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On Coupled System of Langevin Fractional Problems with Different Orders of μ-Caputo Fractional Derivatives

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Abstract: In this paper, we study coupled nonlinear Langevin fractional problems with different orders of μ-Caputo fractional derivatives on arbitrary domains with nonlocal integral boundary conditions. In order to ensure the existence and uniqueness of the solutions to the problem at hand, the tools of the fixed-point theory are applied. An overview of the main results of this study is presented through examples.

Keywords: Langevin problems; coupled system; integral boundary conditions; fixed-point theorems; existence; uniqueness; μ-Caputo fractional derivatives

MSC: 34A08; 34G20; 26A33

Citation: Almaghamsi, L.; Alruwaily, Y.; Karthikeyan, K.; El-hady, E.-s. On Coupled System of Langevin Fractional Problems with Different Orders of μ-Caputo Fractional Derivatives. *Fractal Fract.* **2023**, *7*, 337. https://doi.org/10.3390/fractalfract7040337

Academic Editor: Rodica Luca

Received: 7 March 2023
Revised: 14 April 2023
Accepted: 16 April 2023
Published: 18 April 2023

1. Introduction

Fractional differential equations (FDEs) have gained a lot of attention in recent years due to their numerous applications in engineering, physics, biology, chemistry, and other fields (see, for instance, [1–6], and the references therein for more information on the boundary value issues of FDEs and inclusions subject to diverse boundary conditions). Differential inclusion and differential equations are thought to be particularly helpful when studying dynamical systems and stochastic processes (see [7–10] for some recent related results).

The Langevin equation successfully captures Brownian motion when the random fluctuation force is assumed to be white noise. Otherwise, the extended Langevin equation represents the particle motion (see, e.g., [11,12]). In fractal media, Langevin's equation has become widely used to represent dynamical operations (see [13–18] for more recent interesting results). In [19], the authors utilized the fractional Langevin equation to recreate Brownian motion. By applying both fluctuation–dissipation theorems and fractional calculus techniques, they derived analytical expressions for the correlation functions. The fractional Langevin equation has drawn the attention of numerous researchers due to its wide-ranging applications in various fields such as physics, chemistry, biology, aerodynamics, economics, control theory, biophysics, signal and image processing, fitting of experimental data, blood flow phenomena, and others. Moreover, it has been studied under various conditions. Moreover, it has been studied under various conditions (see e.g., [20–28]). Due to its numerous applications, the coupled system of differential equations with fractional order is regarded as a crucial and worthwhile topic of study (see e.g., [29,30]). It is important to note that the majority of the research on the coupled systems of FDEs focuses on fixed domains. The mixed-order coupled system offers a more comprehensive approach and is a valuable addition to the existing literature.

The ability to describe a variety of physical and technical systems, including viscoelastic materials, diffusion and transport processes, and electromagnetic phenomena, necessitates the study of FDEs on arbitrary domains. The domain on which the system is defined is frequently irregular or has a complicated border rather than a straightforward geometric shape. In contrast to conventional integer-order differential equations, we are able to model these systems more precisely and successfully by using fractional differential operators on arbitrary domains. Additionally, the study of FDEs on arbitrary domains results in the creation of fresh analytical and numerical approaches for resolving these equations, which have applications in materials science, biology, finance, and control theory.

A new class of coupled FDEs of different orders with nonlocal multi-point boundary conditions was studied in [31]. Since then, numerous studies have focused on these types of systems of equations, including [32–34]. In the latter, the authors focused on the study of a coupled system of FDEs of Caputo type with different derivatives orders, which inspired us to study coupled systems of Langevin fractional problems with different orders of μ-Caputo fractional derivatives with nonlocal integral boundary conditions of the form:

$$ {}^{c}D^{\gamma_i,\mu}({}^{c}D^{\sigma_i,\mu}+\alpha_i)\varphi_i(t)=\Xi_i(t,\varphi_1(t),\varphi_2(t)), \qquad t\in[a,b], i=1,2. \tag{1} $$

subject to the specific boundary conditions

$$ \begin{aligned} &\varphi_i(a)=0, \qquad I^{\vartheta_i,\mu}\varphi_i(b)=0, \\ &{}^{c}D^{\sigma_1,\mu}\varphi_1(a)=\kappa\int_a^{\xi}\varphi_2(s)ds. \end{aligned} \tag{2} $$

where for $i=1,2$, ${}^{c}D^{\gamma_i,\mu}$ and ${}^{c}D^{\sigma_i,\mu}$ are μ-Caputo fractional derivatives, $0<\sigma_i,\gamma_2<1$, $1<\gamma_1\leq 2$, $\alpha_i,\kappa\in\mathbb{R}$.

As indicated in the above system, we present this study with a μ-Caputo fractional derivative operator (FDO), which is a generalization of the Riemann–Liouville FDO. Below, we highlight some of its advantages, which have been discussed in various research papers and articles in the field of fractional calculus and its applications (see e.g., [1,2,35–37]):

- Flexibility: The μ-Caputo FDO is more flexible than the Riemann–Liouville FDO because it allows for the use of different kernels (functions that define the fractional derivative), depending on the application.
- Smoothing property: The μ-Caputo FDO has a smoothing property that can be used to eliminate noise from a signal or image. This property makes it useful in image processing, signal processing, and other applications where noise reduction is important.
- Nonlocality: The μ-Caputo FDO is nonlocal, meaning that the value of the derivative at a point depends on the values of the function at all other points. This property allows for the detection of long-range correlations in data, which can be useful for studying complex systems.
- Fractional order: The μ-Caputo FDO allows for the use of non-integer orders, which can be used to model phenomena that do not conform to integer-order models. This property makes it efficient for various applications such as physics, engineering, and other fields where non-integer orders are needed to accurately model systems.
- Numerical methods: The μ-Caputo FDO can be efficiently computed using numerical methods, which makes it useful for computer simulations and other applications where analytical solutions are not available.

The structure of the rest of this paper is as follows. Section 2 outlines the fundamental principles of fractional calculus and defines the key terms and symbols. In Section 3, we present the main finding for fractional differential derivatives. Section 4 discusses the use of the Leray–Schauder alternative and Krasnoselskii's theorem to establish the existence of a solution. In contrast, in Section 5, we prove that there is a unique solution based on Banach's contraction mapping principle. Section 6 includes examples that illustrate the key points of our study and the last section represents the conclusions.

2. Preliminaries

In this section, we introduce a number of fundamental concepts and relevant lemmas in fractional calculus. Let $\mathfrak{J} = [a, b]$. We define $\mathfrak{C} = \mathcal{C}(\mathfrak{J}, \mathbb{R})$ as the Banach space of all continuous functions $g : \mathfrak{J} \to \mathbb{R}$ with the norm

$$\|g\| = \sup\{|g(t)| = t \in \mathfrak{J}\},$$

and we represent the Banach space of Lebesgue-integrable functions $g : \mathfrak{J} \to \mathbb{R}$ by $\mathbf{L^1}(\mathfrak{J}, \mathbb{R})$ with the norm

$$\|g\|_{\mathbf{L^1}} = \int_{\mathfrak{J}} |g(t)| dt.$$

Assume that $g : \mathfrak{J} \to \mathbb{R}$ is integrable and that $\mu \in \mathcal{C}^m(\mathfrak{J}, \mathbb{R})$ is increasing such that $\mu'(t) \neq 0$ for every t included in $\mathfrak{J}$.

Definition 1 ([38])**.** The q-μ-Riemann–Liouville integral of a function g is defined as

$$I_{a+}^{q;\mu} g(t) = \frac{1}{\Gamma(q)} \int_a^t \mu'(\sigma)(\mu(t) - \mu(\sigma))^{q-1} g(\sigma) d\sigma, \qquad q > 0.$$

Definition 2 ([38])**.** The q-μ-Riemann–Liouville fractional derivative of a function g is

$$D_{a+}^{q;\mu} g(t) = \left(\frac{1}{\mu'(t)} \frac{d}{dt} \right)^m I_{a+}^{(m-q);\mu} g(t),$$

where $m = [q] + 1$.

Definition 3 ([38])**.** For a function $g \in \mathcal{AC}^m(\mathfrak{J}, \mathbb{R})$, the μ-fractional derivative of order q in a Caputo sense is given as follows

$${}^C D_{a+}^{q;\mu} g(t) = I_{a+}^{(m-q);\mu} g^{[m]}(t),$$

where $g^{[m]}(t) = \left(\frac{1}{\mu'(t)} \frac{d}{dt} \right)^m g(t)$ and $m = [q] + 1, m \in .$

Lemma 4 ([38])**.** *Let* $q_1, q_2 > 0$. *Then:*

1. $I_{a+}^{q_1;\mu} (\mu(\sigma) - \mu(a))^{q_2-1}(t) = \frac{\Gamma(q_2)}{\Gamma(q_1+q_2)} (\mu(t) - \mu(a))^{q_1+q_2-1}$
2. ${}^C D_{a+}^{q_1;\mu} (\mu(\sigma) - \mu(a))^{q_2-1}(t) = \frac{\Gamma(q_2)}{\Gamma(q_2-q_1)} (\mu(t) - \mu(a))^{q_2-q_1-1}$

Lemma 5 ([38])**.** *If* $g \in \mathcal{AC}^m(\mathfrak{J}, \mathbb{R})$ *and* $q \in (m-1, m)$,

$$I_{a+}^{q;\mu} {}^C D_{a+}^{q;\mu} g(t) = g(t) - \sum_{k=0}^{m-1} \frac{g^{[m]}(a^+)}{k!} (\mu(t) - \mu(a))^k.$$

Lemma 6. *Let* $h_1, h_2 \in C([a, b], \mathbb{R})$. *Then, for* $\alpha_i, \kappa \in \mathbb{R}; i = 1, 2$ *the linear-type system*

$$\begin{cases} {}^c D^{\gamma_i,\mu}({}^c D^{\sigma_i,\mu} + \alpha_i)\varphi_i(t) = h_i(t), \quad t \in [a, b], 0 < \sigma_i, \gamma_2 < 1, 1 < \gamma_1 \leq 2, \\ \varphi_i(a) = 0, \qquad I^{\vartheta_i,\mu}\varphi_i(b) = 0, \\ {}^c D^{\sigma_1,\mu}\varphi_1(a) = \kappa \int_a^\xi \varphi_2(s) ds, \end{cases}$$

has a unique solution, which is

$$\varphi_1(t) = I^{\sigma_1+\gamma_1,\mu}h_1(t) - \alpha_1 I^{\sigma_1,\mu}\varphi_1(t) - \Lambda_1\frac{\Delta_t^{\sigma_1+1}}{\Gamma(\sigma_1+2)}\Big[I^{\vartheta_1+\sigma_1+\gamma_1,\mu}h_1(b) - \alpha_1 I^{\vartheta_1+\sigma_1,\mu}\varphi_1(b)\Big] + \kappa\frac{\Delta_t^{\sigma_1}}{\Gamma(\sigma_1+1)}\varsigma_1(t)\int_a^{\xi}\varphi_2(s)ds, \tag{3}$$

and

$$\varphi_2(t) = I^{\sigma_2+\gamma_2,\mu}h_2(t) - \alpha_2 I^{\sigma_2,\mu}\varphi_2(t) - \Lambda_2\frac{\Delta_t^{\sigma_2}}{\Gamma(\sigma_2+1)}\Big[I^{\vartheta_2+\sigma_2+\gamma_2,\mu}h_2(b) - \alpha_2 I^{\vartheta_2+\sigma_2,\mu}\varphi_2(b)\Big].$$

where

$$\Delta_t = \mu(t) - \mu(a),$$
$$\Lambda_1 = \frac{\Gamma(\vartheta_1+\sigma_1+2)}{\Delta_b^{\vartheta_1+\sigma_1+1}}, \Lambda_2 = \frac{\Gamma(\vartheta_2+\sigma_2+1)}{\Delta_b^{\vartheta_2+\sigma_2}},$$
$$\varsigma_1(t) = \frac{(\sigma_1+1)\Delta_b - (\vartheta_1+\sigma_1+1)\Delta_t}{(\sigma_1+1)\Delta_b}.$$

Proof. From Lemma [2], we have

$$\varphi_i = I^{\sigma_i+\gamma_i,\mu}h_i(t) - \alpha_i I^{\sigma_i,\mu}\varphi_i(t) + c_{i0} + \sum_{k=1}^{n} c_{ik}\frac{\Delta_t^{\sigma_i+k-1}}{\Gamma(\sigma_i+k)}, \qquad i = 1,2, n = [\gamma_i]+1. \tag{4}$$

From $\varphi_i(a) = 0$ and ${}^cD^{\sigma_1,\mu}\varphi_1(0) = \kappa\int_0^{\xi}\varphi_2(s)ds$, we find that $c_{i0} = 0$, $c_{11} = \kappa\int_0^{\xi}\varphi_2(s)ds$, and the last two conditions enable us to directly obtain

$$c_{12} = -\Lambda_1\Big[I^{\vartheta_1+\sigma_1+\gamma_1,\mu}h_1(b) - \alpha_1 I^{\vartheta_1+\sigma_1,\mu}\varphi_1(b)\Big] - \frac{\kappa(\vartheta_1+\sigma_1+1)}{\Delta_b}\int_a^{\xi}\varphi_2(s)ds,$$
$$c_{21} = -\Lambda_2\Big[I^{\vartheta_2+\sigma_2+\gamma_2,\mu}h_2(b) - \alpha_2 I^{\vartheta_2+\sigma_2,\mu}\varphi_2(b)\Big].$$

By substituting in Equation (4), we obtain the desired result. □

3. Main Results

Let $\mathfrak{C} = C([a,b],\mathbb{R})$ denote the Banach space of all continuous functions from $[a,b]$ to $\mathbb{R}$. Let us introduce the space $\mathfrak{X} = \{\mathfrak{u}(t)|\mathfrak{u}(t) \in C([a,b])\}$ endowed with the norm $\|\mathfrak{u}(t)\| = \sup\{|\mathfrak{u}(t)|, t \in [a,b]\}$. Obviously, $(\mathfrak{X}, \|\cdot\|)$ is a Banach space. The product space $(\mathfrak{X}\times\mathfrak{X}, \|(\mathfrak{u},\mathfrak{v})\|)$ is a Banach space with the norm $\|(\mathfrak{u},\mathfrak{v})\| = \|\mathfrak{u}\| + \|\mathfrak{v}\|$.

According to Lemma 6, an operator $\mathfrak{K} : \mathfrak{X}\times\mathfrak{X} \to \mathfrak{X}\times\mathfrak{X}$ is defined as follows:

$$\mathfrak{K} = \begin{pmatrix} \eth(\varphi_1,\varphi_2)(t) \\ \wp(\varphi_1,\varphi_2)(t) \end{pmatrix}, \tag{5}$$

where

$$\eth(\varphi_1,\varphi_2)(t) = I^{\sigma_1+\gamma_1,\mu}h_1(t,\varphi_1(t),\varphi_2(t)) - \alpha_1 I^{\sigma_1,\mu}\varphi_1(t) - \Lambda_1\frac{\Delta_t^{\sigma_1+1}}{\Gamma(\sigma_1+2)} \Big[I^{\vartheta_1+\sigma_1+\gamma_1,\mu}h_1(b,\varphi_1(b),\varphi_2(b)) - \alpha_1 I^{\vartheta_1+\sigma_1,\mu}\varphi_1(b)\Big] + \kappa\frac{\Delta_t^{\sigma_1}}{\Gamma(\sigma_1+1)}\varsigma_1(t)\int_a^{\xi}\varphi_2(s)ds, \tag{6}$$

and

$$\wp(\varphi_1,\varphi_2)(t) = I^{\sigma_2+\gamma_2,\mu}h_2(t,\varphi_1(t),\varphi_2(t)) - \alpha_2 I^{\sigma_2,\mu}\varphi_2(t) - \Lambda_2\frac{\Delta_t^{\sigma_2}}{\Gamma(\sigma_2+1)}$$
$$\left[I^{\vartheta_2+\sigma_2+\gamma_2,\mu}h_2(b,\varphi_1(b),\varphi_2(b)) - \alpha_2 I^{\vartheta_2+\sigma_2,\mu}\varphi_2(b)\right]. \tag{7}$$

We simplify the notations using the following constants:

$$\Phi_1 = \left[\frac{\Delta_b^{\sigma_1+\gamma_1}}{\Gamma(\sigma_1+\gamma_1+1)} + |\Lambda_1|\frac{\Delta_b^{\vartheta_1+2\sigma_1+\gamma_1+1}}{\Gamma(\sigma_1+2)\Gamma(\vartheta_1+\sigma_1+\gamma_1+1)}\right] \tag{8}$$

$$\Phi_2 = \left[\frac{\Delta_b^{\sigma_1}}{\Gamma(\sigma_1+1)} + |\Lambda_1|\frac{\Delta_b^{\vartheta_1+2\sigma_1+1}}{\Gamma(\sigma_1+2)\Gamma(\vartheta_1+\sigma_1+1)}\right] \tag{9}$$

$$\Phi_3 = \left[\kappa|\varsigma_1|\frac{\Delta_b^{\sigma_1}}{\Gamma(\sigma_1+1)}(\xi-a)\right] \tag{10}$$

$$\Psi_1 = \left[\frac{\Delta_b^{\sigma_2+\gamma_2}}{\Gamma(\sigma_2+\gamma_2+1)} + |\Lambda_2|\frac{\Delta_b^{\vartheta_2+2\sigma_2+\gamma_2}}{\Gamma(\sigma_2+1)\Gamma(\vartheta_2+\sigma_2+\gamma_2+1)}\right] \tag{11}$$

and

$$\Psi_2 = \left[\frac{\Delta_b^{\sigma_2}}{\Gamma(\sigma_2+1)} + |\Lambda_2|\frac{\Delta_b^{\vartheta_2+2\sigma_2}}{\Gamma(\sigma_2+1)\Gamma(\vartheta_2+\sigma_2+1)}\right] \tag{12}$$

4. Existence Results

Fixed-point theorems have recently played a vital role in proving many interesting results (see, e.g., [39–41]).

Lemma 7 ([42])**.** *Let $\mathcal{W}$ be a closed convex and nonempty subset of a Banach space E. Let $\mathfrak{F}1$ and $\mathfrak{F}2$ be two operators such that:*

1. $\mathfrak{F}1\mathcal{X} + \mathfrak{F}2\mathcal{Y} \in \mathcal{W}, \mathcal{X}, \mathcal{Y} \in \mathcal{W}$,
2. *$\mathfrak{F}1$ is compact and continuous on $\mathcal{W}$,*
3. *$\mathfrak{F}2$ is a contraction mapping on $\mathcal{W}$.*

Then, there exists $\mathcal{Z} \in \mathcal{W}$ such that $\mathcal{Z} = \mathfrak{F}1\mathcal{Z} + \mathfrak{F}2\mathcal{Z}$.

Theorem 8. *Suppose that the following conditions are satisfied:*

$(\mathcal{C}_1)$ $|h_1(t,\varphi_1,\varphi_2)| \le \mathfrak{w}_1(t)$
$(\mathcal{C}_2)$ $|h_2(t,\varphi_1,\varphi_2)| \le \mathfrak{w}_2(t)$
If

$$\mathfrak{m} = \max\{|\alpha_1|\Phi_2, \Phi_3 + |\alpha_2|\Psi_2\} \le 1 \tag{13}$$

where Φ_2,Φ_3, and Ψ_2 are defined by (9), (10), and (12). Then, Problems (1) and (2) have at least one solution for $[a,b]$.

Proof. To prove our results, we set $\sup_{t\in[a,b]}|\mathfrak{w}_1(t)| = \|\mathfrak{w}_1\|$, $\sup_{t\in[a,b]}|\mathfrak{w}_2(t)| = \|\mathfrak{w}_2\|$ and chose

$$\mathfrak{r} \ge \frac{\|\mathfrak{w}_1\|\Phi_1 + \|\mathfrak{w}_2\|\Psi_1}{1-\mathfrak{m}} \tag{14}$$

where Φ_1 and Ψ_1 are defined by (8) and (11). Let $\mathfrak{B}_{\mathfrak{r}} = \{(\varphi_1, \varphi_2) \in \mathfrak{X} \times \mathfrak{X} : \|(\varphi_1, \varphi_2)\| \leq \mathfrak{r}\}$. Now, we represent the four operators as follows:

$$\begin{aligned}
\eth_1(\varphi_1, \varphi_2)(t) &= I^{\sigma_1+\gamma_1,\mu}\hat{h}_1(t) - \Lambda_1 \frac{\Delta_t^{\sigma_1+1}}{\Gamma(\sigma_1+2)} I^{\vartheta_1+\sigma_1+\gamma_1,\mu}\hat{h}_1(b),\\
\eth_2(\varphi_1, \varphi_2)(t) &= -\alpha_1 I^{\sigma_1,\mu}\varphi_1(t) + \Lambda_1 \frac{\Delta_t^{\sigma_1+1}}{\Gamma(\sigma_1+2)}\alpha_1 I^{\vartheta_1+\sigma_1,\mu}\varphi_1(b) + \kappa \frac{\Delta_t^{\sigma_1}}{\Gamma(\sigma_1+1)}\varsigma_1(t)\int_a^{\xi}\varphi_2(s)ds,\\
\wp_1(\varphi_1, \varphi_2)(t) &= I^{\sigma_2+\gamma_2,\mu}\hat{h}_2(t) - \Lambda_2 \frac{\Delta_t^{\sigma_2}}{\Gamma(\sigma_2+1)} I^{\vartheta_2+\sigma_2+\gamma_2,\mu}\hat{h}_2(b).\\
\wp_2(\varphi_2)(t) &= -\alpha_2 I^{\sigma_2,\mu}\varphi_2(t) + \Lambda_2 \frac{\Delta_t^{\sigma_2}}{\Gamma(\sigma_2+1)}\alpha_2 I^{\vartheta_2+\sigma_2,\mu}\varphi_2(b).
\end{aligned}$$

where $\hat{h}_i(\tau) = \hat{h}_i(\tau, \varphi_1(\tau), \varphi_2(\tau))$, $i = 1, 2$, and

$$\mathfrak{K}_1 = \begin{pmatrix} \eth_1(\varphi_1, \varphi_2)(t) \\ \wp_1(\varphi_1, \varphi_2)(t) \end{pmatrix}, \qquad \mathfrak{K}_2 = \begin{pmatrix} \eth_2(\varphi_1, \varphi_2)(t) \\ \wp_2(\varphi_2)(t) \end{pmatrix}. \tag{15}$$

Note that $\eth = \eth_1 + \eth_2$, $\wp = \wp_1 + \wp_2$ and $\mathfrak{K} = \mathfrak{K}_1 + \mathfrak{K}_2$:

$$\begin{aligned}
|\eth_1(\varphi_1, \varphi_2) + \eth_2(\varphi_1, \varphi_2)| &= \Big| I^{\sigma_1+\gamma_1,\mu}\hat{h}_1(t) - \Lambda_1 \frac{\Delta_t^{\sigma_1+1}}{\Gamma(\sigma_1+2)} I^{\vartheta_1+\sigma_1+\gamma_1,\mu}\hat{h}_1(b)\\
&\quad - \alpha_1 I^{\sigma_1,\mu}\varphi_1(t) + \Lambda_1 \frac{\Delta_t^{\sigma_1+1}}{\Gamma(\sigma_1+2)}\alpha_1 I^{\vartheta_1+\sigma_1,\mu}\varphi_1(b) + \kappa \frac{\Delta_t^{\sigma_1}}{\Gamma(\sigma_1+1)}\varsigma_1(t)\int_a^{\xi}\varphi_2(s)ds\Big|\\
&\leq \|\mathfrak{w}_1\| \left[\frac{\Delta_b^{\sigma_1+\gamma_1}}{\Gamma(\sigma_1+\gamma_1+1)} + |\Lambda_1| \frac{\Delta_b^{\vartheta_1+2\sigma_1+\gamma_1}}{\Gamma(\sigma_1+2)\Gamma(\vartheta_1+\sigma_1+\gamma_1+1)}\right] + \|\varphi_1\||\alpha_1| \left[\frac{\Delta_b^{\sigma_1}}{\Gamma(\sigma_1+1)}\right.\\
&\quad \left. + |\Lambda_1| \frac{\Delta_b^{\vartheta_1+2\sigma_1}}{\Gamma(\sigma_1+2)\Gamma(\vartheta_1+\sigma_1+1)}\right] + \|\varphi_2\| \left[\kappa|\varsigma_1| \frac{\Delta_b^{\sigma_1}}{\Gamma(\sigma_1+1)}(\xi - a)\right]\\
&\leq \|\mathfrak{w}_1\|\Phi_1 + \|\varphi_1\||\alpha_1|\Phi_2 + \|\varphi_2\|\Phi_3.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
|\wp_1(\varphi_1, \varphi_2) + \wp_2(\varphi_2)| &= \Big| I^{\sigma_2+\gamma_2,\mu}\hat{h}_2(t) - \Lambda_2 \frac{\Delta_t^{\sigma_2}}{\Gamma(\sigma_2+1)} I^{\vartheta_2+\sigma_2+\gamma_2,\mu}\hat{h}_2(b))\\
&\quad - \alpha_2 I^{\sigma_2,\mu}\varphi_2(t) + \Lambda_2 \frac{\Delta_t^{\sigma_2}}{\Gamma(\sigma_2+1)}\alpha_2 I^{\vartheta_2+\sigma_2,\mu}\varphi_2(b)\Big|\\
&\leq \|\mathfrak{w}_2\| \left[\frac{\Delta_b^{\sigma_2+\gamma_2}}{\Gamma(\sigma_2+\gamma_2+1)} + |\Lambda_2| \frac{\Delta_b^{\vartheta_2+2\sigma_2+\gamma_2}}{\Gamma(\sigma_2+1)\Gamma(\vartheta_2+\sigma_2+\gamma_2+1)}\right]\\
&\quad + \|\varphi_2\||\alpha_2| \left[\frac{\Delta_b^{\sigma_2}}{\Gamma(\sigma_2+1)} + |\Lambda_2| \frac{\Delta_b^{\vartheta_2+2\sigma_2}}{\Gamma(\sigma_2+1)\Gamma(\vartheta_2+\sigma_2+1)}\right]\\
&\leq \|\mathfrak{w}_2\|\Psi_1 + \|\varphi_2\||\alpha_2|\Psi_2.
\end{aligned}$$

which implies that $\|\mathfrak{K}_1 + \mathfrak{K}_2\| \leq \mathfrak{r}$. This shows that $\mathfrak{K}_1 + \mathfrak{K}_2 \in \mathfrak{B}_{\mathfrak{r}}$. For $(\varphi_1, \varphi_2), (\overset{*}{\varphi}_1, \overset{*}{\varphi}_2) \in \mathfrak{X} \times \mathfrak{X}$ and $t \in [a, b]$, we have

$$\|\eth_2(\varphi_1, \varphi_2) - \eth_2(\overset{*}{\varphi}_1, \overset{*}{\varphi}_2)\| \leq |\alpha_1|\Phi_2\|\varphi_1 - \overset{*}{\varphi}_1\| + \Phi_3\|\varphi_2 - \overset{*}{\varphi}_2\|,$$

and

$$\|\wp_2(\varphi_2) - \wp_2(\overset{*}{\varphi}_2)\| \leq |\alpha_2|\Psi_2\|\varphi_2 - \overset{*}{\varphi}_2\|.$$

Thus,

$$\|\mathfrak{K}_2(\varphi_1,\varphi_2) - \mathfrak{K}_2(\overset{*}{\varphi}_1,\overset{*}{\varphi}_2)\| \le \mathfrak{m}\|\varphi_1 - \overset{*}{\varphi}_1\| + \mathfrak{m}\|\varphi_2 - \overset{*}{\varphi}_2\| = \mathfrak{m}\|(\varphi_1 - \overset{*}{\varphi}_1, \varphi_2 - \overset{*}{\varphi}_2)\|,$$

which implies that $\mathfrak{K}_2$ is a contraction mapping by (13). The continuity of h_i, $i = 1,2$ implies that the operator $\mathfrak{K}_1$is continuous. In addition, $\mathfrak{K}_1$ is uniformly bounded on $\mathfrak{B}_\mathfrak{r}$ as

$$\|\eth_1(\varphi_1,\varphi_2)\| \le \|\mathfrak{w}_1\|\Phi_1, \quad \text{and } \|\wp_1(\varphi_1,\varphi_2)\| \le \|\mathfrak{w}_2\|\Psi_1.$$

Thus,

$$\|\mathfrak{K}_1(\varphi_1,\varphi_2)\| \le \|\mathfrak{w}_1\|\Phi_1 + \|\mathfrak{w}_2\|\Psi_1.$$

Next, we prove the compactness of the operator $\mathfrak{K}_1$. Let $t_1, t_2 \in [a,b]$ with $t_1 < t_2$. Then, we obtain

$$\begin{aligned}
&|\eth_1(\varphi_1,\varphi_2)(t_2) - \eth_1(\varphi_1,\varphi_2)(t_1)| \\
&\le \left| I^{\sigma_1+\gamma_1,\mu}\hat{h}_1(t_2) - \Lambda_1 \frac{\Delta_{t_2}^{\sigma_1+1}}{\Gamma(\sigma_1+2)} I^{\vartheta_1+\sigma_1+\gamma_1,\mu}\hat{h}_1(b) - I^{\sigma_1+\gamma_1,\mu}\hat{h}_1(t_1) + \Lambda_1 \frac{\Delta_{t_1}^{\sigma_1+1}}{\Gamma(\sigma_1+2)} I^{\vartheta_1+\sigma_1+\gamma_1,\mu}\hat{h}_1(b) \right| \\
&\le \|\mathfrak{w}_1\| \left[\frac{1}{\Gamma(\sigma_1+\gamma_1+1)} (\Delta_{t_2}^{\sigma_1+\gamma_1} - \Delta_{t_1}^{\sigma_1+\gamma_1}) + |\Lambda_1| \frac{\Delta_b^{\vartheta_1+\sigma_1+\gamma_1}}{\Gamma(\sigma_1+2)\Gamma(\vartheta_1+\sigma_1+\gamma_1+1)} (\Delta_{t_2}^{\sigma_1+1} - \Delta_{t_1}^{\sigma_1+1}) \right],
\end{aligned}$$

and

$$\begin{aligned}
&|\wp_1(\varphi_1,\varphi_2)(t_2) - \wp_1(\varphi_1,\varphi_2)(t_1)| \\
&\le \left| I^{\sigma_2+\gamma_2,\mu}\hat{h}_2(t) - \Lambda_2 \frac{\Delta_t^{\sigma_2}}{\Gamma(\sigma_2+1)} I^{\vartheta_2+\sigma_2+\gamma_2,\mu}\hat{h}_2(b) - I^{\sigma_2+\gamma_2,\mu}\hat{h}_2(t) + \Lambda_2 \frac{\Delta_t^{\sigma_2}}{\Gamma(\sigma_2+1)} I^{\vartheta_2+\sigma_2+\gamma_2,\mu}\hat{h}_2(b) \right| \\
&\le \|\mathfrak{w}_2\| \left[\frac{1}{\Gamma(\sigma_2+\gamma_2+1)} (\Delta_{t_2}^{\sigma_2+\gamma_2} - \Delta_{t_1}^{\sigma_2+\gamma_2}) + |\Lambda_1| \frac{\Delta_b^{\vartheta_2+\sigma_2+\gamma_2}}{\Gamma(\sigma_2+1)\Gamma(\vartheta_2+\sigma_2+\gamma_2+1)} (\Delta_{t_2}^{\sigma_2} - \Delta_{t_1}^{\sigma_2}) \right].
\end{aligned}$$

As $t_1 \to t_2$, we have $|\mathfrak{K}_1 - \mathfrak{K}_2| \to 0$. Hence, $\mathfrak{K}_1$ is equicontinuous. By the Arzelá–Ascoli theorem, $\mathfrak{K}_1$ is compact. □

Theorem 9. *Let $h_1, h_2 : [a,b] \times \to \mathbb{R}$ be continuous functions. Suppose that (13) holds. Additionally, we assume that:*

($\mathcal{H}_1$) there exist a non-negative function $\mathfrak{z}_1(t)$, $\mathfrak{z}_1(t) \in C([a,b],\mathbb{R})$ and nondecreasing functions ψ_1, ψ_2

$$|h_1(t,(\varphi_1,\varphi_2))| \le [\|\mathfrak{z}_1\|\psi_1(\mathfrak{r}) + \|\mathfrak{z}_2\|\psi_2(\mathfrak{r})]$$

($\mathcal{H}_2$) there exist a non-negative function $\mathfrak{s}_1(t)$, $\mathfrak{s}_2(t) \in C([a,b],\mathbb{R})$ and nondecreasing functions χ_1, χ_2

$$|h_2(t,(\varphi_1,\varphi_2))| \le [\|\mathfrak{s}_1\|\chi_1(\mathfrak{r}) + \|\mathfrak{s}_2\|\chi_2(\mathfrak{r})]$$

Then, the problem in (1) has at least one solution for $[a,b]$.

Proof. Observe that the continuity of the operator $\mathfrak{K} : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ follows that of the functions h_1 and h_2. Next, let $\Omega_\mathfrak{r} \subset \mathfrak{X} \times \mathfrak{X}$ be bounded so we need to prove some steps.

The set $\mathfrak{K}(\Omega_\mathfrak{r})$ is bounded. We first show that $\mathfrak{K}_1$ is bounded. For any $(\varphi_1,\varphi_2) \in \Omega_\mathfrak{r}$, we have

$$\|\eth_1(\varphi_1,\varphi_2)\| \le [\|\mathfrak{z}_1\|\psi_1(\mathfrak{r}) + \|\mathfrak{z}_2\|\psi_2(\mathfrak{r})]\Phi_1,$$

and

$$|\wp_1(\varphi_1, \varphi_2) \leq \left[\|\mathfrak{s}_1\|\chi_1(\mathfrak{r}) + \|\mathfrak{s}_2\|\chi_2(\mathfrak{r})\right]\Psi_1,$$

This proves that $\mathfrak{K}_1$ is uniformly bounded. Similarly, we have

$$\|\mathfrak{K}_2(\varphi_1, \varphi_2)\| \leq \mathfrak{m}\mathfrak{r}.$$

In this step, to show that $\mathfrak{K}$ is equicontinuous, we only have to prove that $\mathfrak{K}_2$ is equicontinuous (in the previous theorem we proved that $\mathfrak{K}_1$ was equicontinuous). Let t_2, $t_1 \in [a, b]$ with $t_2 < t_1$. Then, we have

$$\begin{aligned}
&|\eth_2(\varphi_1, \varphi_2)(t_2) - \eth_2(\varphi_1, \varphi_2)(t_1)| \\
&= \Bigg| - \alpha_1 I^{\sigma_1,\mu}\varphi_1(t_2) + \Lambda_1 \frac{\Delta_{t_2}^{\sigma_1+1}}{\Gamma(\sigma_1+2)}\alpha_1 I^{\vartheta_1+\sigma_1,\mu}\varphi_1(b) + \kappa\frac{\Delta_{t_2}^{\sigma_1}}{\Gamma(\sigma_1+1)}\varsigma_1(t_2)\int_a^\xi \varphi_2(s)ds \\
&- [-\alpha_1 I^{\sigma_1,\mu}\varphi_1(t_1) + \Lambda_1 \frac{\Delta_{t_1}^{\sigma_1+1}}{\Gamma(\sigma_1+2)}\alpha_1 I^{\vartheta_1+\sigma_1,\mu}\varphi_1(b) + \kappa\frac{\Delta_{t_1}^{\sigma_1}}{\Gamma(\sigma_1+1)}\varsigma_1(t_1)\int_a^\xi \varphi_2(s)ds]\Bigg| \\
&\leq |\alpha_1||\Lambda_1|\frac{\Delta_b^{\sigma_1+1}(\Delta_{t_2}^{\vartheta_1+\sigma_1} - \Delta_{t_1}^{\vartheta_1+\sigma_1})}{\Gamma(\sigma_1+2)\Gamma(\vartheta_1+\sigma_1+1)} + [|\alpha_1| + \kappa|\varsigma_1(t_2) - \varsigma_1(t_1)|(\xi - a)]\frac{(\Delta_{t_2}^{\sigma_1} - \Delta_{t_1}^{\sigma_1})}{\Gamma(\sigma_1+1)}.
\end{aligned}$$

In a similar manner, we can obtain

$$\begin{aligned}
&|\wp_2(\varphi_2)(t_2) - \wp_2(\varphi_2)(t_1)| \\
&= | - \alpha_2 I^{\sigma_2,\mu}\varphi_2(t_2) + \Lambda_2\frac{\Delta_{t_2}^{\sigma_2}}{\Gamma(\sigma_2+1)}\alpha_2 I^{\vartheta_2+\sigma_2,\mu}\varphi_2(b) \\
&- [-\alpha_2 I^{\sigma_2,\mu}\varphi_2(t_1) + \Lambda_2\frac{\Delta_{t_1}^{\sigma_2}}{\Gamma(\sigma_2+1)}\alpha_2 I^{\vartheta_2+\sigma_2,\mu}\varphi_2(b)]| \\
&\leq \left[\frac{|\alpha_2|(\Delta_{t_2}^{\sigma_2} - \Delta_{t_1}^{\sigma_2})}{\Gamma(\sigma_2+1)} + |\Lambda_2|\frac{\Delta_b^{\sigma_2}(\Delta_{t_2}^{\vartheta_2+\sigma_2} - \Delta_{t_1}^{\vartheta_2+\sigma_2})}{\Gamma(\sigma_2+1)\Gamma(\vartheta_2+\sigma_2+1)}\right]
\end{aligned}$$

In the last step, it is verified that the set $\Pi = \{(\varphi_1, \varphi_2) \in \mathfrak{X} \times \mathfrak{X} : (\varphi_1, \varphi_2) = \delta\mathfrak{K}(\varphi_1, \varphi_2)\}$ is bounded. Let $(\varphi_1, \varphi_2) \in \Pi$ with $(\varphi_1, \varphi_2) = \delta\mathfrak{K}$ so we have

$$\varphi_1(t) = \delta[\eth_1(\varphi_1, \varphi_2)(t) + \eth_2(\varphi_1, \varphi_2)(t)],$$

and

$$\varphi_2(t) = \delta[\wp_1(\varphi_1, \varphi_2)(t) + \wp_2(\varphi_2)(t)].$$

Then,

$$\|\varphi_1\| \leq \left[\|\mathfrak{z}_1\|\psi_1(\mathfrak{r}) + \|\mathfrak{z}_2\|\psi_2(\mathfrak{r})\right]\Phi_1 + \|\varphi_1\||\alpha_1|\Phi_2 + \|\varphi_2\|\Phi_3,$$

and

$$\|\varphi_2)\| \leq \left[\|\mathfrak{s}_1\|\chi_1(\mathfrak{r}) + \|\mathfrak{s}_2\|\chi_2(\mathfrak{r})\right]\Psi_1 + \|\varphi_2\||\alpha_2|\Psi_2,$$

As a consequence, this implies that

$$\|(\varphi_1, \varphi_2)\| \leq \frac{\left[\|\mathfrak{z}_1\|\psi_1(\mathfrak{r}) + \|\mathfrak{z}_2\|\psi_2(\mathfrak{r})\right]\Phi_1 + \left[\|\mathfrak{s}_1\|\chi_1(\mathfrak{r}) + \|\mathfrak{s}_2\|\chi_2(\mathfrak{r})\right]\Psi_1}{1 - \mathfrak{m}}$$

By using this result, it can be established that set Π is bounded as a result of the Leray–Schauder alternative [43]. As a result, at least one solution exists for Systems (1) and (2). $\square$

5. Uniqueness Result

Theorem 10. *Suppose that: the functions $h_1, h_2 : [a,b] \times \mathbb{R} \times \mathbb{R}$ are continuous functions and ℓ_1, ℓ_2 are positive constants such that for all $t \in [a,b]$ and $\varphi_i\, \overset{*}{\varphi}_i \in \mathbb{R}$, we have:*

$$\|h_1(t,\varphi_1,\varphi_2) - h_1(t,\overset{*}{\varphi}_1,\overset{*}{\varphi}_2)\| \leq \ell_1(\|\varphi_1 - \overset{*}{\varphi}_1\| + \|\varphi_2 - \overset{*}{\varphi}_2\|),$$
$$\|h_2(t,\varphi_1,\varphi_2) - h_2(t,\overset{*}{\varphi}_1,\overset{*}{\varphi}_2)\| \leq \ell_2(\|\varphi_1 - \overset{*}{\varphi}_1\| + \|\varphi_2 - \overset{*}{\varphi}_2\|),$$

If

$$\mathfrak{m} + \ell_1\Phi_1 + \ell_2\Psi_1 < 1, \tag{16}$$

Systems (1) and (2) have unique solutions for $[a,b]$.

Proof. Consider the two assumptions $\sup_{t\in[a,b]} h_1(t,0,0) = \mathcal{N}_1$ and $\sup_{t\in[a,b]} h_2(t,0,0) = \mathcal{N}_2$. Choose a number $\mathfrak{r}$ that satisfies the condition below.

Here, we prove that $\mathfrak{K}\mathcal{B}_{\mathfrak{r}} \subset \mathcal{B}_{\mathfrak{r}}$, where $\mathcal{B}_{\mathfrak{r}} = \{(\mathfrak{u},\mathfrak{v}) : \|(\mathfrak{u},\mathfrak{v}\| \leq \mathfrak{r}\}$ and $\mathfrak{K}$ is defined by (5). Based on assumption $(\mathcal{L})$, for $(\varphi_1,\varphi_2) \in \mathcal{B}_{\mathfrak{r}}$, we have:

$$\begin{aligned}
|h_1(t,\varphi_1,\varphi_2)| &\leq |h_1(t,\varphi_1,\varphi_2) - h_1(t,0,0)| + |h_1(t,0,0)| \\
&\leq \ell_1(|\varphi_1| + |\varphi_2|) + \mathcal{N}_1 \\
&\leq \ell_1(\|\varphi_1\| + \|\varphi_2\|) + \mathcal{N}_1 \leq \ell_1\mathfrak{r} + \mathcal{N}_1,
\end{aligned}$$

and

$$\begin{aligned}
|h_2(t,\varphi_1,\varphi_2)| &\leq |h_2(t,\varphi_1,\varphi_2) - h_2(t,0,0)| + |h_2(t,0,0)| \\
&\leq \ell_2(|\varphi_1| + |\varphi_2|) + \mathcal{N}_2 \\
&\leq \ell_2(\|\varphi_1\| + \|\varphi_2\|) + \mathcal{N}_2 \leq \ell_2\mathfrak{r} + \mathcal{N}_2.
\end{aligned}$$

As a result, we obtain

$$\begin{aligned}
|\mathfrak{d}_1(\varphi_1,\varphi_2) + \mathfrak{d}_2(\varphi_1,\varphi_2)| &= \Big| I^{\sigma_1+\gamma_1,\mu}\hat{h}_1(t) - \Lambda_1 \frac{\Delta_t^{\sigma_1+1}}{\Gamma(\sigma_1+2)} I^{\vartheta_1+\sigma_1+\gamma_1,\mu}\hat{h}_1(b) \\
&\quad - \alpha_1 I^{\sigma_1,\mu}\varphi_1(t) + \Lambda_1 \frac{\Delta_t^{\sigma_1+1}}{\Gamma(\sigma_1+2)}\alpha_1 I^{\vartheta_1+\sigma_1,\mu}\varphi_1(b) + \kappa \frac{\Delta_t^{\sigma_1}}{\Gamma(\sigma_1+1)}\varsigma_1(t)\int_a^{\xi}\varphi_2(s)ds \Big| \\
&\leq (\ell_1\mathfrak{r} + \mathcal{N}_1)\left[\frac{\Delta_b^{\sigma_1+\gamma_1}}{\Gamma(\sigma_1+\gamma_1+1)} + |\Lambda_1|\frac{\Delta_b^{\vartheta_1+2\sigma_1+\gamma_1}}{\Gamma(\sigma_1+2)\Gamma(\vartheta_1+\sigma_1+\gamma_1+1)}\right] + \|\varphi_1\||\alpha_1|\left[\frac{\Delta_b^{\sigma_1}}{\Gamma(\sigma_1+1)}\right. \\
&\quad \left. + |\Lambda_1|\frac{\Delta_b^{\vartheta_1+2\sigma_1}}{\Gamma(\sigma_1+2)\Gamma(\vartheta_1+\sigma_1+1)}\right] + \|\varphi_2\|\left[\kappa|\varsigma_1|\frac{\Delta_b^{\sigma_1}}{\Gamma(\sigma_1+1)}(\xi - a)\right] \\
&\leq (\ell_1\mathfrak{r} + \mathcal{N}_1)\Phi_1 + \|\varphi_1\||\alpha_1|\Phi_2 + \|\varphi_2\|\Phi_3.
\end{aligned}$$

and

$$
\begin{aligned}
|\wp_1(\varphi_1,\varphi_2)+\wp_2(\varphi_2)| =& \Big| I^{\sigma_2+\gamma_2,\mu}\hat{h_2}(t)-\Lambda_2\frac{\Delta_t^{\sigma_2}}{\Gamma(\sigma_2+1)}I^{\vartheta_2+\sigma_2+\gamma_2,\mu}\hat{h_2}(b))\\
&-\alpha_2 I^{\sigma_2,\mu}\varphi_2(t)+\Lambda_2\frac{\Delta_t^{\sigma_2}}{\Gamma(\sigma_2+1)}\alpha_2 I^{\vartheta_2+\sigma_2,\mu}\varphi_2(b)\Big|\\
\leq& (\ell_2\mathfrak{r}+\mathcal{N}_2)\left[\frac{\Delta_b^{\sigma_2+\gamma_2}}{\Gamma(\sigma_2+\gamma_2+1)}+|\Lambda_2|\frac{\Delta_b^{\vartheta_2+2\sigma_2+\gamma_2}}{\Gamma(\sigma_2+1)\Gamma(\vartheta_2+\sigma_2+\gamma_2+1)}\right]\\
&+\|\varphi_2\||\alpha_2|\left[\frac{\Delta_b^{\sigma_2}}{\Gamma(\sigma_2+1)}+|\Lambda_2|\frac{\Delta_b^{\vartheta_2+2\sigma_2}}{\Gamma(\sigma_2+1)\Gamma(\vartheta_2+\sigma_2+1)}\right]\\
\leq& (\ell_2\mathfrak{r}+\mathcal{N}_2)\Psi_1+\|\varphi_2\||\alpha_2|\Psi_2.
\end{aligned}
$$

so

$$\|\mathfrak{K}(\varphi_1,\varphi_2)\|\leq[\ell_1\Phi_1+\ell_2\Psi_1]\mathfrak{r}+[\mathcal{N}_1\Phi_1+\mathcal{N}_2\Psi_1]+\mathfrak{r}\mathfrak{m},$$

and from (16), we obtain $\|\mathfrak{K}(\varphi_1,\varphi_2)\|\leq\mathfrak{r}$.

Next, for $(\varphi_1,\varphi_2),(\overset{*}{\varphi}_1,\overset{*}{\varphi}_2)$, as we have already established that $\mathfrak{K}_2$ is a contraction mapping, it is similarly easy to find:

$$\|\mathfrak{K}(\varphi_1,\varphi_2)-\mathfrak{K}(\overset{*}{\varphi}_1,\overset{*}{\varphi}_2)\|\leq[\mathfrak{m}+\ell_1\Phi_1+\ell_2\Psi_1]\Big[\|\varphi_1-\overset{*}{\varphi}_1\|+\|\varphi_2-\overset{*}{\varphi}_2\|\Big].$$

Since $\mathfrak{m}+\ell_1\Phi_1+\ell_2\Psi_1<1$, this indicates that $\mathfrak{K}$ is a contraction. Accordingly, Problems (1) and (2) have unique solutions based on Banach's contraction mapping principle. The proof is completed. □

6. Examples

Example 1. *The following fractional Langevin equation system can be considered:*

$$
\begin{cases}
{}^cD^{\frac{8}{7},t}({}^cD^{\frac{1}{2},t}+\frac{1}{10})\varphi_1(t)=\frac{t\sin 2t\,\tan^{-1}\varphi_1(t)}{(t+1)(7|\varphi_1(t)|+2)}+\frac{\cos t\,\sin\varphi_2(t)}{(5t^2+1)(2|\varphi_2(t)|+1)}, & t\in[1,2],\\
{}^cD^{\frac{1}{3},t}({}^cD^{\frac{7}{10},t}+\frac{1}{5})\varphi_2(t)=\frac{2t^2(2\varphi_1(t))}{(3t+1)(5|\varphi_1(t)|+1)}+\frac{(2\varphi_2(t)+3)}{(3|\varphi_2(t)|+\frac{9}{2})}, & t\in[1,2],\\
\varphi_1(1)=0,\qquad I^{\frac{3}{2},t}\varphi_1(2)=0,{}^cD^{\frac{1}{2},t}\varphi_1(1)=\frac{1}{4}\int_1^{\frac{6}{5}}\varphi_2(s)ds,\\
\varphi_2(1)=0,\qquad I^{\frac{9}{8},t}\varphi_2(2)=0.
\end{cases}
\tag{17}
$$

Here, $\gamma_1=\frac{8}{7},\gamma_2=\frac{1}{3}\ \sigma_1=\frac{1}{2},\sigma_2=\frac{7}{10}\ \vartheta_1=\frac{3}{2},\vartheta_2=\frac{9}{8},\alpha_1=\frac{1}{10},\alpha_2=\frac{1}{5},\xi_1=\frac{6}{5},\kappa=\frac{1}{4}$, *and*
$h_1(t,\varphi_1,\varphi_2)=\frac{t\sin 2t\,\tan^{-1}\varphi_1(t)}{(t+1)(7|\varphi_1(t)|+2)}+\frac{\cos t\,\sin\varphi_2(t)}{(5t^2+1)(2|\varphi_2(t)|+1)}$,
$h_2(t,\varphi_1,\varphi_2)=\frac{2t^2(2\varphi_1(t))}{(3t+1)(5|\varphi_1(t)|+1)}+\frac{(2\varphi_2(t)+3)}{(3|\varphi_2(t)|+\frac{9}{2})}$.

Since $h_1(t,\varphi_1,\varphi_2)\leq\frac{t\sin 2t}{(7t+7)}+\frac{\cos t}{(10t^2+2)}$, $h_2(t,\varphi_1,\varphi_2)\leq\frac{4t^2}{(15t+5)}+\frac{2}{3}$. *The Maple program can be used to determine the following values:*

$$
\begin{aligned}
\Phi_2 =& \frac{\Delta_b^{\sigma_1}}{\Gamma(\sigma_1+1)}+|\Lambda_1|\frac{\Delta_b^{\vartheta_1+2\sigma_1+1}}{\Gamma(\sigma_1+2)\Gamma(\vartheta_1+\sigma_1+1)}\\
\approx& 3.385137501,
\end{aligned}
$$

$$
\begin{aligned}
\Phi_3 =& \kappa|\varsigma_1|\frac{\Delta_b^{\sigma_1}}{\Gamma(\sigma_1+1)}(\xi-a)\\
\approx& 0.1692568750,
\end{aligned}
$$

and

$$\Psi_2 = \frac{\Delta_b^{\sigma_2}}{\Gamma(\sigma_2+1)} + |\Lambda_2| \frac{\Delta_b^{\vartheta_2+2\sigma_2}}{\Gamma(\sigma_2+1)\Gamma(\vartheta_2+\sigma_2+1)}$$

$$\approx 2.201094810.$$

Thus, $\mathfrak{m} \approx 0.6094758370 < 1$. For $[1,2]$, System (17) must have at least one solution according to Theorem 8.

Example 2. *The following fractional Langevin equation system can be considered.*

$$\begin{cases} {}^cD^{\frac{5}{3},e^t}({}^cD^{\frac{2}{7},e^t} + \frac{5}{9})\varphi_1(t) = \frac{5}{(3t^2+9)}\left(\frac{\varphi_1^2(t)}{7(|\varphi_1(t)|+1)} + \frac{3}{8}\right) + \frac{1}{(3+t)}\left(\frac{\varphi_2}{3(|\varphi_2(t)|+1)}\right), & t \in [0, \ln 2], \\ {}^cD^{\frac{2}{9},e^t}({}^cD^{\frac{3}{4},e^t} + \frac{1}{6})\varphi_2(t) = \frac{\sin t}{(3+t)}\left(\frac{3\varphi_1}{|\varphi_1(t)|+7}\right) + \frac{1}{(5+t^2)}\left(\frac{\varphi_2}{2|\varphi_2(t)|+\frac{5}{3}}\right), & t \in [0, \ln 2], \\ \varphi_1(0) = 0, \quad I^{\frac{3}{5},e^t}\varphi_1(\ln 2) = 0, {}^cD^{\frac{2}{3},e^t}\varphi_1(0) = \frac{1}{9}\int_0^{\frac{5}{8}} \varphi_2(s)ds, \\ \varphi_2(0) = 0, \quad I^{\frac{7}{8},e^t}\varphi_2(\ln 2) = 0. \end{cases} \tag{18}$$

Here, $\gamma_1 = \frac{5}{3}, \gamma_2 = \frac{2}{9}\ \sigma_1 = \frac{2}{7}, \sigma_2 = \frac{3}{4}\ \vartheta_1 = \frac{3}{5}, \vartheta_2 = \frac{7}{8}, \alpha_1 = \frac{5}{9}, \alpha_2 = \frac{1}{6}, \xi_1 = \frac{5}{8}, \kappa = \frac{1}{9}$, *and*
$h_1(t,\varphi_1,\varphi_2) = \frac{5}{(3t^2+9)}\left(\frac{\varphi_1^2(t)}{7(|\varphi_1(t)|+1)} + \frac{3}{8}\right) + \frac{1}{(3+t)}\left(\frac{\varphi_2}{3(|\varphi_2(t)|+1)}\right)$,
$h_2(t,\varphi_1,\varphi_2) = \frac{\sin t}{(3+t)}\left(\frac{3\varphi_1}{|\varphi_1(t)|+7}\right) + \frac{1}{(5+t^2)}\left(\frac{\varphi_2}{2|\varphi_2(t)|+\frac{5}{3}}\right)$.

Since $h_1(t,\varphi_1,\varphi_2) \leq \frac{1}{9}(\frac{5}{7}|\varphi_1| + |\varphi_2| + \frac{3}{8})$, $h_2(t,\varphi_1,\varphi_2) \leq \frac{1}{7}|\varphi_1| + \frac{3}{25}|\varphi_2|$. *The Maple program can be used to determine that* $\mathfrak{m} = 0.5190783766 < 1$.

Thus, System (18) must have at least one solution according to Theorem 9.

Example 3. *The following fractional Langevin equation system can be considered.*

$$\begin{cases} {}^cD^{\frac{11}{7},t^2}({}^cD^{\frac{2}{3},t^2} + \frac{2}{11})\varphi_1(t) = \frac{1}{10}\frac{\varphi_1^2+|\varphi_1|}{(|\varphi_1|+3)}\sin^2 t + \frac{5}{9}(1 + t\cos t\varphi_2), & t \in [0,1], \\ {}^cD^{\frac{2}{3},t^2}({}^cD^{\frac{1}{2},t^2} + \frac{3}{10})\varphi_2(t) = \frac{2}{5}\left(\frac{|\varphi_1|}{(|\varphi_1|+5)}\cos^2 t + 1\right) + \frac{3}{7}(1 + \frac{t}{1+t}\sin t\varphi_2), & t \in [0,1], \\ \varphi_1(0) = 0, \quad I^{\frac{5}{4},t}\varphi_1(1) = 0, {}^cD^{\frac{1}{2},t}\varphi_1(1) = \frac{1}{7}\int_1^{\frac{2}{5}} \varphi_2(s)ds, \\ \varphi_2(0) = 0, \quad I^{\frac{3}{2},t}\varphi_2(1) = 0. \end{cases} \tag{19}$$

Here, $\gamma_1 = \frac{11}{7}, \gamma_2 = \frac{2}{3}\ \sigma_1 = \frac{2}{3}, \sigma_2 = \frac{1}{2}\ \vartheta_1 = \frac{5}{4}, \vartheta_2 = \frac{3}{2}, \alpha_1 = \frac{2}{11}, \alpha_2 = \frac{3}{10}, \xi_1 = \frac{2}{5}, \kappa = \frac{1}{7}$, *and*
$h_1(t,\varphi_1,\varphi_2) = \frac{1}{10}\frac{\varphi_1^2+|\varphi_1|}{(|\varphi_1|+3)}\sin^2 t + \frac{5}{9}(1 + t\cos t\varphi_2)$,
$h_2(t,\varphi_1,\varphi_2) = \frac{2}{5}\left(\frac{|\varphi_1|}{(|\varphi_1|+5)}\cos^2 t + 1\right) + \frac{3}{7}(1 + \frac{t}{1+t}\sin t\varphi_2)$.

Since $|h_1(t,\varphi_1,\varphi_2) - h_1(t,\overset{*}{\varphi}_1,\overset{*}{\varphi}_2)| \leq \frac{5}{9}\left(|\varphi_1 - \overset{*}{\varphi}_1| + |\varphi_2 - \overset{*}{\varphi}_2|\right)$, $|h_2(t,\varphi_1,\varphi_2) - h_2(t,\overset{*}{\varphi}_1,\overset{*}{\varphi}_2)| \leq \frac{3}{7}\left(|\varphi_1 - \overset{*}{\varphi}_1| + |\varphi_2 - \overset{*}{\varphi}_2|\right)$. *The Maple program can be used to obtain:*

$$\mathfrak{m} + \ell_1\Phi_1 + \ell_2\Psi_1 < 0.8496699665 < 1,$$

which means (based on Theorem 10) that the given system has only one solution for $[0,1]$.

7. Conclusions

In this work, we investigate coupled nonlinear Langevin fractional problems with different orders of μ-Caputo fractional derivatives on arbitrary domains with nonlocal integral boundary conditions. We address the original problem by transforming it into an equivalent fixed-point problem and applying the standard tools of modern functional analysis to determine its existence and uniqueness. Our results are not only new in this setting but also provided some special cases that we obtained by fixing certain parameters or giving a function-specific definition to the appropriate interval, for example:

(1) $\kappa = 0$

$$^{c}D^{\gamma_i,\mu}(^{c}D^{\sigma_i,\mu} + \alpha_i)\varphi_i(t) = \Xi_i(t, \varphi_1(t), \varphi_2(t)), \qquad t \in [a,b], i = 1,2.$$

Subjected to the specific boundary conditions

$$\varphi_i(a) = 0, \qquad I^{\vartheta_i,\mu}\varphi_i(b) = 0,$$
$$^{c}D^{\sigma_1,\mu}\varphi_1(a) = \kappa \int_a^{\xi} \varphi_2(s)ds.$$

(2) $\mu(t) = t$

$$^{c}D^{\gamma_i}(^{c}D^{\sigma_i} + \alpha_i)\varphi_i(t) = \Xi_i(t, \varphi_1(t), \varphi_2(t)), \qquad t \in [a,b], i = 1,2.$$

Subjected to the specific boundary conditions

$$\varphi_i(a) = 0, \qquad I^{\vartheta_i,\mu}\varphi_i(b) = 0,$$
$$^{c}D^{\sigma_1,\mu}\varphi_1(a) = \kappa \int_a^{\xi} \varphi_2(s)ds.$$

(3) $\mu(t) = t^{\rho}$

$$^{c}D^{\gamma_i,\rho}(^{c}D^{\sigma_i,\rho} + \alpha_i)\varphi_i(t) = \Xi_i(t, \varphi_1(t), \varphi_2(t)), \qquad t \in [a,b], i = 1,2.$$

Subjected to the specific boundary conditions

$$\varphi_i(a) = 0, \qquad I^{\vartheta_i,\mu}\varphi_i(b) = 0,$$
$$^{c}D^{\sigma_1,\mu}\varphi_1(a) = \kappa \int_a^{\xi} \varphi_2(s)ds.$$

(4) $\mu(t) = \log t$

$$^{c}D^{\gamma_i,\mathcal{H}}(^{c}D^{\sigma_i,\mathcal{H}} + \alpha_i)\varphi_i(t) = \Xi_i(t, \varphi_1(t), \varphi_2(t)), \qquad t \in [a,b], a > 0, i = 1,2.$$

Subjected to the specific boundary conditions

$$\varphi_i(a) = 0, \qquad I^{\vartheta_i,\mu}\varphi_i(b) = 0,$$
$$^{c}D^{\sigma_1,\mu}\varphi_1(a) = \kappa \int_a^{\xi} \varphi_2(s)ds.$$

In future work, we could investigate our results based on other FDs such as the Abu-Shady–Kaabar FD, Katugampola derivative, or conformable derivative.

Author Contributions: Conceptualization, Y.A., K.K., L.A. and E.-s.E.-h.; methodology, L.A., E.-s.E.-h. and K.K.; software, K.K., Y.A., L.A. and E.-s.E.-h.; validation, K.K., Y.A., L.A. and E.-s.E.-h.; formal analysis, E.-s.E.-h. and K.K.; investigation, K.K., Y.A., L.A. and E.-s.E.-h.; data curation, K.K., Y.A., L.A. and E.-s.E.-h.; writing—original draft preparation, K.K.; writing—review and editing, K.K., Y.A., L.A. and E.-s.E.-h.; visualization, K.K., Y.A., L.A. and E.-s.E.-h.; supervision, K.K.; project administration, K.K., Y.A., L.A. and E.-s.E.-h. All authors have read and agreed to the published version of the manuscript.

Funding: This work was funded by the University of Jeddah, Jeddah, Saudi Arabia, under grant No. (UJ-22-DR-113). The author, therefore, acknowledges with thanks the University of Jeddah for its technical and financial support.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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