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Soliton Solutions of Fractional Stochastic Kraenkel–Manna–Merle Equations in Ferromagnetic Materials

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Abstract: In this study, we take into account the fractional stochastic Kraenkel–Manna–Merle system (FSKMMS). The mapping approach may be used to produce various type of stochastic fractional solutions, such as elliptic, hyperbolic, and trigonometric functions. Solutions to the Kraenkel–Manna–Merle system equation, which explains the propagation of a magnetic field in a zero-conductivity ferromagnet, may provide insight into a variety of fascinating scientific phenomena. Moreover, we construct a variety of 3D and 2D graphics in MATLAB to illustrate the influence of the stochastic term and the conformable derivative on the exact solutions of the FSKMMS.

Keywords: stochastic KMM; fractional KMM; exact solutions; mapping method

MSC: 60H15; 26A33; 34A08; 34A34; 83C15; 35Q51; 60H10

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1. Introduction

Due to the strong progress in information technology and the need for massive data and high-density storage, a considerable amount of research on ferromagnetic materials has been publicly accessible in recent decades. Tiny ferromagnetic particles may now be produced due to recent technical advancements. It is crucial to comprehend the characteristics of microstructures and super microstructures in nanoscale ferrous metals. A magnetic moment may be used to conceptually describe the magnetization of a population of nanoparticles if they are all around the same size. Magnetic moments in ferromagnetic particles move in opposite directions, allowing them to exchange information with one another. These interactions perpetually generate solitons. As a result, many different types of events related to the spread of solitary waves have been studied.

The exact solution of the differential equation must be discovered in order to determine whether the soliton is destroyed after the impact. However, it has long been a difficult but important task to solve nonlinear partial differential equations. Numerous powerful methods for finding exact solutions, such as the generalized Kudryashov approach [1], the (G'/G)-expansion [2,3], the extended simple equation [4], the modified stretched mapping technique [5], the improved $\tan(\varphi/2)$-expansion [6], the Bernoulli subequation function [7], the Ricatti equation expansion [8], Lie symmetry [9], the Exp-function [10], and the sine-Gordon expansion [11,12], have been created by experts in the fields of science and engineering.

Recently, stochastic partial differential equations (SPDEs) have been employed to analyze chemical, biological, and physical systems that are affected by random factors.

The importance of taking random impacts into account when modeling complex systems has been emphasized. SPDEs are used more and more in information systems, condensed matter physics, finance, biophysics, mechanical and electrical engineering, materials sciences, and climate system modeling to create mathematical models of complicated processes [13,14]. As a result, finding exact solutions to fractional or stochastic differential equations is crucial. For the purpose of solving these equations, several analytical and numerical techniques, such as the modified $F-$expansion method [15], the extended tanh–coth method [16], the Riccati–Bernoulli sub-ODE [17], the mapping method [18], the (G'/G)-expansion method [19], etc., have been developed. On the other side, many branches of physics, including solid-state physics, optical fibers, fluid mechanics, plasma physics, neural physics, quantum field theory, and mathematical biology [20–23], make use of fractional differential equations (FDEs). Furthermore, the concept of a fractional derivative has been used to define a wide range of phenomena in fields as diverse as nuclear physics, plasma physics, optical fiber communication, photonics, chaotic systems, wave propagation, electromagnetism, signal processing, ocean waves, fluid dynamics, and porous media.

In this study, we take into account the fractional stochastic Kraenkel–Manna–Merle system (FSKMMS):

$$\begin{cases} \mathfrak{D}_x^\alpha \Phi_t - \Phi \mathfrak{D}_x^\alpha \Psi + \kappa \mathfrak{D}_x^\alpha \Psi = \sigma \mathfrak{D}_x^\alpha \Phi \mathcal{B}_t, \\ \mathfrak{D}_x^\alpha \Psi_t - \Phi \mathfrak{D}_x^\alpha \Phi = \sigma \mathfrak{D}_x^\alpha \Psi \mathcal{B}_t, \end{cases} \tag{1}$$

where the magnetization, represented by $\Phi = \Phi(x,t)$, and the external magnetic fields, represented by $\Psi = \Psi(x,t)$, are related to the ferrite, $\mathfrak{D}_x^\alpha$ is the conformable derivative operator for $\alpha \in (0,1]$, κ represents the damping coefficient, σ is the noise intensity, $\mathcal{B}$ is the Brownian motion, and $\mathcal{B}_t = \frac{\partial \mathcal{B}}{\partial t}$.

If we put $\sigma = 0$ and $\alpha = 1$, then the Kraenkel–Manna–Merle system (KMMS) is obtained:

$$\begin{cases} \Phi_{xt} - \Phi \Psi_x + \kappa \Psi_x = 0, \\ \Psi_{xt} - \Phi \Phi_x = 0, \end{cases} \tag{2}$$

which may characterize the zero-conductivity nonlinear propagation of short waves in saturated ferromagnetic materials. Equation (2) is integrable, and there are Lax pairings when the damping is ignored ($\kappa = 0$). Many authors have created many methods in order to obtain the solutions of KMMS (2) with $\kappa = 0$, such as the auxiliary equation method [24], the (G'/G)-expansion method [25], the inverse scattering method [26], the bilinear method [27], etc. This is the first time that KMMS (2) has involved the presence of both multiplicative noise and fractional derivatives.

The goal of this work is to establish the exact fractional stochastic solutions of the FSKMMS (1) with $\kappa = 0$ by applying the mapping technique. In explaining crucial physical phenomena, the solutions presented would be of great use to physicists. Moreover, we provide many graphical representations using the MATLAB program to investigate the influence of the fractional derivative on the exact solution of the FSKMMS (1).

The structure of the paper is as follows: In Section 2, we define the conformable derivative (CD) and Brownian motion (BM) and discuss some of their features. In Section 3, we find the wave equation for the FSKMMS (1), while the description of the mapping method is given. In Section 5, the mapping method is used to provide an exact solution to the FSKMMS (1). In Section 6, we examine how the fractional derivative and noise influence the obtained solutions of the FSKMMS. Finally, the conclusions of the paper are presented.

2. The CD and BM

Different forms of fractional derivatives have been presented by several mathematicians. The most well-known are the ones proposed by Riesz, Marchaud, Kober, Riemann–Liouville, Erdelyi, Hadamard, Grunwald–Letnikov, and Caputo [28–31]. Recently, Khalil et al. [32] developed a novel fractional derivative identified as the conformable derivative (CD). Over the classical fractional derivatives, the CD offers two benefits. First, the CD definition is natural, and it fulfills the majority of the features of the classical integral derivative, including the mean value theorem, Rolle's theorem, the chain rule, the power rule, linearity, the quotient rule, the product rule, and vanishing derivatives for constant functions. Second, the CD is useful for modeling many physical problems because differential equations with CD are simpler to solve numerically than those with Riemann–Liouville or Caputo fractional derivatives.

In the following, we define the conformable fractional derivative and discuss some of its key characteristics.

Definition 1 ([32]). *For $\alpha \in (0,1]$, the CD of $u : \mathbb{R}^+ \to \mathbb{R}$ is defined as*

$$\mathfrak{D}_x^\alpha u(y) = \lim_{h\to 0} \frac{u(x+hx^{1-\alpha}) - u(x)}{h}.$$

The CD has the following features, if we assume that $u, \varphi : \mathbb{R}^+ \to \mathbb{R}$ are differentiable and α differentiable functions:

1. $\mathfrak{D}_x^\alpha[c_1 u(y) + c_2\varphi(y)] = c_1\mathfrak{D}_x^\alpha u(y) + c_2\mathfrak{D}_x^\alpha\varphi(y)$,
2. $\mathfrak{D}_x^\alpha[c_1] = 0$,
3. $\mathfrak{D}_x^\alpha(u\circ\varphi)(x) = y^{1-\alpha}\varphi'(x)u(\varphi(x))$,
4. $\mathfrak{D}_x^\alpha[x^n] = nx^{n-\alpha}$,
5. $\mathfrak{D}_x^\alpha u(x) = x^{1-\alpha}\frac{du}{dx}$,

for any real constants c_1, c_2.

Moreover, the BM $\mathcal{B}$ is defined as follows [33]:

Definition 2. *The BM $\{\mathcal{B}(\tau)\}_{\tau\geq 0}$ is a stochastic process and fulfills:*

1. $\mathcal{B}(0) = 0$,
2. $\mathcal{B}(t)$ *is continuous for* $t \geq 0$,
3. $\mathcal{B}(t_2) - \mathcal{B}(t_1)$ *is independent for* $t_2 > t_1$,
4. $\mathcal{B}(t_2) - \mathcal{B}(t_1)$ *has a normal distribution* $N(0, t_2 - t_1)$.

We need the following lemma:

Lemma 1 ([33]). $\mathbb{E}(e^{\rho\mathcal{B}(t)}) = e^{\frac{1}{2}\rho^2 t}$ *for* $\rho \geq 0$.

3. The Traveling Wave Equation for the FSKMMS

Considering the zero dumping effect ($\kappa = 0$), we utilize the next wave transformation

$$\Phi(x,t) = \varphi(\xi)e^{(\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t)},\ \Psi(x,t) = \psi(\xi)e^{(\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t)},\ \text{and } \xi = \frac{1}{\alpha}\xi_1 x^\alpha + \xi_2 t, \tag{3}$$

where $\varphi(\xi)$ and $\psi(\xi)$ are real functions, ξ_1 and ξ_2 are nonzero constants, and we are able to obtain the wave equation of the FSKMMS (1). We note that

$$\mathfrak{D}_x^\alpha\Phi = \xi_1\varphi' e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]},\ \mathfrak{D}_x^\alpha\Phi_t = [\xi_1\xi_2\varphi'' + \sigma\xi_1\varphi'\mathcal{B}_t]e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]}, \tag{4}$$

and

$$\begin{aligned}\mathfrak{D}_x^\alpha\Phi_t &= [\xi_1\xi_2\varphi'' + \frac{1}{2}\xi_1\sigma^2\varphi' + \sigma\xi_1\varphi'\mathcal{B}_t - \frac{1}{2}\xi_1\sigma^2\varphi']e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]} \\ &= [\xi_1\xi_2\varphi'' + \sigma\xi_1\varphi'\mathcal{B}_t]e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]},\end{aligned}$$

$$\mathfrak{D}_x^\alpha\Psi = \xi_1\psi' e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]},\ \mathfrak{D}_x^\alpha\Psi_t = [\xi_1\xi_2\psi'' + \sigma\xi_1\psi'\mathcal{B}_t]e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]}. \tag{5}$$

Inserting Equation (3) into Equation (1) and utilizing (4) and (5), we obtain

$$\begin{cases} \xi_1\xi_2\varphi'' - \xi_1\Phi\psi' e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]} = 0, \\ \xi_1\xi_2\psi'' - \xi_1\varphi\varphi' e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]} = 0. \end{cases} \tag{6}$$

Taking the expectation on both sides, we obtain

$$\begin{cases} \xi_2\varphi'' - \varphi\psi' e^{-\frac{1}{2}\sigma^2 t}\mathbb{E}e^{[\sigma\mathcal{B}(t)]} = 0, \\ \xi_2\psi'' - \varphi\varphi' e^{-\frac{1}{2}\sigma^2 t}\mathbb{E}e^{[\sigma\mathcal{B}(t)]} = 0. \end{cases} \tag{7}$$

Using Lemma 2, where $\mathcal{B}(t)$ is a normal process with $\mathbb{E}(e^{\sigma\mathcal{B}(t)}) = e^{\frac{1}{2}\sigma^2 t}$, Equation (7) becomes

$$\begin{cases} \xi_2\varphi'' - \varphi\psi' = 0, \\ \xi_2\psi'' - \varphi\varphi' = 0. \end{cases} \tag{8}$$

Integrating the second equation in (8), we have

$$\psi' = \frac{1}{2\xi_2}\varphi^2 + \frac{c_0}{\xi_2}. \tag{9}$$

Substituting Equation (9) into the first equation in (8), we obtain

$$\varphi'' + \ell_1\varphi^3 + \ell_2\varphi = 0, \tag{10}$$

where

$$\ell_1 = \frac{-1}{2\xi_2^2} \text{ and } \ell_2 = \frac{-c_0}{\xi_2^2}.$$

4. Description of the Mapping Method

Here, let us describe the mapping method mentioned in [34]. We suppose that the solutions to Equation (10) are

$$\varphi(\xi) = \sum_{k=0}^{M} \hbar_k \mathcal{G}^k(\xi), \tag{11}$$

where $\hbar_i$, for $i = 1, 2, \ldots \hbar_M$, are unknown constants, and $\mathcal{G}$ is the solution of

$$\mathcal{G}' = \sqrt{\gamma_1 + \gamma_3\mathcal{G}^2 + \gamma_2\mathcal{G}^4}, \tag{12}$$

where the parameters γ_1, γ_3, and γ_2 are real numbers.

We can observe that Equation (12) has multiple solutions dependent on γ_1, γ_2, and γ_3, as shown in the following Table 1.

Table 1. All solutions for Equation (12) for different values of γ_1, γ_2, and γ_3 .

Case	γ_1	γ_2	γ_3	$\mathcal{G}(\xi)$
1	1	$\mathbf{m}^2$	$-(1+\mathbf{m}^2)$	$sn(\xi)$
2	$-\mathbf{m}^2(1-\mathbf{m}^2)$	1	$2\mathbf{m}^2-1$	$ds(\xi)$
3	$(1-\mathbf{m}^2)$	1	$2-\mathbf{m}^2$	$cs(\xi)$
4	$(1-\mathbf{m}^2)$	$-\mathbf{m}^2$	$2\mathbf{m}^2-1$	$cn(\xi)$
5	$(\mathbf{m}^2-1)$	-1	$2-\mathbf{m}^2$	$dn(\xi)$
6	$\frac{1}{4}$	$\frac{\mathbf{m}^2}{4}$	$\frac{(\mathbf{m}^2-2)}{2}$	$\frac{sn(\xi)}{1\pm dn(\xi)}$
7	$\frac{\mathbf{m}^2}{4}$	$\frac{\mathbf{m}^2}{4}$	$\frac{(\mathbf{m}^2-2)}{2}$	$\frac{sn(\xi)}{1\pm dn(\xi)}$
8	$\frac{-(1-\mathbf{m}^2)^2}{4}$	$\frac{-1}{4}$	$\frac{(\mathbf{m}^2+1)}{2}$	$\mathbf{m}cn(\xi) \pm dn(\xi)$
9	$\frac{(\mathbf{m}^2-1)}{4}$	$\frac{\mathbf{m}^2-1}{4}$	$\frac{(\mathbf{m}^2+1)}{2}$	$\frac{dn(\xi)}{1\pm sn(\xi)}$

Table 1. *Cont.*

Case	γ_1	γ_2	γ_3	$\mathcal{G}(\xi)$
10	$\frac{(1-\mathbf{m}^2)}{4}$	$\frac{1-\mathbf{m}^2}{4}$	$\frac{(1-\mathbf{m}^2)}{2}$	$\frac{cn(\xi)}{1\pm sn(\xi)}$
11	$\frac{1}{4}$	$\frac{(1-\mathbf{m}^2)^2}{4}$	$\frac{(1-\mathbf{m}^2)^2}{2}$	$\frac{sn(\xi)}{dn\pm cn(\xi)}$
12	0	1	0	$\frac{c}{\xi}$
13	0	0	1	ce^{ξ}

where $sn(\xi) = sn(\xi, \mathbf{m})$, $dn(\xi, \mathbf{m}) = dn(\xi, \mathbf{m})$, and $cn(\xi) = cn(\xi, \mathbf{m})$, for $0 < \mathbf{m} < 1$, represent the Jacobi elliptic functions (JEFs). The JEFs become the following hyperbolic functions when $\mathbf{m} \to 1$:

$$\begin{aligned} dn(\xi) &\to \operatorname{sech}(\xi),\ sn(\xi) \to \tanh(\xi),\ ds \to \operatorname{csch}(\xi) \\ cn(\xi) &\to \operatorname{sech}(\xi),\ cs(\xi) \to \operatorname{csch}(\xi). \end{aligned}$$

The JEFs become the following trigonometric functions when $\mathbf{m} \to 0$:

$$\begin{aligned} cn(\xi) &\to \cos(\xi),\quad sn(\xi) \to \sin(\xi),\ dn(\xi) \to 1, \\ ds &\to \csc(\xi),\quad cs(\xi) \to \cot(\xi). \end{aligned}$$

5. Exact Solutions of the FSKMMS

Now, we balance φ'' with φ^3 in Equation (10) to calculate M as follows

$$M + 3 = 2M \ \Rightarrow\ M = 1.$$

We rewrite Equation (12) with $M = 1$ as

$$\varphi(\xi) = \hbar_0 + \hbar_1 \mathcal{G}(\xi). \tag{13}$$

Differentiating Equation (13) twice and using (12), we obtain

$$\varphi'' = \hbar_1 \gamma_3 \mathcal{G} + \hbar_1 \gamma_2 \mathcal{G}^3. \tag{14}$$

Putting Equations (13) and (14) into Equation (10), we have

$$(\hbar_1\gamma_2 + \ell_1\hbar_1^3)\mathcal{G}^3 + 3\hbar_0\hbar_1^2\ell_1\mathcal{G}^2 + (\hbar_1\gamma_3 + 3\ell_1\hbar_0^2\hbar_1 + \ell_2\hbar_1)\mathcal{G} + (\ell_1\hbar_0^3 + \ell_2\hbar_0) = 0.$$

Comparing each coefficient of $\mathcal{G}^j$ with zero for $j = 0, 1, 2, 3$, we obtain

$$\hbar_1\gamma_2 + \ell_1\hbar_1^3 = 0,$$

$$3\hbar_0\hbar_1^2\ell_1 = 0,$$

$$\hbar_1\gamma_3 + 3\ell_1\hbar_0^2\hbar_1 + \ell_2\hbar_1 = 0,$$

and

$$\ell_1\hbar_0^3 + \ell_2\hbar_0 = 0.$$

When we solve these equations, we obtain

$$\hbar_0 = 0,\ \ \hbar_1 = \pm\sqrt{\frac{2\gamma_2 c_0}{\gamma_3}},\ \ \xi_2 = \pm\sqrt{\frac{c_0}{\gamma_3}}.$$

Thus, the solution of the wave Equation (10) is:

$$\varphi(\zeta) = \pm\sqrt{\frac{2\gamma_2 c_0}{\gamma_3}}\mathcal{G}(\zeta). \tag{15}$$

There are many cases relying on γ_2, γ_3, and c_0, such that $\gamma_2 > 0$ and $\frac{c_0}{\gamma_3} > 0$, by using Table 1 as follows:

Case 1: With $\gamma_2 = \mathbf{m}^2$, $\gamma_3 = -(\mathbf{m}^2 + 1)$, and $c_0 < 0$, $\mathcal{G}(\zeta) = sn(\zeta)$, Equation (15) becomes

$$\varphi(\zeta) = \pm\sqrt{\frac{-2\mathbf{m}^2 c_0}{(\mathbf{m}^2+1)}}sn(\zeta).$$

Therefore, the solution of FSKMMS (1) is

$$\Phi(x,t) = \pm\sqrt{\frac{-2\mathbf{m}^2 c_0}{(\mathbf{m}^2+1)}}sn(\zeta)e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]}, \tag{16}$$

$$\Psi(x,t) = [\mathbf{m}^2\sqrt{\frac{-c_0}{(\mathbf{m}^2+1)}}\int sn^2(\zeta)d\zeta - \frac{1}{2}\zeta(\mathbf{m}^2+1)]e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]}, \tag{17}$$

where $\zeta = \frac{\zeta_1}{\alpha}x^\alpha \pm \sqrt{\frac{-c_0}{(\mathbf{m}^2+1)}}t$. If $\mathbf{m} \to 1$, then Equations (16) and (17) transfer into

$$\Phi(x,t) = \pm\sqrt{-c_0}\tanh(\zeta)e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]}, \tag{18}$$

$$\Psi(x,t) = [(\sqrt{\frac{-c_0}{2}} - 1)\zeta - \sqrt{-c_0}\tanh(\zeta)]e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]}, \tag{19}$$

where $\zeta = \frac{\zeta_1}{\alpha}x^\alpha \pm \sqrt{\frac{-c_0}{2}}t$.

Case 2: If $\gamma_2 = 1$, $\gamma_3 = (2-\mathbf{m}^2)$, and $c_0 > 0$, then $\mathcal{G}(\zeta) = cs(\zeta)$, and Equation (15) becomes

$$\varphi(\zeta) = \pm\sqrt{\frac{2c_0}{2-\mathbf{m}^2}}cs(\zeta).$$

Therefore, the solution of FSKMMS (1) is

$$\Phi(x,t) = \pm\sqrt{\frac{2c_0}{2-\mathbf{m}^2}}cs(\zeta)e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]}, \tag{20}$$

$$\Psi(x,t) = [\frac{1}{2}(2-\mathbf{m}^2)\xi + \sqrt{\frac{c_0}{2-\mathbf{m}^2}}\int cs^2(\zeta)d\xi]e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]}, \tag{21}$$

where $\zeta = \frac{\zeta_1}{\alpha}x^\alpha \pm \sqrt{\frac{c_0}{2-\mathbf{m}^2}}t$.

If $\mathbf{m} \to 1$, then Equations (20) and (21) transfer into

$$\Phi(x,t) = \pm\sqrt{2c_0}\operatorname{csch}(\zeta)e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]}, \tag{22}$$

$$\Psi(x,t) = [(\sqrt{c_0} + \frac{1}{2})\zeta - \sqrt{c_0}\coth(\zeta)]e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]}, \tag{23}$$

where $\zeta = \frac{\zeta_1}{\alpha}x^\alpha \pm \sqrt{c_0}t$.

Case 3: If $\gamma_2 = 1,\ \gamma_3 = 2\mathbf{m}^2 - 1$, and either $c_0 > 0$ for $m < \frac{1}{\sqrt{2}}$ or $c_0 < 0$ for $m > \frac{1}{\sqrt{2}}$, then $\mathcal{G}(\zeta) = ds(\zeta)$, and Equation (15) becomes

$$\varphi(\zeta) = \pm\sqrt{\frac{2c_0}{2\mathbf{m}^2 - 1}}ds(\zeta).$$

Therefore, the solution of FSKMMS (1) is

$$\Phi(x,t) = \pm\sqrt{\frac{2c_0}{2\mathbf{m}^2 - 1}}ds(\zeta)e^{[\sigma\mathcal{B}(t) - \frac{1}{2}\sigma^2 t]}, \tag{24}$$

$$\Psi(x,t) = [\frac{1}{2}(2\mathbf{m}^2 - 1)\xi + \sqrt{\frac{c_0}{2\mathbf{m}^2 - 1}}\int ds^2(\zeta)d\xi]e^{[\sigma\mathcal{B}(t) - \frac{1}{2}\sigma^2 t]}, \tag{25}$$

where $\zeta = \frac{\zeta_1}{\alpha}x^\alpha \pm \sqrt{\frac{c_0}{2\mathbf{m}^2-1}}t$. If $\mathbf{m} \to 1$, then Equations (24) and (25) transfer into

$$\Phi(x,t) = \pm\sqrt{2c_0}\text{csch}(\zeta)e^{[\sigma\mathcal{B}(t) - \frac{1}{2}\sigma^2 t]}, \tag{26}$$

$$\Psi(x,t) = [(\frac{1}{2} + \sqrt{c_0})(\zeta) - \sqrt{c_0}\coth(\zeta)]e^{[\sigma\mathcal{B}(t) - \frac{1}{2}\sigma^2 t]}, \tag{27}$$

with $\zeta = \frac{\zeta_1}{\alpha}x^\alpha \pm \sqrt{c_0}t$ for $c_0 > 0$. If $\mathbf{m} \to 0$, then Equations (24) and (25) become

$$\Phi(x,t) = \pm\sqrt{-2c_0}\csc(\zeta)e^{[\sigma\mathcal{B}(t) - \frac{1}{2}\sigma^2 t]}, \tag{28}$$

$$\Psi(x,t) = [(\frac{1}{2} + \sqrt{-c_0})(\zeta) + \sqrt{-c_0}\cot(\zeta)]e^{[\sigma\mathcal{B}(t) - \frac{1}{2}\sigma^2 t]}, \tag{29}$$

where $\zeta = \frac{\zeta_1}{\alpha}x^\alpha \pm \sqrt{-c_0}t$ for $c_0 < 0$.

Case 4: With $\gamma_2 = \frac{\mathbf{m}^2}{4},\ \gamma_3 = \frac{(\mathbf{m}^2-2)}{2}$, and $c_0 < 0$, then $\mathcal{G}(\zeta) = ns(\zeta) + ds(\zeta)$, and Equation (15) becomes

$$\varphi(\zeta) = \pm\sqrt{\frac{c_0\mathbf{m}^2}{\mathbf{m}^2 - 2}}(ns(\zeta) + ds(\zeta)).$$

Therefore, the solution of FSKMMS (1) is

$$\Phi(x,t) = \pm\sqrt{\frac{c_0\mathbf{m}^2}{\mathbf{m}^2 - 2}}[ns(\zeta) + ds(\zeta)]e^{[\sigma\mathcal{B}(t) - \frac{1}{2}\sigma^2 t]}, \tag{30}$$

$$\Psi(x,t) = [(\frac{(\mathbf{m}^2 - 2)}{4}\zeta + \frac{\mathbf{m}^2}{4}\sqrt{\frac{2c_0}{\mathbf{m}^2 - 2}}\int [ns(\zeta) + ds(\zeta)]^2 d\zeta]e^{[\sigma\mathcal{B}(t) - \frac{1}{2}\sigma^2 t]}, \tag{31}$$

where $\zeta = \frac{\zeta_1}{\alpha}x^\alpha \pm \sqrt{\frac{2c_0}{\mathbf{m}^2-2}}t$. If $\mathbf{m} \to 1$, then Equations (30) and (31) transfer into

$$\Phi(x,t) = \pm\sqrt{-c_0}[\coth(\zeta) - \text{csch}(\zeta)]e^{[\sigma\mathcal{B}(t) - \frac{1}{2}\sigma^2 t]}, \tag{32}$$

$$\Psi(x,t) = [\frac{\mathbf{1}}{4}(\sqrt{-2c_0} - 1)\zeta + \frac{\mathbf{1}}{4}\sqrt{-2c_0}(\text{csch}(\zeta) + \coth(\zeta))]e^{[\sigma\mathcal{B}(t) - \frac{1}{2}\sigma^2 t]}, \tag{33}$$

with $\zeta = \frac{\zeta_1}{\alpha}x^\alpha \pm \sqrt{-2c_0}t$.

Case 5: If $\gamma_2 = \frac{(1-\mathbf{m}^2)^2}{4},\ \gamma_3 = \frac{(1-\mathbf{m}^2)^2}{2}$, and $c_0 > 0$, then $\mathcal{G}(\zeta) = \frac{sn(\zeta)}{dn \pm cn(\zeta)}$, and Equation (15) becomes

$$\varphi(\zeta) = \pm\sqrt{c_0}\frac{sn(\zeta)}{dn \pm cn(\zeta)}.$$

Therefore, the solution of FSKMMS (1) is

$$\Phi(x,t) = \pm\sqrt{c_0}\frac{sn(\zeta)}{dn \pm cn(\zeta)}e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]}, \tag{34}$$

$$\Psi(x,t) = [\frac{(1-\mathbf{m}^2)^2}{4}\zeta - \frac{(1-\mathbf{m}^2)}{4}\sqrt{2c_0}\int\frac{sn^2(\zeta)}{(dn+cn(\zeta))^2}d\zeta]e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]}, \tag{35}$$

where $\zeta = \frac{\zeta_1}{\alpha}x^\alpha \pm \sqrt{\frac{2c_0}{(1-\mathbf{m}^2)^2}}t$. If $\mathbf{m} \to 0$, then Equations (34) and (35) become

$$\Phi(x,t) = \pm\sqrt{c_0}[\csc(\zeta) - \cot(\zeta)]e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]}, \tag{36}$$

$$\Psi(x,t) = [\frac{1}{4}(1-\sqrt{2c_0})\zeta - \frac{1}{2}\sqrt{2c_0}(\cot(\zeta) - \csc(\zeta))]e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]}, \tag{37}$$

with $\zeta = \frac{\zeta_1}{\alpha}x^\alpha \pm \sqrt{2c_0}t$.

6. The Influences of the CD and BM

Next, we address the impact of the CD and BM on the exact solution of FSKMMS (1). We provide a number of graphs to demonstrate the current state of these solutions. For the obtained solutions, such as (16), (18), and (19), we simulate these graphs using the parameters $\xi_1 = 1,\ x \in [0,6]$, and $t \in [0,6]$.

Influences of the CD: The following graphs illustrate how the CD affected the obtained solutions when $\sigma = 0$:

From Figures 1–3, we see that the solution curves do not intersect for any given set of values of α.

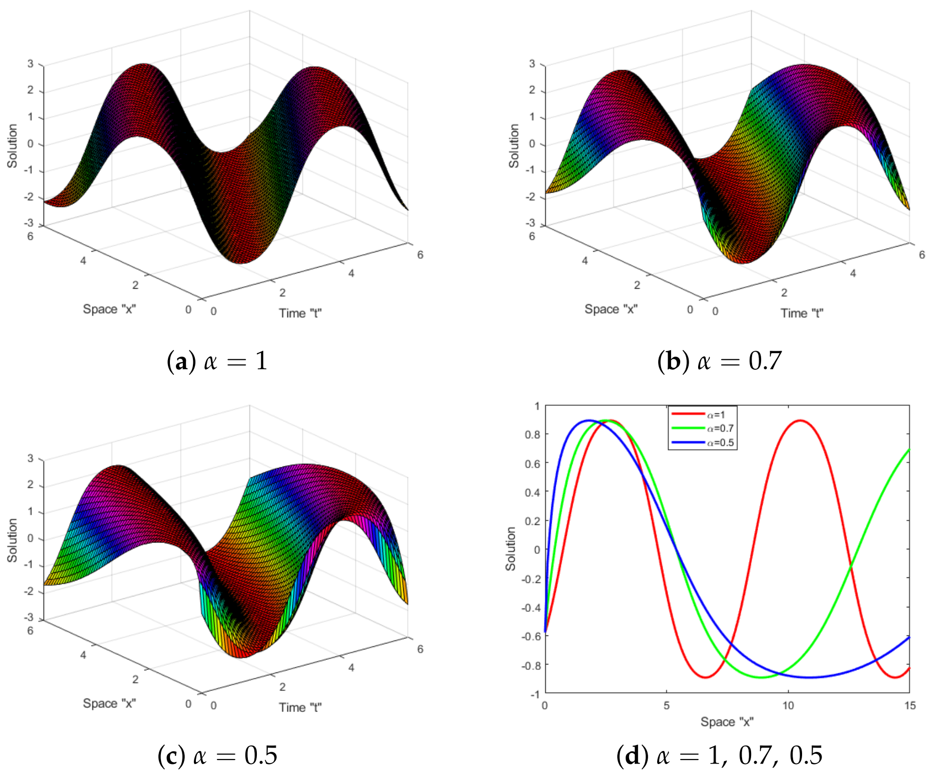
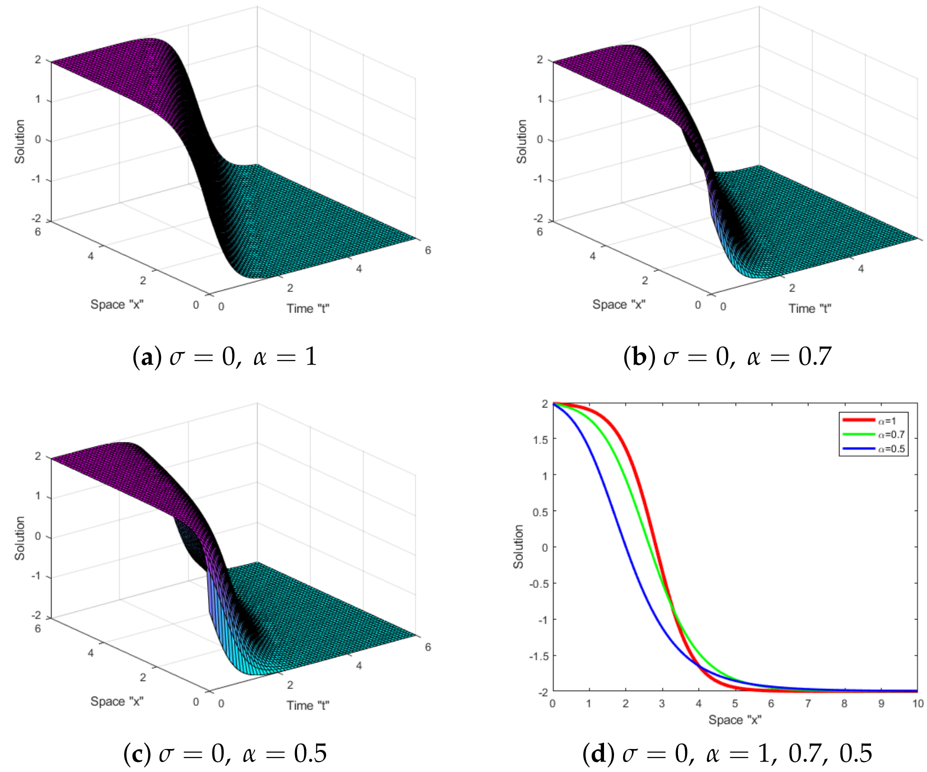


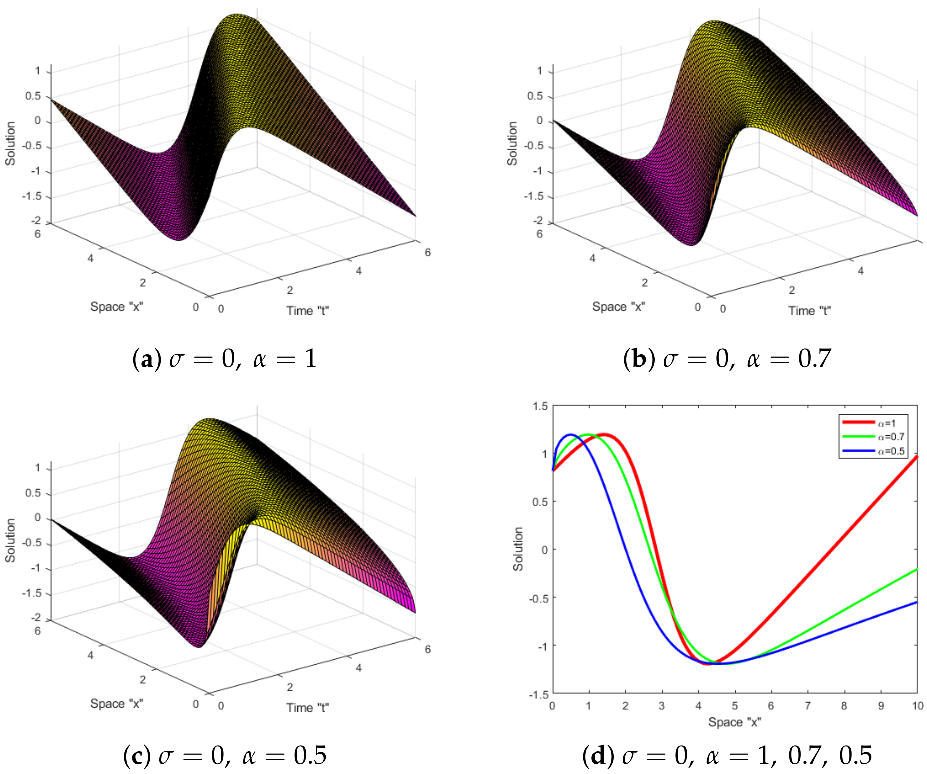
Figure 1. (**a**–**c**) show the 3D profiles of Equation (16) with $\sigma = 0,\ c_0 = 3$, and different values of α. (**d**) denotes the 2D profile of Equation (16).



(**a**) $\sigma = 0,\ \alpha = 1$ (**b**) $\sigma = 0,\ \alpha = 0.7$

(**c**) $\sigma = 0,\ \alpha = 0.5$ (**d**) $\sigma = 0,\ \alpha = 1,\ 0.7,\ 0.5$

Figure 2. (**a**–**c**) show the 3D profiles of Equation (18) with $\sigma = 0$, $c_0 = -4$, and different values of α. (**d**) indicates the 2D profile of Equation (18).



(**a**) $\sigma = 0,\ \alpha = 1$ (**b**) $\sigma = 0,\ \alpha = 0.7$

(**c**) $\sigma = 0,\ \alpha = 0.5$ (**d**) $\sigma = 0,\ \alpha = 1,\ 0.7,\ 0.5$

Figure 3. (**a**–**c**) show the 3D profiles of Equation (19) with $\sigma = 0$, $c_0 = -4$, and different values of α. (**d**) indicates the 2D profile of Equation (19).

Noise Influences: The following graphs illustrate how the noise affected the obtained solutions:

From Figures 4–6, we may deduce that there are several types of solutions, including periodic ones, kinked ones, and others, when the noise is disregarded (i.e., at $\sigma = 0$). The surface becomes much flatter after a few minor transits once noise is added and its intensity is increased. Evidently, BM stabilizes the FSKMMS solutions.

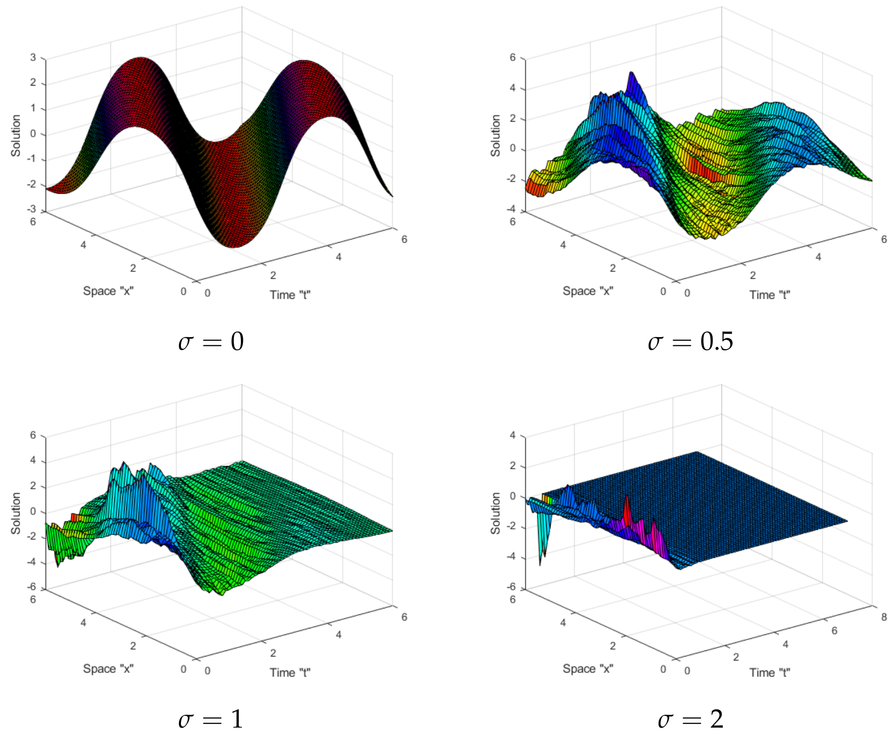


Figure 4. The 3D-profiles of solution $\Phi(x, t)$ in Equation (16).

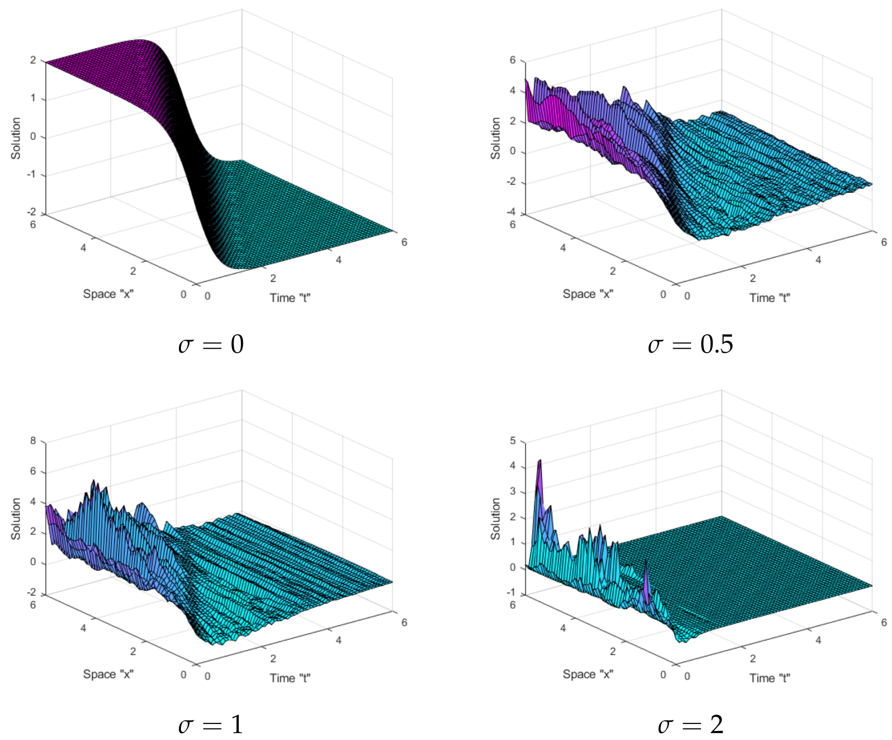


Figure 5. The 3D-profiles of solution $\Phi(x, t)$ in Equation (18) for various values of σ.

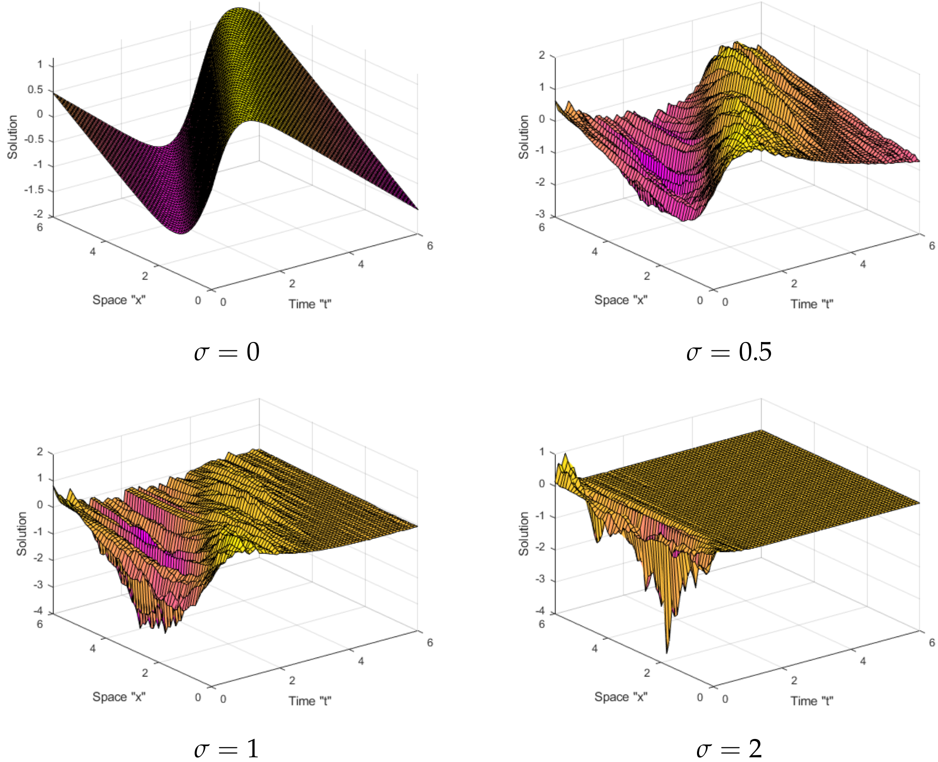


Figure 6. The 3D-profiles of solution $\Psi(x, t)$ in Equation (19) for various values of σ.

7. Conclusions

In this study, we examined the fractional Kraenkel–Manna–Merle System (1) in ferromagnetic materials. Using the mapping approach, we obtained the exact solutions for the FSKMMS without taking into account the damping term. Since Equation (1) is essential for explaining the propagation of a magnetic field in a ferromagnet with zero conductivity, the solutions it yields are fundamental in understanding a wide range of intriguing and complex physical phenomena. In addition, the MATLAB program was utilized to demonstrate how the conformable derivative and Brownian motion affect the analytical solution to the FSKMMS (1). We deduced that the Brownian motion stabilizes the solutions of the FSKMMS. In future work, we can consider the FSKMMS with additive noise or with multiplicative color noise.

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