tau - r_2) - u_j(\tau - r_2)) + 2\Delta x_j(\tau) - Fu_0(\tau) \\
&\quad -f(\tau, u_0(\tau)) - G\int_{-r_1}^{0} u_0(\tau+\phi)d\phi \\
{}_{\tau_0}^{C}D_{\tau}^{\alpha}[u_j(\tau) - u_0(\tau)] &= F[u_j(\tau) - u_0(\tau)] + f(\tau, u_j(\tau)) - f(\tau, u_0(\tau)) + G\int_{-r_1}^{0}[u_j(t+\phi) \\
&\quad -u_0(\tau+\phi)]d\phi + K\sum_{k=1}^{M} c_{jk}[u_k(\tau - r_2) - u_0(\tau - r_2) + u_0(\tau - r_2) \\
&\quad -u_j(\tau - r_2)] - Kc_{j0}[u_j(\tau - r_2) - u_0(\tau - r_2)] + 2\Delta x_j(\tau).
\end{aligned}$$

$$\begin{aligned}
{}_{\tau_0}^{C}D_{\tau}^{\alpha}[u_j(\tau) - u_0(\tau)] &= F[u_j(\tau) - u_0(\tau)] + f(\tau, u_j(\tau)) - f(\tau, u_0(\tau)) + G\int_{-r_1}^{0}[u_j(t+\phi) - u_0(\tau+\phi)]d\phi \\
&\quad +K\sum_{k=1}^{M} c_{jk}[u_k(\tau - r_2) - u_0(\tau - r_2)] - K\sum_{k=1}^{M} c_{jk}[u_j(\tau - r_2) - u_0(\tau - r_2)] \\
&\quad -Kc_{j0}[u_j(\tau - r_2) - u_0(\tau - r_2)] + 2\Delta x_j(\tau).
\end{aligned} \tag{16}$$

Suppose

$$\omega_j(\tau) = u_j(\tau) - u_0(\tau), j = 1, 2, ..., M. \tag{17}$$

Then, Equation (13) becomes

$$\begin{aligned}
{}_{\tau_0}^{C}D_{\tau}^{\alpha}\omega_j(\tau) &= F\omega_j(\tau) + f(\tau, u_j(\tau)) - f(\tau, u_0(\tau)) + G\int_{-r_1}^{0}\omega_j(\tau+\phi)d\phi + K\sum_{k=1}^{M} c_{jk}(\omega_k(\tau - r_2) \\
&\quad -\omega_j(\tau - r_2)) - Kc_{j0}\omega_j(\tau - r_2) + 2\Delta x_j(\tau).
\end{aligned} \tag{18}$$

Choose a quadratic Lyapunov function

$$J(\tau) = \sum_{j=1}^{M}\omega_j^T(\tau)\omega_j(\tau). \tag{19}$$

From Lemma 3 and along the solutions of (18), find the α-order derivative of $J(\tau)$.

$${}_{\tau_0}^{C}D_{\tau}^{\alpha}J(\tau) \le 2\sum_{j=1}^{M}\omega_j^T(\tau){}_{\tau_0}^{c}D_{\tau}^{\alpha}\omega_j(\tau). \tag{20}$$

Putting Equation (18) in Equation (20).

$$\begin{aligned}
{}_{\tau_0}^{C}D_{\tau}^{\alpha}J(\tau) &\le 2\sum_{j=1}^{M}\omega_j^T(\tau)[F\omega_j(\tau) + f(\tau, u_j(\tau)) - f(\tau, u_0(\tau)) + G\int_{-r_1}^{0}\omega_j(\tau+\phi)d\phi \\
&\quad +K\sum_{k=1}^{M} c_{jk}(\omega_k(\tau - r_2) - \omega_j(\tau - r_2)) - Kc_{j0}\omega_j(\tau - r_2) + 2\Delta x_j(\tau)].
\end{aligned} \tag{21}$$

From (H1), we have

$$\omega_j^T(\tau)[f(\tau, u_j(\tau)) - f(\tau, u_0(\tau))] - \omega_j^T(\tau)\Delta\omega_j(\tau) \le -e\omega_j^T(\tau)\omega_j(\tau).$$

Then,

$$\omega_j^T(\tau)[f(\tau,u_j(\tau))-f(\tau,u_0(\tau))]\leq\omega_j^T(\tau)\Delta\omega_j(\tau)-eI_m\omega_j^T(\tau)\omega_j(\tau).$$

which implies that

$$\omega_j^T(\tau)[f(\tau,u_j(\tau))-f(\tau,u_0(\tau)]\leq\omega_j^T(\tau)(\Delta-eI_m)\omega_j(\tau). \tag{22}$$

Notice that

$$I_{jk}=-c_{jk}, j\neq k \; and \; I_{jj}=\sum_{k=1,k\neq j}^{M}c_{jk},$$

We obtain

$$\begin{aligned}\sum_{k=1}^{M}c_{jk}(\omega_k(\tau)-\omega_j(\tau)) &= \sum_{k=1,k\neq j}^{M}c_{jk}(\omega_k(\tau)-\omega_j(\tau))=\sum_{k=1,k\neq j}^{M}c_{jk}\omega_k(\tau)-\sum_{k=1,k\neq j}^{M}c_{jk}\omega_j(\tau)\\ &= -\sum_{k=1,k\neq j}^{M}I_{jk}\omega_k(\tau)-I_{jj}\omega_j(\tau).\end{aligned}$$

Then, we obtain

$$\sum_{k=1}^{M}c_{jk}(\omega_k(\tau)-\omega_j(\tau))=-\sum_{k=1}^{M}I_{jk}\omega_k(\tau). \tag{23}$$

Now, putting values in Equation (21)

$$\begin{aligned}{}_{\tau_0}^{C}D_{\tau}^{\alpha}J(\tau) &\leq 2\omega_j^T(\tau)\sum_{j=1}^{M}[F\omega_j(\tau)+f(\tau,u_j(\tau))-f(\tau,u_0(\tau))+G\int_{-r_1}^{0}\omega_j(\tau+\phi)d\phi\\ &\quad +K\sum_{k=1}^{M}c_{jk}(\omega_k(\tau-r_2)-\omega_j(\tau-r_2))-Kc_{j0}\omega_j(\tau-r_2)+2\Delta x_j(\tau)]\\ &\leq 2\sum_{j=1}^{M}[\omega_j^T(\tau)F\omega_j(\tau)+\omega_j^T(\tau)(f(\tau,u_j(\tau))-f(\tau,u_0(\tau)))\\ &\quad +2\int_{-r_1}^{0}\sum_{j=1}^{M}\omega_j^T(\tau)G\omega_j(\tau+\phi)d\phi+2\sum_{j=1}^{M}\omega_j^T(\tau)K\sum_{k=1}^{M}c_{jk}(\omega_k(\tau-r_2)-\omega_j(\tau-r_2))\\ &\quad -2\sum_{j=1}^{M}\omega_j^T(\tau)Kc_{j0}\omega_j(\tau-r_2)+2\sum_{j=1}^{M}\omega_j^T(\tau)[2\Delta x_j(\tau)]].\end{aligned}$$

By using Equation (22),

$$\begin{aligned}{}_{\tau_0}^{C}D_{\tau}^{\alpha}J(\tau) &\leq 2\sum_{j=1}^{M}[\omega_j^T(\tau)F\omega_j(\tau)+\omega_j^T(\tau)(\Delta-2eI_m)\omega_j(\tau)]\\ &\quad +2\int_{-r_1}^{0}\sum_{j=1}^{M}\omega_j^T(\tau)G\omega_j(\tau+\phi)d\phi+2\sum_{j=1}^{M}\omega_j^T(\tau)K\sum_{k=1}^{M}c_{jk}(\omega_k(\tau-r_2)-\omega_j(\tau-r_2))\\ &\quad -2\sum_{j=1}^{M}\omega_j^T(\tau)Kc_{j0}\omega_j(\tau-r_2)+4\Delta x(\tau)\omega^T(\tau).\end{aligned}$$

$$\begin{aligned}
{}^{C}_{\tau_0}D^{\alpha}_{\tau}J(\tau) &\leq 2\sum_{j=1}^{M}\omega_j^T(\tau)(F+\Delta-eI_m)\omega_j(\tau)+2\int_{-r_1}^{0}\sum_{j=1}^{M}\omega_j^T(\tau)G\omega_j(\tau+\phi)d\phi \\
&+2\sum_{j=1}^{M}\omega_j^T(\tau)K(-\sum_{k=1}^{M}I_{jk}\omega_k(\tau-r_2))-2\sum_{j=1}^{M}\omega_j^T(\tau)Kc_{j0}\omega_j(\tau-r_2) \\
&+\Delta x_j(\tau)\omega_j(\tau-r_2)+4\Delta x(\tau)\omega^T(\tau) \\
&\leq 2\sum_{j=1}^{M}\omega_j^T(\tau)(F+\Delta-eI_m)\omega_j(\tau)+2\int_{-r_1}^{0}\sum_{j=1}^{M}\omega_j^T(\tau)\omega_j(\tau+\phi)d\phi \\
&-2\sum_{j=1}^{M}\omega_j^T(\tau)K\sum_{k=1}^{M}I_{jk}e_k(\tau-r_2)-2\sum_{j=1}^{M}\omega_j^T(\tau)Kc_{j0}\omega_j(\tau-r_2)+4\Delta x\omega^T(\tau). \\
&\leq 2\omega^T(\tau)(I_M\otimes(F+\Delta-eI_m)\omega(\tau)+2\int_{-r_1}^{0}\omega^T(\tau)(I_M\otimes G)\omega(\tau+\phi)d\phi \\
&-2\omega^T(\tau)(D\otimes K)\omega(\tau-r_2)-2\omega^T(\tau)(A_0\otimes K)\omega(\tau-r_2)+4\Delta x(\tau)\omega^T(\tau) \\
&\leq 2\omega^T(\tau)(I_M\otimes(F+\Delta-eI_m)\omega(\tau)+2\int_{-r_1}^{0}\omega^T(\tau)(I_M\otimes G)\omega(\tau+\phi)d\phi \\
&-2\omega^T(\tau)(D+A_0)\otimes K\omega(\tau-r_2)+4\Delta x(\tau)\omega^T(\tau).
\end{aligned}$$

By using Lemma 6,

$$\begin{aligned}
{}^{C}_{\tau_0}D^{\alpha}_{\tau}J(\tau) &\leq 2\omega^T(\tau)(I_M\otimes(F+\Delta-eI_m)\omega(\tau)+2\int_{-r_1}^{0}\omega^T(\tau)(I_M\otimes G)\omega(\tau+\phi)d\phi \\
&-\omega^T(\tau)(L\otimes K)\omega(\tau-r_2)+4\Delta x(\tau)\omega^T(\tau) \\
&\leq \omega^T(\tau)(I_M\otimes(2F+2\Delta-2eI_m)\omega(\tau)+2\int_{-r_1}^{0}\omega^T(\tau)(I_M\otimes G)\omega(\tau+\phi)d\phi \\
&-2\omega^T(\tau)(L\otimes K)\omega(\tau-r_2)+4\Delta x(\tau)\omega^T(\tau) \\
&\leq \omega^T(\tau)(I_M\otimes(F+F^T+2\Delta-2eI_m)\omega(\tau)+2\int_{-r_1}^{0}\omega^T(\tau)(I_M\otimes G)\omega(\tau+\phi)d\phi \\
&-2\omega^T(\tau)(L\otimes K)\omega(\tau-r_2)+4\Delta x(\tau)\omega^T(\tau).
\end{aligned}$$

where

$$\omega^T(\tau)=(\omega_1^T(\tau),...,\omega_M^T(\tau))^T.$$

Whenever

$$J(\tau+\varpi,u(\tau+\varpi))<\eta J(\tau,u(\tau)), for\ \ all-r\leq\varpi<0$$

here $r=\max\{r_1,r_2\}$,for any $\beta>0$ and for some $\eta>1$

$$\begin{aligned}
{}^{C}_{\tau_0}D^{\alpha}_{\tau}J(\tau) &\leq \omega^T(\tau)(I_M\otimes(F+F^T+2\Delta-2eI_m)\omega(\tau)+2\int_{-r_1}^{0}\omega^T(\tau)(I_M\otimes G)\omega(\tau+\phi)d\phi \\
&-2\omega^T(\tau)(L\otimes K)\omega(\tau-r_2)+4\Delta x(\tau)\omega^T(\tau)+\beta[\eta\omega^T(\tau)(I_M\otimes I_m)\omega(\tau) \\
&-\omega^T(\tau-r_2)(I_M\otimes I_m)\omega(\tau-r_2)]+\int_{-r_1}^{0}\beta[\eta\omega^T(\tau)(I_M\otimes I_m)\omega(\tau) \\
&-\omega^T(\tau+\phi)(I_M\otimes I_m)\omega\omega(\tau+\phi)]d\phi.
\end{aligned}$$

Which implies that

$$
\begin{aligned}
{}^{C}_{\tau_0}D^{\alpha}_{\tau}J(\tau) \leq\ & \omega^T(\tau)[I_M \otimes (F + F^T + 2\Delta - 2eI_m + \beta\eta I_m + R)]\omega(\tau) - 2\omega^T(\tau)(L \otimes K)\omega(\tau - r_2) \\
& -\beta\omega^T(\tau - r_2)(I_M \otimes I_m)\omega(\tau - r_2) + \int_{-r_1}^{0} [\omega^T(\tau)(I_M \otimes (\beta\eta I_m - \frac{1}{r_1}R)\omega(\tau))) + 2\omega^T(\tau)(I_M \otimes G) \\
& \omega(\tau + \phi) - \beta\omega^T(\tau + \phi)(I_M \otimes I_m)\omega(\tau + \phi)]d\phi + \frac{4\Delta x(\tau)\omega^T(\tau)\omega(\tau)}{\omega(\tau)}.
\end{aligned}
$$

$$
\begin{aligned}
{}^{C}_{\tau_0}D^{\alpha}_{\tau}J(\tau) \leq \chi_{r2}^T & \\
& \begin{pmatrix} I_M \otimes (F + F^T + 2\Delta - 2eI_m + \beta\eta I_m + R + \frac{4\Delta x(\tau)}{\omega(\tau)}) & -L \otimes K \\ -L^T \otimes K^T & -\beta I_M \otimes I_m \end{pmatrix}\chi_{r2} \\
& + \int_{-r1}^{0} \chi_\phi^T \begin{pmatrix} I_M \otimes (\beta\eta I_m - \frac{1}{r_1}R & I_M \otimes G \\ I_M \otimes G^T & \beta I_M \otimes I_m \end{pmatrix}\chi_\phi d\phi.
\end{aligned} \tag{24}
$$

where $\chi_{r2} = (\omega^T(\tau), \omega^T(\tau - r_2))^T$,$\chi_\phi = (\omega^T(\tau), \omega^T(\tau + \phi))^T$,$\sigma = \frac{1}{\omega(\tau)}$ and $\phi \in [-r_1, 0]$. Suppose $\eta \to +1$

$$
\begin{aligned}
{}^{C}_{\tau_0}D^{\alpha}_{\tau}J(\tau) \leq \chi_{r2}^T & \\
& \begin{pmatrix} I_M \otimes (F + F^T + 2\Delta - 2eI_m + \beta I_m + R + \frac{4\Delta x(\tau)}{\sigma}) & -L \otimes K \\ -L^T \otimes K^T & -\beta I_M \otimes I_m \end{pmatrix}\chi_{r2} + \int_{-r1}^{0} \chi_\phi^T \\
& \begin{pmatrix} I_M \otimes (\beta I_m - \frac{1}{r_1}R) & I_M \otimes G \\ I_M \otimes G^T & -\beta I_M \otimes I_m \end{pmatrix}\chi_\phi d\phi.
\end{aligned}
$$

By using inequalities (13) and (14) and Lemma 2

$$
\begin{pmatrix} I_M \otimes (F + F^T + 2\Delta - 2eI_m + \beta I_m + R + \frac{4\Delta x(\tau)}{\sigma}) & -L \otimes K \\ -L^T \otimes K^T & -\beta I_M \otimes I_m \end{pmatrix} < 0, \tag{25}
$$

and

$$
\begin{pmatrix} I_m \otimes (\beta I_m - \frac{1}{r_1}R) & I_M \otimes G \\ I_M \otimes G^T & -\beta I_M \otimes I_m \end{pmatrix} < 0. \tag{26}
$$

Systems (13) and (14) are satisfied in the sense of Lemma 2. It shows that if $J(\tau + \varpi, u(\tau + \varpi)) < \eta J(\tau, u(\tau))$, then ${}^{c}_{\tau_0}D^{\alpha}_{\tau}J(\tau) < 0$, for some $\varpi \in [-r, 0]$ and $\eta > 1$. Hence, according to Lemma 4, the system (15) is asymptotically stable. Under the control law (11), for systems (9) and (10), leader–follower consensus is obtained. □

Corollary 1. *If (H1) and (H2) are satisfied, and there are a scalar $\beta > 0$, a scalar $\sigma > 0$ and a non-negative definite matrix $R > 0$, then under control law (11), leader–follower consensus of (9) and (10) can be obtained.*

$$
\lambda_{max}[I_M \otimes (F + F^T + 2\Delta - 2eI_m + \beta I_m + R + \frac{4\Delta x(\tau)}{\sigma}) + \frac{1}{\beta}(L^T \otimes K^T)(L \otimes K)] < 0, \tag{27}
$$

and

$$
\lambda_{max}(\beta I_m - \frac{1}{r_1}R + \frac{1}{\beta}G^T G) < 0. \tag{28}
$$

Lemma 9. *By using our criteria, it is more suitable to calculate a multi-agent system's consensus, in spite of criteria used in [33]. In this paper, the system which is being considered not only contains* *input and distributed delay, but it leads to a weighted directed graph too. Our criteria can be used for more general multi-agent systems because criteria used in [33] lead to a directed graph only. If "$\omega^T(\tau)(I_M \otimes R)\omega(\tau)$" is replaced with $r_1\omega^T(\tau)(I_M \otimes R)\omega(\tau)'$ in the first equation of Formula (24), then we obtain the following theorem.*

Lemma 10. *If (H1) and (H2) are satisfied and if there are scalars σ and $\beta > 0$ and a non-negative definite matrix $R > 0$, then under the control law (11), we can obtain leader–follower consensus of (9) and (10).*

$$I_M \otimes (F + F^T + 2\Delta - 2eI_m + \beta I_m + r_1 R + \frac{4\Delta x(\tau)}{\sigma}) + \frac{1}{\beta}(L^T \otimes K^T)(L \otimes K) < 0, \tag{29}$$

and

$$I_M \otimes (\beta I_m - R) + \frac{1}{\beta}(I_M \otimes G^T)(I_M \otimes G) < 0. \tag{30}$$

The results that are obtained must be suitable for FOMAS having undirected topology.

Lemma 11 ([34])**.** *A graph A is considered connected if and only if its corresponding Laplacian matrix D is a non-negative semi-definite matrix in an undirected graph A. The Laplacian matrix D has a single, nonrepeated eigenvalue of zero, and all other eigenvalues are non-negative.*

Lemma 12. *The system leads to a directed weighted graph in Lemmas 8 and 11. If the system's topology is a connected undirected graph, then only Lemma 1 is replaced, and the proof is continued in the same way.*

4. Numerical Examples

Consider some examples for determining the convenience of results.

Example 1. *Under the control law (11), consider nonlinear system (9) and (10) with four followers and one leader, as shown in Figure 1. Consider the matrices F and G.*

$$F = \begin{pmatrix} -5.3 & 0 & 1.1 \\ 0 & -3.2 & 0 \\ 0 & 0 & -2.5 \end{pmatrix},$$

and

$$G = \begin{pmatrix} 0.19 & 0 & 0 \\ 0.12 & 0.35 & 0 \\ 0 & 0 & 0.17 \end{pmatrix}$$

From Figure 1

$$E = \begin{pmatrix} 0 & 0 & 0 & 0.5 \\ 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.4 \\ 0 & 0 & 0.6 & 0 \end{pmatrix}$$

$$A_0 = \begin{pmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9 \end{pmatrix}$$

$$D = \begin{pmatrix} 0.5 & 0 & 0 & -0.5 \\ -0.2 & 0.2 & 0 & 0 \\ 0 & 0 & 1.4 & -1.4 \\ 0 & 0 & -0.6 & 0.6 \end{pmatrix}$$

$$L = \begin{pmatrix} 0.5 & 0 & 0 & -0.5 \\ -0.2 & 0.2 & 0 & 0 \\ 0 & 0 & 1.4 & -1.4 \\ 0 & 0 & -0.6 & 0.6 \end{pmatrix} + \begin{pmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9 \end{pmatrix}$$

which implies that

$$L = \begin{pmatrix} 1 & 0 & 0 & -0.5 \\ -0.2 & 0.2 & 0 & 0 \\ 0 & 0 & 1.4 & -1.4 \\ 0 & 0 & -0.6 & 0.75 \end{pmatrix}$$

here $f(\tau, u_j(\tau)) = \cos(u_j(\tau))$, $j = 0, 1, ..., 4$, $\alpha = 0.6$, $r_1 = 0.7$.

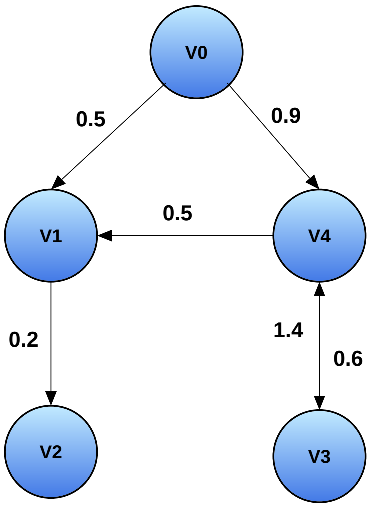

Figure 1. Directed spanning tree shown by nonlinear multi-agent system topology.

Consider $e = 0.3$ *by using assumption (H1).*

$$\Delta = \begin{pmatrix} 0.7 & 0 & 0 \\ 0 & 0.7 & 0 \\ 0 & 0 & 0.7 \end{pmatrix}$$

Take $\beta = 0.4$ *, we have according to (14)*

$I_M \otimes (\beta I_m - \frac{1}{r_1}R) + \frac{1}{\beta}(I_M \otimes G^T)(I_M \otimes G)$.

$= I_M \otimes (\beta I_m - \frac{1}{r_1}R) + I_M \otimes \frac{1}{\beta}G^T G$.

$= I_M \otimes (\beta I_m + \frac{1}{\beta}G^T G - \frac{1}{r_1}R) < 0$. *That is,*

$$\beta r_1 I_m + \frac{r_1}{\beta}G^T G - R < 0. \tag{31}$$

Therefore,

$$R = \begin{pmatrix} 0.72 & 0 & 0.08 \\ 0 & 0.56 & 0 \\ 0.08 & 0 & 0.64 \end{pmatrix}$$

for satisfying (30) can be chosen.

Similarly, (13) implies

$$\beta I_M \otimes (F + F^T + 2\Delta - 2eI_m + R + \frac{4\Delta x(\tau)}{\sigma}) + L^T L \otimes K^T K < 0. \tag{32}$$

Here $\Delta x(\tau) = 20\cos(0.2)$, *and* $\sigma = 0.1$. *Hence,*

$$K = \begin{pmatrix} 0.2 & 0 & 0 \\ 0.3 & 0.5 & 0 \\ 0 & 0 & 0.1 \end{pmatrix}$$

for satisfying (31) can be chosen. According to Lemma 8, systems' (9) and (10) leader–follower consensuses under the control law (11) are obtained, and in Figures 2–5 the error states $\omega_j(\tau)$ *are discussed.*

Example 2. *Let nonlinear systems (9) and (10) under control law (11) with a leader and four followers be as shown in Figure 3. Consider matrices F and G*

$$F = \begin{pmatrix} -3.8 & 0.4 & 0.9 \\ 0 & -4.5 & 0 \\ 1 & 0.3 & -3.7 \end{pmatrix}, G = \begin{pmatrix} 0.19 & 0 & 0.1 \\ 0.16 & 0.3 & 0.1 \\ 0 & 0.2 & 0.1 \end{pmatrix} \tag{33}$$

From Figure 6,

$$E = \begin{pmatrix} 0 & 0.4 & 0 & 0 \\ 0 & 0 & 1.1 & 0.3 \\ 0 & 1.1 & 0 & 0 \\ 0 & 0.3 & 0 & 0 \end{pmatrix}$$

and

$$A_0 = \begin{pmatrix} 0.2 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 0.4 & -0.4 & 0 & 0 \\ -0.4 & 1.8 & -1.1 & -0.3 \\ 0 & -1.1 & 1.1 & 0 \\ 0 & -0.3 & 0 & 0.3 \end{pmatrix}$$

and

$$L = \begin{pmatrix} 0.4 & -0.4 & 0 & 0 \\ -0.4 & 1.8 & -1.1 & -0.3 \\ 0 & -1.1 & 1.1 & 0 \\ 0 & -0.3 & 0 & 0.3 \end{pmatrix} + \begin{pmatrix} 0 & 0.4 & 0 & 0 \\ 0 & 0 & 1.1 & 0.3 \\ 0 & 1.1 & 0 & 0 \\ 0 & 0.3 & 0 & 0 \end{pmatrix}$$

which implies that

$$L = \begin{pmatrix} 0.4 & 0 & 0 & 0 \\ -0.4 & 1.8 & 0 & 0 \\ 0 & 0 & 1.1 & 0 \\ 0 & 0 & 0 & 0.3 \end{pmatrix}$$

here $f(\tau, u_j(\tau)) = \frac{1}{4}\tanh u_j(\tau)$, $j = 0, 1, ..., 4$,$\alpha = 0.5$, $r_1 = 0.3$.

Let $e = 0.3$, $\Delta x(\tau) = 20\sin(0.3)$ *and*

$$\Delta = \begin{pmatrix} 0.6 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.6 \end{pmatrix}$$

Take $\alpha = 0.9$. *For fulfilling the consensus's conditions*

$$R = \begin{pmatrix} 0.48 & 0 & 0.04 \\ 0.2 & 0.52 & 0 \\ 0.04 & 0 & 0.56 \end{pmatrix}$$

and

$$K = \begin{pmatrix} 0.3 & 0 & 0 \\ 0.2 & 0.5 & 0 \\ 0 & 0 & 0.4 \end{pmatrix}$$

can be taken. According to Lemma 8 under control law (11), the leader–follower consensus of systems (9) and (10) with undirected topology is obtained. In Figures 7–10, the error states $\omega_j(\tau)$ *are explained.*

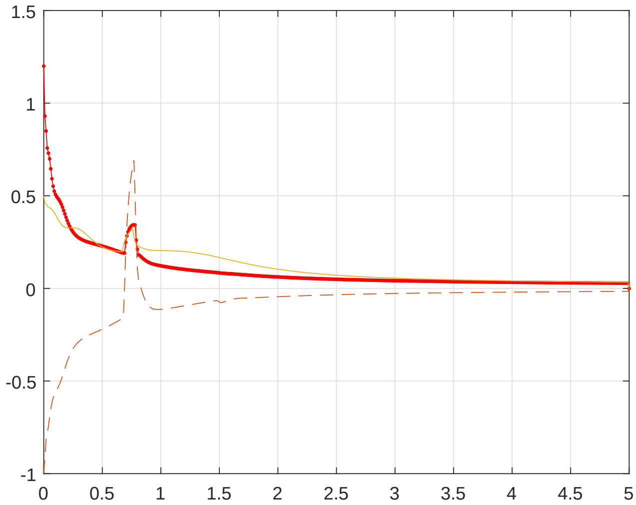

Figure 2. The graph represents the error state $\omega_j(\tau)$ of a multi-agent system following leader.

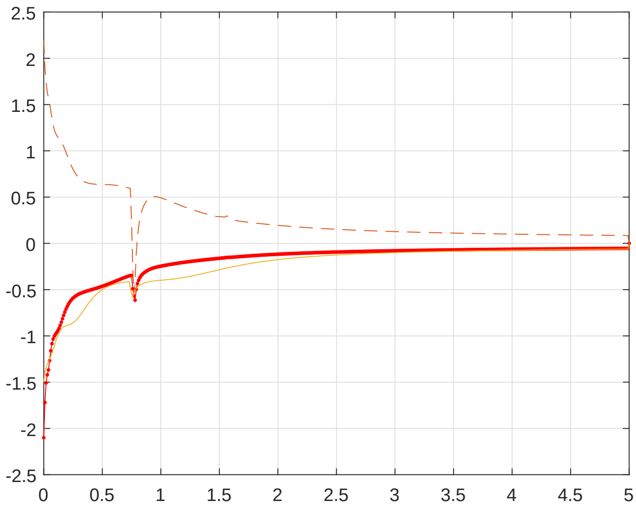

Figure 3. Leader–follower multi-agent system's error state $\omega_j(\tau)$.

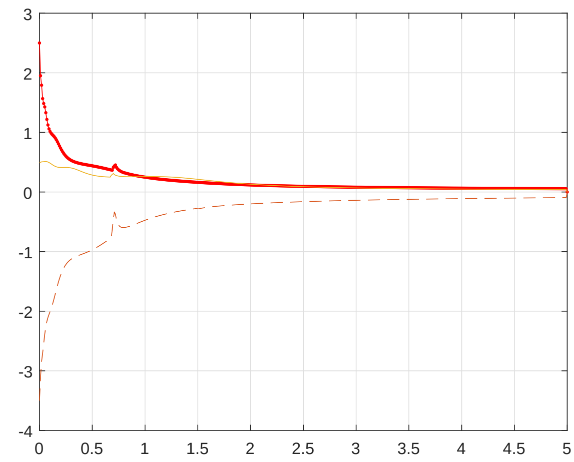

Figure 4. The leader–follower multi-agent system error states $\omega_j(\tau)$.

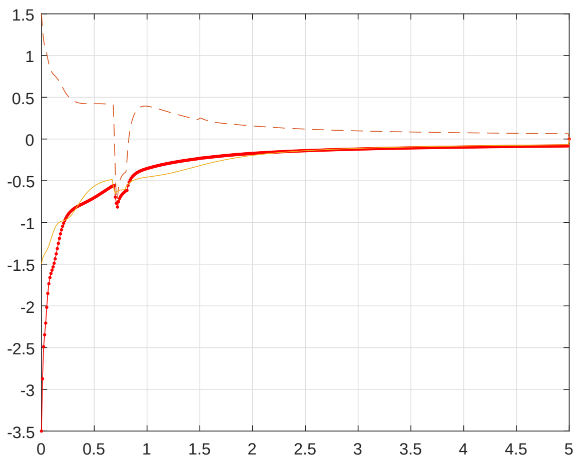

Figure 5. The leader–follower multi-agent system error states $\omega_j(\tau)$.

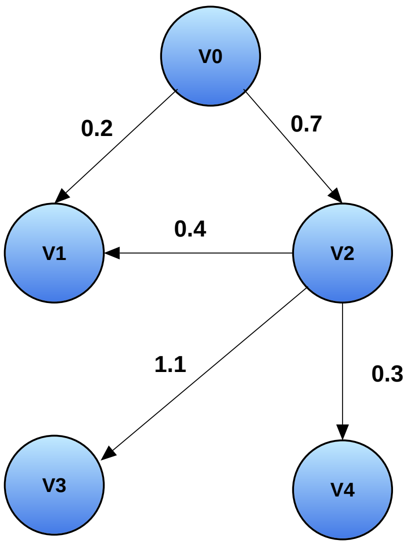

Figure 6. Undirected connected graph shown by topology of a nonlinear multi-agent system.

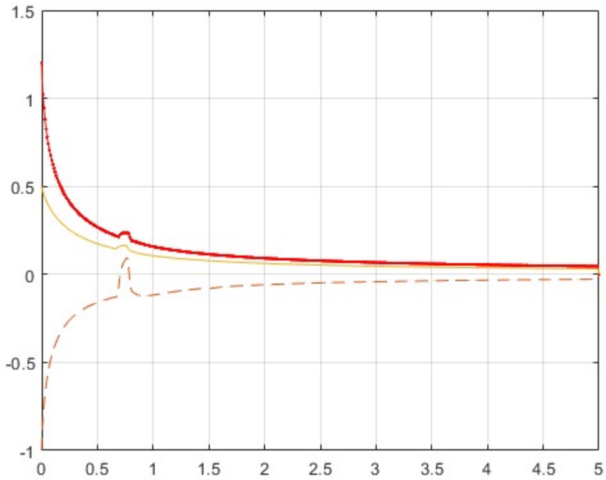

Figure 7. Graph represents error state $\omega_j(\tau)$ of a multi-agent system following leader.

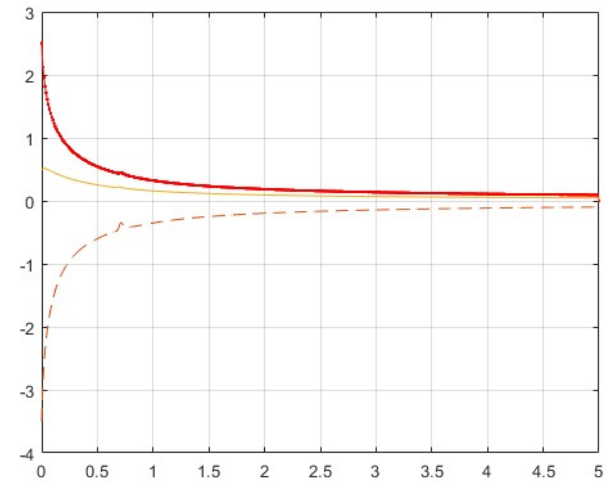

Figure 8. Leader–follower multi-agent system's error state $\omega_j(\tau)$.

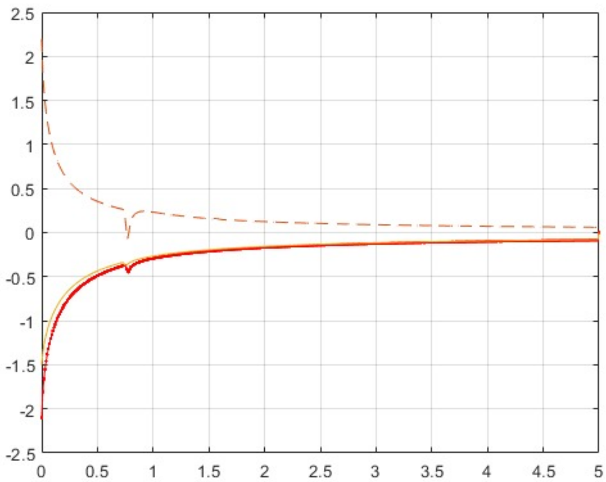

Figure 9. The leader–follower multi-agent system error states $\omega_j(\tau)$.

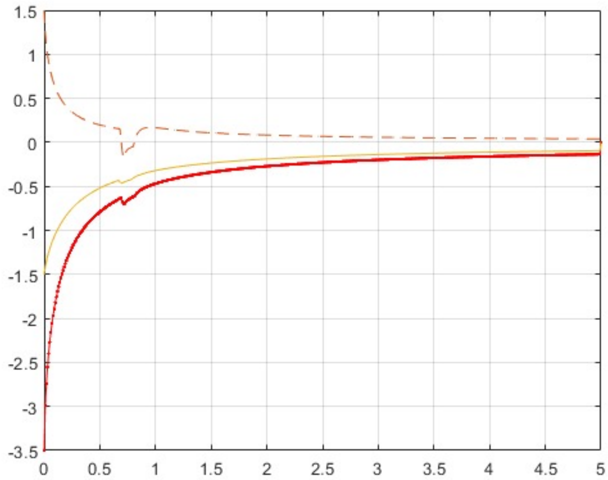

Figure 10. The leader–follower multi-agent system error states $\omega_j(\tau)$.

5. Conclusions

Fractional-order nonlinear leader and follower systems with input and distributed delay-resilient-based consensus were studied. By using the Razumikhin approach, some suitable conditions were achieved. The criteria were expressed as linear matrix inequalities, providing a suitable way to calculate consensus. This multi-agent system leads to a weighted directed graph, and the results obtained are convenient for an undirected graph. This shows that our criteria is more convenient for vast leader and follower systems. Our upcoming research focuses on fractional-order singular multi-agent systems with delayed consensus.

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