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Fractional Itô–Doob Stochastic Differential Equations Driven by Countably Many Brownian Motions

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Abstract: This article is devoted to showing the existence and uniqueness (EU) of a solution with non-Lipschitz coefficients (NLC) of fractional Itô-Doob stochastic differential equations driven by countably many Brownian motions (FIDSDECBMs) of order $\varkappa \in (0,1)$ by using the Picard iteration technique (PIT) and the semimartingale local time (SLT).

Keywords: stochastic system; fractional integral; existence; uniqueness

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1. Introduction

The concept of the fractional derivative of order a, where $a \in (0,1)$, has been introduced by many scientists including Joseph Liouville and Bernhard Riemann in the 19th century. Recently, fractional calculus is a useful mathematical tool for applied sciences. Fractional calculus is a natural generalization of differential calculus. Later, Fourier, Abel, Liouville, Riemann, Riesz, and Caputo, among others, contributed to its development. They defined derivatives and integrals of noninteger order.

The importance of fractional calculus is an essential tool in the modeling of real phenomena. One might think that this area for fractional calculus is a question of "pure" mathematics without interest for the applications. However, a simple example from fluid mechanics shows how the derivative of order $n = \frac{1}{2}$ appears quite naturally when one wants to explain a flow of heat coming out laterally from a fluid flow according to the temporal evolution of the internal source (see [1–7]).

One of the most famous class of the fractional equations are the fractional Itô–Doob stochastic differential equations (FIDSDEs). In the literature, there are a few papers on the FIDSDE (see [8–11]). In [9], the authors discuss the averaging principle of FIDSDE with NLC. In [8], the authors examine the EU and mean square stability of solutions to the non-Lipschitz FIDSDE.

Motivated by several works in the literature, we extend, the results from the ordinary stochastic differential equations in [12,13] to the fractional Itô–Doob sense. The main contributions of this article are as follows:

- To investigate the EU of solutions to FIDSDECBM with NLC;
- To use the PIT and the SLT in our results.

The contents of the article are as follows: Section 2 is devoted to presenting the principal notions. Section 3 outlines the EU of the solution of FIDSDECBM using the PIT and the SLT.

2. Preliminaries and Definitions

Let $\mathcal{M} = \{\mathbb{X}, \widetilde{\mathfrak{F}}, \widetilde{\mathbb{F}} = (\widetilde{\mathfrak{F}}_\iota)_{\iota \geq 0}, \widetilde{\mathfrak{P}}\}$ be a complete probability space and $(\mathbb{W}^q(\iota))_{q \in \mathbb{N}^*}$ an infinite sequence of independent standard Brownian motions defined on the space $\mathcal{M}$.

Let $\mathbb{X}_\iota^2 = L^2(\mathbb{X}, \mathfrak{F}_\iota, \widetilde{\mathfrak{P}})$ be the space of all $\mathbb{F}_\iota$-measurable and mean square integrable functions $\theta : \mathbb{X} \to \mathbb{R}$ with

$$\|\theta\|_2 = \left(\mathbb{E}[\theta^2]\right)^{\frac{1}{2}}.$$

For more details about the basic notions of stochastic calculus, see [14].

Definition 1 ([15])**.** *Let $\varkappa \in (0,1)$ and let $e(\kappa)$ be a continuous function; then, the integral of $e(\kappa)$ with respect to $(d\iota)^\varkappa$ is given by*

$$\int_0^\iota e(s)(ds)^\varkappa = \varkappa \int_0^\iota (\iota - s)^{\varkappa-1} e(s)ds. \tag{1}$$

Consider the following FIDSDE:

$$d\pi_\iota = b(\pi_\iota)d\iota + \sum_{q=1}^{\infty} \vartheta_q(\pi_\iota)d\mathbb{W}^q(\iota) + \eta(\pi_\iota)(d\iota)^\varkappa, \tag{2}$$

where $0 < \varkappa < 1$, $\pi(0) = \xi \in \mathbb{R}$ is the initial condition, b, $(\vartheta_q)_{q\in\mathbb{N}^*}, \eta : [0,\Theta] \times \mathbb{R} \to \mathbb{R}$ are two given functions, and $\iota \in [0,\Theta]$.

Let $\vartheta(\pi) = (\vartheta_1(\pi), \vartheta_2(\pi), \dots)$ and $|\vartheta(\pi)| = \left(\sum_{q=1}^{\infty} \vartheta_q^2(\pi)\right)^{\frac{1}{2}}$, where $\vartheta(\pi) \in \mathcal{L}^2$ for all $\pi \in \mathbb{R}$, and

$$\mathcal{L}^2 = \left\{\phi = (\phi_q)_{q\in\mathbb{N}^*} : \mathbb{R} \to \mathbb{R} : |\phi(x)|^2 = \sum_{q=1}^{\infty} \phi_q^2(x) < \infty, \forall x \in \mathbb{R}\right\}.$$

The associated integral equation of (2) is given by the following:

$$\pi_\iota = \xi + \int_0^\iota b(\pi_s)ds + \sum_{q=1}^{\infty} \int_0^\iota \vartheta_q(\pi_s)d\mathbb{W}^q(s) + \varkappa \int_0^\iota (\iota - s)^{\varkappa-1} \eta(\pi_s)ds. \tag{3}$$

3. Existence and Uniqueness Results

Theorem 1. *Let λ, α. and ς be nondecreasing continuous concave functions on $[0, +\infty)$, satisfying $\lambda(0) = \alpha(0) = \varsigma(0) = 0$.*
For all $\pi, \overline{\pi} \in \mathbb{R}$,

$$|b(\pi) - b(\overline{\pi})| \le \lambda(|\pi - \overline{\pi}|),$$
$$|\vartheta(\pi) - \vartheta(\overline{\pi})| \le \alpha(|\pi - \overline{\pi}|),$$
$$|\eta(\pi) - \eta(\overline{\pi})| \le \varsigma(|\pi - \overline{\pi}|).$$

If there is a number $p \ge 2$ satisfying the following:

$$\int_{0^+} \frac{\pi^{p-1}}{\lambda^p(\pi) + \alpha^p(\pi) + \varsigma^p(\pi)} d\pi = \infty,$$

then, for any $\xi \in \mathbb{X}_0^2$, Equation (2) has a unique solution.

To show our main result, we design an approximation sequence using a PIT. Let $\pi^0 = \xi$ and π^n be a sequence defined by $\pi_0^n = \xi$, $\forall n \ge 1$,

$$d\pi_\iota^n = b(\pi_\iota^{n-1})d\iota + \sum_{q=1}^{\infty} \vartheta_q(\pi_\iota^{n-1})d\mathbb{W}^q(\iota) + \eta(\pi_\iota^{n-1})(d\iota)^\varkappa, \ (\iota > 0),$$

and $\forall n \geq 1$, $\forall \iota \geq 0$

$$\pi_\iota^n = \xi + \int_0^\iota b(\pi_s^{n-1})ds + \sum_{q=1}^{\infty} \int_0^\iota \vartheta_q(\pi_s^{n-1})d\mathbb{W}^q(s) + \varkappa \int_0^\iota (\iota - s)^{\varkappa-1}\eta(\pi_s^{n-1})ds, \quad (4)$$

where $\pi_\iota^0 = \xi$.

Let $\mathcal{H}_{loc}^2$ be the space of all adapted $(\widetilde{\mathfrak{F}}_\iota)$ processes ζ satisfying

$$\int_0^\iota \mathbb{E}|\zeta(s)|^2 ds < \infty,$$

where $|\zeta(s)|^2 = \sum_{q=1}^{\infty} \zeta_q^2(s)$.

Lemma 1. *π^n is well defined in (4),is continuous, $|\vartheta(\pi^n)| \in \mathcal{H}_{loc}^2$, and is a $(\widetilde{\mathfrak{F}}_\iota)$ semimartingale for all $n \geq 1$.*

Proof. Using (4), Lemma 2.1 and Corollary 3.4 in [13], we can derive that there exists $K_2 > 0$ such that $\forall n \geq 1$, and $0 \leq \iota \leq \Delta$,

$$\begin{aligned}
\mathbb{E}(\pi_\iota^n)^2 \leq\ & 4\xi^2 + 4\mathbb{E}\left(\int_0^\iota b(\pi_s^{n-1})ds\right)^2 + 4\mathbb{E}\left(\sum_{q=1}^{\infty}\int_0^\iota \vartheta_q(\pi_s^{n-1})d\mathbb{W}^q(s)\right)^2 \\
+\ & 4\varkappa^2\mathbb{E}\left(\int_0^\iota (\iota - s)^{\varkappa-1}\eta(\pi_s^{n-1})ds\right)^2 \\
\leq\ & 4\xi^2 + 4\Delta\int_0^\iota \mathbb{E}\left(b^2(\pi_s^{n-1})\right)ds + 4\int_0^\iota \mathbb{E}|\vartheta(\pi_s^{n-1})|^2 ds \\
+\ & 4\varkappa^2\frac{\Delta^{2\varkappa-1}}{(2\varkappa-1)}\int_0^\iota \mathbb{E}\left(\eta^2(\pi_s^{n-1})\right)ds \\
\leq\ & 4\xi^2 + 4K_2\left(\Delta + 1 + \varkappa^2\frac{\Delta^{2\varkappa-1}}{(2\varkappa-1)}\right)\int_0^\iota\left(1 + \mathbb{E}(\pi_s^{n-1})^2\right)ds \\
\leq\ & 4\xi^2 + 4K_2\Delta\left(\Delta + 1 + \varkappa^2\frac{\Delta^{2\varkappa-1}}{(2\varkappa-1)}\right)\left(1 + \sup_{0\leq\iota\leq\Delta}\mathbb{E}(\pi_\iota^{n-1})^2\right), \quad (5)
\end{aligned}$$

Thus, we can derive the following:

$$\sup_{0\leq\iota\leq\Delta}\mathbb{E}(\pi_\iota^n)^2 \leq 4\xi^2 + 4K_2\Delta\left(\Delta + 1 + \varkappa^2\frac{\Delta^{2\varkappa-1}}{(2\varkappa-1)}\right)\left(1 + \sup_{0\leq\iota\leq\Delta}\mathbb{E}(\pi_\iota^{n-1})^2\right).$$

Using the fact that $\sup_{0\leq\iota\leq\Delta}\mathbb{E}\left(\pi_\iota^0\right)^2 = \xi^2 < \infty$, thus,

$$\sup_{0\leq\iota\leq\Delta}\mathbb{E}(\pi_\iota^n)^2 < \infty, \ \forall n \geq 1.$$

According to Corollary 3.4 in [13], we can obtain

$$\int_0^\iota \mathbb{E}|\vartheta(\pi_s^n)|^2 ds \leq K_2\Delta\left(1 + \sup_{0\leq\iota\leq\Delta}\mathbb{E}(\pi_\iota^n)^2\right) < \infty,$$

which implies that $|\vartheta(\pi^n)| \in \mathcal{H}_{loc}^2$.

Then, π^n is well defined by Lemma 2.1 in [13], and it is a continuous $(\widetilde{\mathfrak{F}}_\iota)$ semimartingale, $\forall n \geq 1$. $\square$

Lemma 2. *Suppose that all the assumptions of Theorem 1 hold. Then, for any fixed $\Delta > 0$, there are some positive numbers $K_{r,\Delta}$ satisfying*

$$\mathbb{E} \sup_{0\leq v\leq \iota} |\pi_v^n|^r \leq K_{r,\Delta}, \tag{6}$$

$$\mathbb{E} \sup_{0\leq v\leq \iota} |\pi_v^n - \pi_v^z|^r \leq K_{r,\Delta}, \tag{7}$$

$\forall n, z \geq 1, r \geq 2, 0 \leq \iota \leq \Delta$.

Proof. For any fixed $z \geq 1$, when $1 \leq n \leq z$, $0 \leq \iota \leq \Delta$, by using the Burkholder–Davis–Gundy Inequality and Corollary 3.4 in [13], one can derive

$$\begin{aligned}
\mathbb{E} \sup_{0\leq v\leq \iota} |\pi_v^n|^r \leq\ & 4^{r-1}|\xi|^r + 4^{r-1}\mathbb{E} \sup_{0\leq v\leq \iota}\left|\int_0^v b(\pi_s^{n-1})ds\right|^r + 4^{r-1}\mathbb{E}\sup_{0\leq v\leq \iota}\left|\sum_{q=1}^{\infty}\int_0^v \vartheta_q(\pi_s^{n-1})d\mathbb{W}^q(s)\right|^r \\
+\ & 4^{r-1}\varkappa^r \mathbb{E}\sup_{0\leq v\leq \iota}\left|\int_0^v (v-s)^{\varkappa-1}\eta(\pi_s^{n-1})ds\right|^r \\
\leq\ & 4^{r-1}|\xi|^r + 4^{r-1}\iota^{r-1}\int_0^\iota \mathbb{E}\left|b(\pi_s^{n-1})\right|^r ds + 4^{r-1}\iota^{\frac{r}{2}-1}M\int_0^\iota \mathbb{E}|\vartheta(\pi_s^{n-1})|^r ds \\
+\ & 4^{r-1}\varkappa^r\left(\frac{r-1}{r\varkappa-1}\right)^{r-1}\iota^{r\varkappa-1}\int_0^\iota \mathbb{E}\left|\eta(\pi_s^{n-1})\right|^r ds \\
\leq\ & 4^{r-1}|\xi|^r + 4^{r-1}K_r\left(\Delta^{r-1} + \Delta^{\frac{r}{2}-1}M + \left(\frac{r-1}{r\varkappa-1}\right)^{r-1}\varkappa^r\Delta^{r\varkappa-1}\right)\int_0^\iota\left(1+\mathbb{E}\left|\pi_s^{n-1}\right|^r ds\right) \\
\leq\ & 4^{r-1}|\xi|^r + 4^{r-1}K_r\left(\Delta^r + \Delta^{\frac{r}{2}}M + \left(\frac{r-1}{r\varkappa-1}\right)^{r-1}\varkappa^r\Delta^{r\varkappa}\right) \\
+\ & 4^{r-1}K_r\left(\Delta^{r-1} + \Delta^{\frac{r}{2}-1}M + \left(\frac{r-1}{r\varkappa-1}\right)^{r-1}\varkappa^r\Delta^{r\varkappa-1}\right)\int_0^\iota \mathbb{E}\sup_{0\leq v\leq s}\left|\pi_v^{n-1}\right|^r ds \\
\leq\ & 4^{r-1}|\xi|^r + 4^{r-1}K_r\left(\Delta^r + \Delta^{\frac{r}{2}}M + \left(\frac{r-1}{r\varkappa-1}\right)^{r-1}\varkappa^r\Delta^{r\varkappa}\right) \\
+\ & 4^{r-1}K_r\left(\Delta^{r-1} + \Delta^{\frac{r}{2}-1}M + \left(\frac{r-1}{r\varkappa-1}\right)^{r-1}\varkappa^r\Delta^{r\varkappa-1}\right)\int_0^\iota \sup_{0\leq\kappa\leq z}\mathbb{E}\sup_{0\leq v\leq s}\left|\pi_v^{\kappa}\right|^r ds,
\end{aligned} \tag{8}$$

where $M > 0$. We know that $\mathbb{E}\sup_{0\leq v\leq s}\left|\pi_v^0\right|^r = |\xi|^r$; then, we obtain

$$\begin{aligned}
\sup_{0\leq\kappa\leq z}\mathbb{E}\sup_{0\leq v\leq \iota}|\pi_v^\kappa|^r \leq\ & 4^{r-1}|\xi|^r + 4^{r-1}K_r\left(\Delta^r + \Delta^{\frac{r}{2}}M + \left(\frac{r-1}{r\varkappa-1}\right)^{r-1}\varkappa^r\Delta^{r\varkappa}\right) \\
+\ & 4^{r-1}K_r\left(\Delta^{r-1} + \Delta^{\frac{r}{2}-1}M + \left(\frac{r-1}{r\varkappa-1}\right)^{r-1}\varkappa^r\Delta^{r\varkappa-1}\right)\int_0^\iota \sup_{0\leq\kappa\leq z}\mathbb{E}\sup_{0\leq v\leq s}\left|\pi_v^{\kappa}\right|^r ds.
\end{aligned}$$

Using Gronwall–Bellman's inequality, we derive that for all $z \geq 1$,

$$\begin{aligned} \sup_{0\leq\kappa\leq z} \mathbb{E} \sup_{0\leq v\leq\iota} |\pi_v^\kappa|^r &\leq \left(4^{r-1}|\xi|^r + 4^{r-1}K_r\left(\Delta^r + \Delta^{\frac{r}{2}}M + \left(\frac{r-1}{r\varkappa-1}\right)^{r-1}\varkappa^r\Delta^{r\varkappa}\right)\right) \\ &\times \exp\left\{4^{r-1}K_r\left(\Delta^{r-1} + \Delta^{\frac{r}{2}-1}M + \left(\frac{r-1}{r\varkappa-1}\right)^{r-1}\varkappa^r\Delta^{r\varkappa-1}\right)\iota\right\} \\ &\leq K_{r,\Delta}, \end{aligned}$$

where $K_{r,\Delta}$ is a positive number such that

$$\begin{aligned} K_{r,\Delta} &= \left(4^{r-1}|\xi|^r + 4^{r-1}K_r\left(\Delta^r + \Delta^{\frac{r}{2}}M + \left(\frac{r-1}{r\varkappa-1}\right)^{r-1}\varkappa^r\Delta^{r\varkappa}\right)\right) \\ &\times \exp\left\{4^{r-1}K_r\left(\Delta^{r-1} + \Delta^{\frac{r}{2}-1}M + \left(\frac{r-1}{r\varkappa-1}\right)^{r-1}\varkappa^r\Delta^{r\varkappa-1}\right)\Delta\right\}. \end{aligned}$$

Since z is arbitrary, then, for all $n \geq 1$, we have

$$\mathbb{E} \sup_{0\leq v\leq\iota} |\pi_v^n|^r \leq K_{r,\Delta}.$$

In the same manner, we can prove Inequality (7). $\square$

Lemma 3. *Suppose that all the assumptions of Theorem 1 hold. Then, for any fixed $\Delta > 0$, there are some positive numbers $\widetilde{K}_{r,\Delta}$ satisfying*

$$\mathbb{E} \sup_{0\leq v\leq\iota} |\pi_v^n - \pi_v^z|^r \leq \widetilde{K}_{r,\Delta} \int_0^\iota \phi\left(\mathbb{E} \sup_{0\leq v\leq\iota} |\pi_v^{n-1} - \pi_v^{z-1}|^r\right) ds, \tag{9}$$

$$\mathbb{E}|\pi_\iota^n - \pi_\iota^z|^r \leq \widetilde{K}_{r,\Delta} \int_0^\iota \phi\left(\mathbb{E}|\pi_s^{n-1} - \pi_s^{z-1}|^r\right) ds, \tag{10}$$

$\forall n, z \geq 1$, $r \geq 2$, $0 \leq \iota \leq \Delta$, with $\phi(\rho) = \lambda^r(\rho^{\frac{1}{r}}) + \alpha^r(\rho^{\frac{1}{r}}) + \varsigma^r(\rho^{\frac{1}{r}})$, $\forall \rho \geq 0$.

Proof. For all $n, z \geq 1$, $0 \leq v \leq \iota \leq \Delta$,

$$\begin{aligned} \pi_v^n - \pi_v^z &= \int_0^\iota \left(b(\pi_s^{n-1}) - b(\pi_s^{z-1})\right) ds + \sum_{q=1}^{\infty} \int_0^\iota \left(\vartheta_q(\pi_s^{n-1}) - \vartheta_q(\pi_s^{z-1})\right) d\mathbb{W}^q(s) \\ &+ \varkappa \int_0^\iota (\iota - s)^{\varkappa-1}\left(\eta(\pi_s^{n-1}) - \eta(\pi_s^{z-1})\right) ds. \end{aligned}$$

Using the Burkholder-Davis-Gundy Inequality, we have

$$\begin{aligned}
\mathbb{E}\sup_{0\le v\le\iota}|\pi_v^n-\pi_v^z|^r &\le 3^{r-1}\mathbb{E}\sup_{0\le v\le\iota}|\int_0^v\Big(b(\pi_s^{n-1})-b(\pi_s^{z-1})\Big)ds|^r\\
&+ 3^{r-1}\mathbb{E}\sup_{0\le v\le\iota}|\sum_{q=1}^{\infty}\int_0^v\Big(\vartheta_q(\pi_s^{n-1})-\vartheta_q(\pi_s^{z-1})\Big)d\mathbb{W}^q(s)|^r\\
&+ 3^{r-1}\varkappa^r\mathbb{E}\sup_{0\le v\le\iota}|\int_0^v(v-s)^{\varkappa-1}\Big(\eta(\pi_s^{n-1})-\eta(\pi_s^{z-1})\Big)ds|^r\\
&\le 3^{r-1}\iota^{r-1}\int_0^\iota\mathbb{E}\Big|b(\pi_s^{n-1})-b(\pi_s^{z-1})\Big|^r ds\\
&+ 3^{r-1}\iota^{\frac{r}{2}-1}M\int_0^\iota\mathbb{E}|\vartheta(\pi_s^{n-1})-\vartheta(\pi_s^{z-1})|^r ds\\
&+ 3^{r-1}\varkappa^r\left(\frac{r-1}{r\varkappa-1}\right)^{r-1}\iota^{r\varkappa-1}\int_0^\iota\mathbb{E}\Big|\eta(\pi_s^{n-1})-\eta(\pi_s^{z-1})\Big|^r ds\\
&\le 3^{r-1}\Delta^{r-1}\int_0^\iota\mathbb{E}\Big(\lambda^r\Big(\Big|\pi_s^{n-1}-\pi_s^{z-1}\Big|\Big)\Big)ds\\
&+ 3^{r-1}\Delta^{\frac{r}{2}-1}M\int_0^\iota\mathbb{E}\Big(\alpha^r\Big(\Big|\pi_s^{n-1}-\pi_s^{z-1}\Big|\Big)\Big)ds\\
&+ 3^{r-1}\varkappa^r\left(\frac{r-1}{r\varkappa-1}\right)^{r-1}\Delta^{r\varkappa-1}\int_0^\iota\mathbb{E}\Big(\varsigma^r\Big(\Big|\pi_s^{n-1}-\pi_s^{z-1}\Big|\Big)\Big)ds,
\end{aligned}\tag{11}$$

where $M>0$. Using lemma 3.5 in [13] and the Jensen inequality, we can derive the following:

$$\begin{aligned}
\mathbb{E}\sup_{0\le v\le\iota}|\pi_v^n-\pi_v^z|^r &\le 3^{r-1}\Delta^{r-1}\int_0^\iota\lambda^r\left(\left(\mathbb{E}\Big|\pi_s^{n-1}-\pi_s^{z-1}\Big|^r\right)^{\frac{1}{r}}\right)ds\\
&+ 3^{r-1}\Delta^{\frac{r}{2}-1}M\int_0^\iota\alpha^r\left(\left(\mathbb{E}\Big|\pi_s^{n-1}-\pi_s^{z-1}\Big|^r\right)^{\frac{1}{r}}\right)ds\\
&+ 3^{r-1}\varkappa^r\left(\frac{r-1}{r\varkappa-1}\right)^{r-1}\Delta^{r\varkappa-1}\int_0^\iota\varsigma^r\left(\left(\mathbb{E}\Big|\pi_s^{n-1}-\pi_s^{z-1}\Big|^r\right)^{\frac{1}{r}}\right)ds\\
&\le \widetilde{K}_{r,\Delta}\int_0^\iota\phi\Big(\mathbb{E}\Big|\pi_s^{n-1}-\pi_s^{z-1}\Big|^r\Big)ds\\
&\le \widetilde{K}_{r,\Delta}\int_0^\iota\phi\left(\mathbb{E}\sup_{0\le v\le s}\Big|\pi_v^{n-1}-\pi_v^{z-1}\Big|^r\right)ds,
\end{aligned}\tag{12}$$

where $\widetilde{K}_{r,\Delta}=\Big(3^{r-1}\Delta^{r-1}\Big)\vee\Big(3^{r-1}\Delta^{\frac{r}{2}-1}M\Big)\vee\left(3^{r-1}\varkappa^r\left(\frac{r-1}{r\varkappa-1}\right)^{r-1}\Delta^{r\varkappa-1}\right)$.

We know that n,z are arbitrary; then, (9) is proven. Using the same procedure, we can prove (10). □

Proof of Theorem 1. Let Δ be a fixed positive constant.
Existence: Using lemma 3, one derive

$$\mathbb{E}\sup_{0\le v\le\iota}|\pi_v^n-\pi_v^z|^p\le\widetilde{K}_{p,\Delta}\int_0^\iota\phi\left(\mathbb{E}\sup_{0\le v\le s}\Big|\pi_v^{n-1}-\pi_v^{z-1}\Big|^p\right)ds,$$

for all $n,z\ge1$, $0\le\iota\le\Delta$, where $\phi(\rho)=\lambda^r(\rho^{\frac{1}{r}})+\alpha^r(\rho^{\frac{1}{r}})+\varsigma^r(\rho^{\frac{1}{r}})$, $\forall\rho\ge0$. Let

$$X_\iota=\lim_{n,z\to\infty}\sup\mathbb{E}\sup_{0\le v\le\iota}|\pi_v^n-\pi_v^z|^p,$$

Thus, X is a continuous and nonnegative function on $[0, \Delta]$. Consequently, using Lemma 2 and Fatou's lemma, one can obviously deduce the following:

$$X_\iota \leq \widetilde{K}_{p,\Delta} \int_0^\iota \phi(X_s) ds.$$

According to Corollary 3.6 and Lemma 3.7 in [13], one finds that $X \equiv 0$, which implies that, for all $0 \leq \iota \leq \Delta$,

$$\lim_{n,z \to \infty} \mathbb{E} \sup_{0 \leq v \leq \iota} |\pi_v^n - \pi_v^z|^p = 0.$$

Therefore, π^n is a Cauchy sequence under the norm $\left(\mathbb{E} \sup_{0 \leq \iota \leq \Delta} (\,\cdot\,)^p \right)^{\frac{1}{p}}$ for any fixed $\Delta > 0$. Let π be the limit; it is a continuous $(\widetilde{\mathfrak{F}}_\iota)$ semimartingale by continuity of π^n. Let $n \to \infty$ in (4); proceeding as the proof of Lemma 3, one can obtain

$$\mathbb{E} \sup_{0 \leq v \leq \iota} |\mathcal{I}(v)|^p \leq \widetilde{K}_{p,\Delta} \int_0^\Delta \phi\left(\mathbb{E} \sup_{0 \leq v \leq s} \left| \pi_v^{n-1} - \pi_v \right|^p \right) ds \to 0,$$

where

$$\begin{aligned} \mathcal{I}(v) \quad &= \quad \int_0^v \left(b\left(\pi_s^{n-1}\right) - b\left(\pi_s^{z-1}\right) \right) ds + \sum_{q=1}^{\infty} \int_0^v \left(\vartheta_q\left(\pi_s^{n-1}\right) - \vartheta_q\left(\pi_s^{z-1}\right) \right) d\mathbb{W}^q(s) \\ &+ \quad \varkappa \int_0^v (v-s)^{\varkappa - 1} \left(\eta\left(\pi_s^{n-1}\right) - \eta\left(\pi_s^{z-1}\right) \right) ds. \end{aligned}$$

Then, π_ι verifies Equation (3) for all $0 \leq \iota \leq \Delta$, which proves the existence result.
Uniqueness: Let π^1 and π^2 be two solutions of Equation (3); thus. proceeding as the proof of Lemma 3, we derive, for all $0 \leq \iota \leq \Delta$,

$$\mathbb{E}\left| \pi_\iota^1 - \pi_\iota^2 \right|^p \leq \widetilde{K}_{p,\Delta} \int_0^\iota \phi\left(\mathbb{E}\left| \pi_s^1 - \pi_s^2 \right|^p \right) ds.$$

Noting that $\iota \to \mathbb{E}\left|\pi_\iota^1 - \pi_\iota^2\right|^p$ is a nonnegative continuous function on $[0, \infty)$, applying Lemma 3.7 in [13], we can deduce the following: $\mathbb{E}\left|\pi_\iota^1 - \pi_\iota^2\right|^p \equiv 0$, for all $0 \leq \iota \leq \Delta$, which implies that $\pi_\iota^1 = \pi_\iota^2$ for a.s. $\iota \geq 0$ since Δ is arbitrary. Consequently, using π^1 and π^2 as continuous stochastic processes on $[0, \infty)$, we obtain

$$\mathbb{P}\left(\pi_\iota^1 = \pi_\iota^2,\ \forall \iota \geq 0 \right) = 1.$$

Therefore, the uniqueness of the solution is proven, as desired. □

4. Conclusions

This paper examines the EU of the solution with NLC of FIDSDECBM of order $\varkappa \in (0, 1)$ according to the Picard iteration technique (PIT) and the semimartingale local time (SLT). Combining our results in this paper with those of [16], we can discuss the EU of the solution with NLC of FIDSDE driven by countably many G-Brownian motions.

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