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Some Families of Differential Equations Associated with Multivariate Hermite Polynomials

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Abstract: In this article, the recurrence relations and shift operators for multivariate Hermite polynomials are derived using the factorization approach. Families of differential equations, including differential, integro–differential, and partial differential equations, are obtained using these operators. The Volterra integral for these polynomials is also discovered.

Keywords: Hermite polynomials; recurrence relation; shift operators; differential equations; integral equation

MSC: 26A33; 33B10; 33C45; 45J05; 65Q30

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Citation: Alkahtani, B.S.T.; Alazman, I.; Wani, S.A. Some Families of Differential Equations Associated with Multivariate Hermite Polynomials. *Fractal Fract.* **2023**, *7*, 390. https://doi.org/10.3390/fractalfract7050390

Academic Editors: Viorel-Puiu Paun, Asifa Tassaddiq, Muhammad Yaseen and Sania Qureshi

Received: 7 March 2023
Revised: 11 April 2023
Accepted: 27 April 2023
Published: 8 May 2023

1. Introduction and Preliminaries

To demonstrate the main idea underlying the factorization approach, let us briefly explore a subject that is commonly discussed: the contrast between Maxwell's and Dirac's equations. Both of these linear systems of equations have first-order partial derivatives, making them useful tools for addressing important questions related to eigenvalue problems for physicists. Both the Maxwell and Dirac equations are Lorentz invariant. It should be emphasized that difficulties with infinite self-energies may arise from oversimplifying the linearity of Maxwell's equations.

Therefore, the factorization method is a powerful technique used in mathematical analyses to derive recurrence relations and differential equations for special functions, including the Appell polynomials and their members. The main idea of the factorization method is to express a given function in terms of a sequence of multiplicative and derivative operators, which enables the derivation of recurrence relations and differential equations satisfied by the function. The factorization method, described in the work of Infield [1], provides a practical technique for solving eigenvalue problems that are of great importance to physicists. The key concept involves considering two first-order differential equations, the solution of which leads to a second-order differential equation of the same magnitude. The computation of transition probabilities also takes the production process into account. This approach is designed to handle perturbation issues.

Let a polynomial sequence of degree n with n with $n = 0, 1, 2, \cdots$ be denoted by $\{Q_n(q_1)\}_{n=0}^{\infty}$. The sequences of the differential operators Ψ_n^- and Ψ_n^+ operating on $\{Q_n(q_1)\}_{n=0}^{\infty}$ satisfy the following properties:

$$\Psi_n^-(Q_n(q_1)) = Q_{n-1}(q_1) \tag{1}$$

and

$$\Psi_n^+(Q_n(q_1)) = Q_{n+1}(q_1). \tag{2}$$

A significant property, named differential equation

$$(\Psi_{n+1}^{-}\Psi_{n}^{+})\{Q_n(q_1)\} = Q_n(q_1), \tag{3}$$

is derived by using Ψ_n^- and Ψ_n^+ operators. The factorization approach is the procedure used to construct differential equations from Equation (3). To ensure that Equation (3) holds, the primary goal of the factorization approach is to obtain the multiplicative operator Ψ_n^+ and the derivative operator Ψ_n^-.

The factorization method has been widely used in various areas of mathematics, including the special functions theory, orthogonal polynomials, and differential equations, to name a few. It provides a systematic approach to deriving recurrence relations and differential equations for special functions, which can be helpful in solving mathematical problems and analyzing the properties of these functions.

The operational rule for the 2-V Hermite Kampé de Fériet polynomials (2VHKdFP) $\mathcal{Y}_n^{[2]}(q_1, q_2)$ [2] is provided below:

$$\exp\left(q_2 \frac{\partial^2}{\partial {q_1}^2}\right) {q_1}^n = \mathcal{Y}_n^{[2]}(q_1, q_2) \tag{4}$$

and the generating function shown below defines these polynomials:

$$\exp(q_1\xi + q_2\xi^2) = \sum_{n=0}^{\infty} \mathcal{Y}_n^{[2]}(q_1, q_2) \frac{\xi^n}{n!}. \tag{5}$$

The following series also defines these polynomials:

$$\mathcal{Y}_n^{[2]}(q_1, q_2) = n! \sum_{k=0}^{[\frac{n}{2}]} \frac{q_2^k q_1^{n-2k}}{k!(n-2k)!}. \tag{6}$$

Further, the multivariate Hermite polynomials (MVHP) $\mathcal{Y}_n^{[m]}(q_1, q_2, \cdots, q_m)$ [3] are given by the following operational rule:

$$\exp\left(q_2 \frac{\partial^2}{\partial {q_1}^2} + q_3 \frac{\partial^3}{\partial {q_1}^3} + \cdots + q_m \frac{\partial^m}{\partial {q_1}^m}\right) q_1^n = \mathcal{Y}_n^{[m]}(q_1, q_2, \cdots, q_m). \tag{7}$$

These polynomials are given by generating expression:

$$\exp(q_1\xi + q_2\xi^2 + \cdots + q_m\xi^m) = \sum_{n=0}^{\infty} \mathcal{Y}_n^{[m]}(q_1, q_2, \cdots, q_m) \frac{\xi^n}{n!} \tag{8}$$

and are defined by the series:

$$\mathcal{Y}_n^{[m]}(q_1, q_2, \cdots q_m) = n! \sum_{r=0}^{[n/m]} \frac{p_m^r \mathcal{Y}_{n-mr}^{[m]}(q_1, q_2, \cdots, p_{m-1})}{r!(n-mr)!}. \tag{9}$$

As a result of the significance of the factorization approach in comparing Maxwell's and Dirac's equations, it was first developed by He and Ricci [4]. Both of these linear systems of equations involve first-order partial derivatives, making them valuable resources for physicists seeking solutions to important eigenvalue problem-related inquiries. This approach helps derive the differential equation for the Appell polynomials. In [5], the multidimensional extensions of the Bernoulli and Appell polynomials are described. The extended 2*D* Bernoulli and Euler polynomials are utilized to produce the differential, integro–differential, and partial differential equations in the [6]. This method is also used to generate the hybrid form, 2*D* extended, and mixed-type polynomial integro–differential equations for the Appell family; for instance, see [7–15]. Additionally, in [16], the Appell polynomials are used to build a collection of finite order differential equations by extending the factorization strategy using k-times shift operators. These findings inspire the creation of families of differential equations, shift operators, and recurrence relations for multivariate Hermite polynomials represented by (8); because of their importance in a variety of fields, such as in quantum mechanics, they arise in the solution of the Schrodinger equation for the harmonic oscillator. They are also useful in the study of random walks and Brownian motion, as well as in the approximation theory and numerical analysis. Additionally, the Hermite polynomials have applications in the biological and medical sciences, such as in the analysis of EEG and ECG signals. These polynomials have many useful properties that make them valuable in various areas of mathematics and physics. For instance, they are complete, meaning that any square-integrable function on the real line can be expressed as a series of Hermite polynomials. Moreover, they have a three-term recurrence relation, which makes their computation efficient. The Hermite polynomials are also orthogonal with respect to the weight function $e^{-q_1^2}$, which arises frequently in problems involving Gaussian integrals.

In this article, the families of differential equations connected to the multivariate Hermite polynomials are derived using the factorization method. The recurrence relation for these multivariate Hermite polynomials is first derived in Section 2. Shift operators are also generated for these polynomials. Section 3 develops families of differential equations for these polynomials, including differential, integro–differential, and partial differential equations. The Volterra integral equation for these polynomials is developed in Section 4.

2. Recurrence Relations and Shift Operators

Recurrence relations are mathematical equations that define the terms of a sequence based on previous terms in the sequence. In the context of polynomials, recurrence relations can be utilized to express the coefficients of a polynomial in terms of its preceding coefficients. Recurrence relations in polynomials serve as a powerful tool for solving polynomial equations, generating polynomial sequences, and examining the properties of polynomials in various mathematical contexts. They offer a concise means of expressing the connection between the coefficients of a polynomial and often lead to efficient algorithms for polynomial computations. Moreover, recurrence relations in polynomials can also describe other types of polynomial sequences, such as those encountered in the study of orthogonal polynomials. In these cases, the coefficients are determined by recurrence relations involving inner products or other mathematical characteristics of the polynomials. For the MVHP $\mathcal{Y}_n^{[m]}(q_1, q_2, \cdots, q_m)$, we developed recurrence relations and shift operators in this section. First, by demonstrating the following conclusion, we construct the recurrence relation for the MVHP $\mathcal{Y}_n^{[m]}(q_1, q_2, \cdots, q_m)$:

Theorem 1. *The multivariate Hermite polynomials $\mathcal{Y}_n^{[m]}(q_1, q_2, \cdots, q_m)$ fulfil the following recurrence relation:*

$$\mathcal{Y}_{n+1}^{[m]}(q_1, q_2, \cdots, q_m) = q_1 \mathcal{Y}_n^{[m]}(q_1, q_2, \cdots, q_m) + 2nq_2 \mathcal{Y}_{n-1}^{[m]}(q_1, q_2, \cdots, q_m) + 3n(n-1)q_3$$
$$\times \mathcal{Y}_{n-2}^{[m]}(q_1, q_2, \cdots, q_m) + \cdots + n(n-1)(n-2)\cdots(n-m+1)q_m \mathcal{Y}_{n-m}^{[m]}(q_1, q_2, \cdots, q_m). \tag{10}$$

Proof. Differentiating expression (8) w.r.t. ξ, we have

$$\frac{\partial}{\partial \xi}\Big\{ \exp(q_1\xi + q_2\xi^2 + \cdots + q_m\xi^m) \Big\} = \Big\{ q_1 + 2q_2\xi + 3q_3\xi^2 + \cdots + mq_m\xi^{m-1} \Big\}$$
$$\times \Big\{ \exp(q_1\xi + q_2\xi^2 + \cdots + q_m\xi^m) \Big\}. \tag{11}$$

Using the r.h.s. of expression (8) in expression (11), we have

$$\frac{\partial}{\partial \xi}\Big\{\sum_{n=0}^{\infty}\mathcal{Y}_n^{[m]}(q_1,q_2,\cdots,q_m)\frac{\xi^n}{n!}\Big\} = \Big\{q_1+2q_2\xi+3q_3\xi^2+\cdots+mq_m\xi^{m-1}\Big\} \times\Big\{\sum_{n=0}^{\infty}\mathcal{Y}_n^{[m]}(q_1,q_2,\cdots,q_m)\frac{\xi^n}{n!}\Big\}, \tag{12}$$

which can further be written as

$$\sum_{n=0}^{\infty}n\mathcal{Y}_n^{[m]}(q_1,q_2,\cdots,q_m)\frac{\xi^{n-1}}{n!} = q_1\sum_{n=0}^{\infty}\mathcal{Y}_n^{[m]}(q_1,q_2,\cdots,q_m)\frac{\xi^n}{n!}+2q_2\sum_{n=0}^{\infty}\mathcal{Y}_n^{[m]}(q_1,q_2,\cdots,q_m) \times\frac{\xi^{n+1}}{n!}+3q_3\sum_{n=0}^{\infty}\mathcal{Y}_n^{[m]}(q_1,q_2,\cdots,q_m)\frac{\xi^{n+2}}{n!}+\cdots+mq_m\sum_{n=0}^{\infty}\mathcal{Y}_n^{[m]}(q_1,q_2,\cdots,q_m)\frac{\xi^{n+m}}{n!}. \tag{13}$$

Replacing $n\to n+1$ in the l.h.s. of the previous equation, $n\to n-1$, $n\to n-2$, $\cdots$ $n\to n-m$ in the r.h.s. of the second, third, and last terms of the previous equation, we have

$$\sum_{n=0}^{\infty}\mathcal{Y}_{n+1}^{[m]}(q_1,q_2,\cdots,q_m)\frac{\xi^n}{n!} = q_1\sum_{n=0}^{\infty}\mathcal{Y}_n^{[m]}(q_1,q_2,\cdots,q_m)\frac{\xi^n}{n!}+2q_2\sum_{n=0}^{\infty}n\mathcal{Y}_{n-1}^{[m]}(q_1,q_2,\cdots,q_m)\frac{\xi^n}{n!} +3q_3\sum_{n=0}^{\infty}n(n-1)\mathcal{Y}_{n-2}^{[m]}(q_1,q_2,\cdots,q_m)\frac{\xi^n}{n!}+\cdots+mq_mn(n-1)(n-2)\cdots(n-m+1) \times\sum_{n=0}^{\infty}\mathcal{Y}_{n-m}^{[m]}(q_1,q_2,\cdots,q_m)\frac{\xi^n}{n!}.$$

which further can be written as

$$\sum_{n=0}^{\infty}\mathcal{Y}_{n+1}^{[m]}(q_1,q_2,\cdots,q_m)\frac{\xi^n}{n!} = \sum_{n=0}^{\infty}\Bigg[q_1\mathcal{Y}_n^{[m]}(q_1,q_2,\cdots,q_m)+2q_2n\mathcal{Y}_{n-1}^{[m]}(q_1,q_2,\cdots,q_m) +3q_3n(n-1)\mathcal{Y}_{n-2}^{[m]}(q_1,q_2,\cdots,q_m)+\cdots+mq_mn(n-1)(n-2)\cdots(n-m+1) \times\mathcal{Y}_{n-m}^{[m]}(q_1,q_2,\cdots,q_m)\Bigg]\frac{\xi^n}{n!}. \tag{14}$$

Assertion (10) is proven by equating the coefficients of the same exponents of $\frac{\xi^n}{n!}$ on b/s of Equation (14). □

The shift operators and recurrence relation are valuable tools for analyzing the properties of polynomials. The shift operators enable the translation of polynomials in different directions and dimensions, while the recurrence relation allows for the computation of polynomial values at higher degrees based on their values at lower degrees. Shift operators in polynomials are linear operators that shift the indices of a polynomial's coefficients by a fixed amount. Let us consider a polynomial of degree n with coefficients $a_0, a_1, a_2, ..., a_{n-1}, a_n$. The left shift operator shifts the coefficients to the left by k positions, resulting in a new polynomial of the same degree n but with coefficients $b_0, b_1, b_2, ..., b_{n-k-1}, b_{n-k}, ..., b_{n-1}$. The right shift operator does the opposite, shifting the coefficients to the right by k positions. Next, by demonstrating the following conclusion, we construct the shift operators for the MVHP $\mathcal{Y}_n^{[m]}(q_1,q_2,\cdots,q_m)$, listed as:

Theorem 2. *The MVHP $\mathcal{Y}_n^{[m]}(q_1,q_2,\cdots,q_m)$ satisfies the listed shift operators:*

$$_{q_1}\pounds_n^- := \frac{1}{n}D_{q_1}, \tag{15}$$

$$ {}_{q_2}\pounds_n^- := \frac{1}{n} D_{q_1}^{-1} D_{q_2}, \tag{16} $$

$$ {}_{q_3}\pounds_n^- := \frac{1}{n} D_{q_1}^{-2} D_{q_3}, \tag{17} $$

$$ \vdots $$

$$ {}_{q_m}\pounds_n^- := \frac{1}{n} D_{q_1}^{-(m-1)} D_{q_m}, \tag{18} $$

$$ {}_{q_1}\pounds_n^+ := q_1 + 2q_2 D_{q_1} + 3q_3 D_{q_1}^2 + \cdots + m q_m D_{q_1}^{m-1} \tag{19} $$

$$ {}_{q_2}\pounds_n^+ := q_1 + 2q_2 D_{q_1}^{-1} D_{q_2} + 3q_3 D_{q_1}^{-2} D_{q_2}^2 + \cdots + m q_m D_{q_1}^{-(m-1)} D_{q_2}^{m-1} \tag{20} $$

and

$$ {}_{q_3}\pounds_n^+ := q_1 + 2q_2 D_{q_1}^{-2} D_{q_3} + 3q_3 D_{q_1}^{-4} D_{q_3}^2 + \cdots + m q_m D_{q_1}^{-2(m-1)} D_{q_3}^{m-1}, \tag{21} $$

$$ \vdots $$

$$ {}_{q_m}\pounds_n^+ := q_1 + 2q_2 D_{q_1}^{-(m-1)} D_{q_m} + 3q_3 D_{q_1}^{-2(m-1)} D_{q_m}^2 + \cdots + m q_m D_{q_1}^{-(m-1)^2} D_{q_m}^{m-1}, \tag{22} $$

where

$$ D_{q_1} := \frac{\partial}{\partial q_1}, D_{q_2} := \frac{\partial}{\partial q_2}, D_{q_3} := \frac{\partial}{\partial q_3} and D_{q_1}^{-1} := \int_0^{q_1} f(\eta) d\eta. $$

Proof. Taking the derivatives of (8) w.r.t. q_1 and, thus, equating the coefficients of similar exponents of ξ on b/s of the resultant equation yields the following expression

$$ \frac{\partial}{\partial q_1} \{\mathcal{Y}_n^{[m]}(q_1, q_2, \cdots, q_m)\} = n \mathcal{Y}_{n-1}^{[m]}(q_1, q_2, \cdots, q_m), \tag{23} $$

Consequently, we have

$$ {}_{q_1}\pounds_n^- \{\mathcal{Y}_n^{[m]}(q_1, q_2, \cdots, q_m)\} = \frac{1}{n} D_{q_1} \{\mathcal{Y}_n^{[m]}(q_1, q_2, \cdots, q_m)\} = \mathcal{Y}_{n-1}^{[m]}(q_1, q_2, \cdots, q_m), \tag{24} $$

which proves assertion (15).

Next, taking the derivatives of (8) w.r.t. q_2 and, thus, equating the coefficients of the same exponents of ξ on b/s of the resultant equation, we have

$$ \frac{\partial}{\partial q_2} \{\mathcal{Y}_n^{[m]}(q_1, q_2, \cdots, q_m)\} = n(n-1) \mathcal{Y}_{n-2}^{[m]}(q_1, q_2, \cdots, q_m). \tag{25} $$

The above equation can also be written as

$$ \frac{\partial}{\partial q_2} \{\mathcal{Y}_n^{[m]}(q_1, q_2, \cdots, q_m)\} = n \frac{\partial}{\partial q_1} \{\mathcal{Y}_{n-1}^{[m]}(q_1, q_2, \cdots, q_m)\}, \tag{26} $$

which finally gives

$$ {}_{q_2}\pounds_n^- \{\mathcal{Y}_n^{[m]}(q_1, q_2, \cdots, q_m)\} = \frac{1}{n} D_{q_1}^{-1} D_{q_2} \{\mathcal{Y}_n^{[m]}(q_1, q_2, \cdots, q_m)\} = \mathcal{Y}_{n-1}^{[m]}(q_1, q_2, \cdots, q_m). \tag{27} $$

Thus, assertion (16) is proven.

Again, taking the derivatives of (8) w.r.t. q_3 and, thus, equating the coefficients of the same exponents of ξ on b/s of the resultant equation, we have

$$ \frac{\partial}{\partial q_3} \{\mathcal{Y}_n^{[m]}(q_1, q_2, \cdots, q_m)\} = n(n-1)(n-2) \mathcal{Y}_{n-3}^{[m]}(q_1, q_2, \cdots, q_m). \tag{28} $$

The above equation can also be written as

$$\frac{\partial}{\partial q_3}\{\mathcal{Y}_n^{[m]}(q_1,q_2,\cdots,q_m)\} = n\frac{\partial^2}{\partial q_1^2}\{\mathcal{Y}_{n-1}^{[m]}(q_1,q_2,\cdots,q_m)\}, \tag{29}$$

which finally gives

$$ {}_{q_3}\pounds_n^{-}\{\mathcal{Y}_n^{[m]}(q_1,q_2,\cdots,q_m)\} = \tfrac{1}{n}D_{q_1}^{-2}D_{q_3}\{\mathcal{Y}_n^{[m]}(q_1,q_2,\cdots,q_m)\} = \mathcal{Y}_{n-1}^{[m]}(q_1,q_2,\cdots,q_m). \tag{30}$$

Thus, it yields assertion (17).

Finally, taking the derivatives of (8) w.r.t. q_m and, thus, equating the coefficients of the same exponents of ξ on b/s of the resultant equation, we have

$$\frac{\partial}{\partial q_m}\{\mathcal{Y}_n^{[m]}(q_1,q_2,\cdots,q_m)\} = n(n-1)(n-2)(n-m+1)\mathcal{Y}_{n-m}^{[m]}(q_1,q_2,\cdots,q_m). \tag{31}$$

The above equation can also be written as

$$\frac{\partial}{\partial q_m}\{\mathcal{Y}_n^{[m]}(q_1,q_2,\cdots,q_m)\} = n\frac{\partial^{m-1}}{\partial q_1^{m-1}}\{\mathcal{Y}_{n-1}^{[m]}(q_1,q_2,\cdots,q_m)\}, \tag{32}$$

which finally gives

$$ {}_{q_m}\pounds_n^{-}\{\mathcal{Y}_n^{[m]}(q_1,q_2,\cdots,q_m)\} = \tfrac{1}{n}D_{q_1}^{-(m-1)}D_{q_m}\{\mathcal{Y}_n^{[m]}(q_1,q_2,\cdots,q_m)\} = \mathcal{Y}_{n-1}^{[m]}(q_1,q_2,\cdots,q_m). \tag{33}$$

Therefore, assertion (18) is proven.

To demonstrate the equation for the raising operator (19), we use the expression:

$$\mathcal{Y}_{n-m}^{[m]}(q_1,q_2,\cdots,q_m) = ({}_{q_1}\pounds_{n-m+1}^{-}{}_{q_1}\pounds_{n-m+2}^{-}\cdots{}_{q_1}\pounds_{n-1}^{-}{}_{q_1}\pounds_n^{-})\{\mathcal{Y}_n^{[m]}(q_1,q_2,\cdots,q_m)\}, \tag{34}$$

considering (24), the above expression can be simplified as:

$$\mathcal{Y}_{n-m}^{[m]}(q_1,q_2,\cdots,q_m) = \frac{(n-m)!}{n!}D_{q_1}^{m}\{\mathcal{Y}_n^{[m]}(q_1,q_2,\cdots,q_m)\}. \tag{35}$$

Inserting (35) in the recurrence relation (10), it follows that

$$\mathcal{Y}_{n+1}^{[m]}(q_1,q_2,\cdots,q_m) = \Big(q_1 + 2q_2D_{q_1} + 3q_3D_{q_1}^2 + \cdots + mq_mD_{q_1}^{m-1}\Big)\mathcal{Y}_n^{[m]}(q_1,q_2,\cdots,q_m). \tag{36}$$

Thus, (19) for the raising operator ${}_{q_1}\pounds_n^{+}$ is proven.

Now, to demonstrate the raising operator (20), the relation listed below is considered:

$$\mathcal{Y}_{n-m}^{[m]}(q_1,q_2,\cdots,q_m) = ({}_{q_2}\pounds_{n-m+1}^{-}{}_{q_2}\pounds_{n-m+2}^{-}\cdots{}_{q_2}\pounds_{n-1}^{-}{}_{q_2}\pounds_n^{-})\{\mathcal{Y}_n^{[m]}(q_1,q_2,\cdots,q_m)\}, \tag{37}$$

considering (27), the above expression can be expanded as follows:

$$\mathcal{Y}_{n-m}^{[m]}(q_1,q_2,\cdots,q_m) = \frac{(n-m)!}{m!}D_{q_1}^{-(m-1)}D_{q_2}^{(m-1)}\{\mathcal{Y}_n^{[m]}(q_1,q_2,\cdots,q_m)\}. \tag{38}$$

Inserting (38) in the recurrence relation (10), it follows that

$$\mathcal{Y}_{n+1}^{[m]}(q_1,q_2,\cdots,q_m) = \Big(q_1 + 2q_2D_{q_1}^{-1}D_{q_2} + 3q_3D_{q_1}^{-2}D_{q_2}^2 + \cdots + mq_mD_{q_1}^{-(m-1)}D_{q_2}^{m-1}\Big). \tag{39}$$

Thus, assertion (20) of the raising operator ${}_{q_2}\pounds_n^{+}$ is proven.

Next, to demonstrate the raising operator ${}_{q_3}\pounds_n^{+}$, the expression listed below is considered:

$$\mathcal{Y}_{n-m}^{[m]}(q_1,q_2,\cdots,q_m) = ({}_{q_3}\pounds_{n-m+1}^{-}{}_{q_3}\pounds_{n-m+2}^{-}\cdots{}_{q_3}\pounds_{n-1}^{-}{}_{q_3}\pounds_n^{-})\{\mathcal{Y}_n^{[m]}(q_1,q_2,\cdots,q_m)\}, \tag{40}$$

considering (30), the above expression can be expanded as:

$$\mathcal{Y}^{[m]}_{n-m}(q_1, q_2, \cdots, q_m) = \frac{(n-m)!}{m!} D_{q_1}^{-2(m-1)} D_{q_3}^{(m-1)} \{\mathcal{Y}^{[m]}_n(q_1, q_2, \cdots, q_m)\}. \tag{41}$$

Inserting (41) in the recurrence relation (10), it follows that

$$\mathcal{Y}^{[m]}_{n+1}(q_1, q_2, \cdots, q_m) = \Big(q_1 + 2q_2 D_{q_1}^{-2} D_{q_3} + 3q_3 D_{q_1}^{-4} D_{q_3}^2 + \cdots + mq_m D_{q_1}^{-2(m-1)} D_{q_3}^{m-1}\Big). \tag{42}$$

Thus, assertion (21) of the raising operator ${}_{q_3}\pounds^+_n$ is proven.

Finally, to demonstrate the raising operator ${}_{q_m}\pounds^+_n$, the expression listed below is considered:

$$\mathcal{Y}^{[m]}_{n-m}(q_1, q_2, \cdots, q_m) = ({}_{q_m}\pounds^-_{n-m+1}\,{}_{q_m}\pounds^-_{n-m+2} \cdots {}_{q_m}\pounds^-_{n-1}\,{}_{q_m}\pounds^-_n)\{\mathcal{Y}^{[m]}_n(q_1, q_2, \cdots, q_m)\}, \tag{43}$$

considering (33), the above expression can be expanded as:

$$\mathcal{Y}^{[m]}_{n-m}(q_1, q_2, \cdots, q_m) = \frac{(n-m)!}{m!} D_{q_1}^{-(m-1)^2} D_{q_m}^{(m-1)} \{\mathcal{Y}^{[m]}_n(q_1, q_2, \cdots, q_m)\}. \tag{44}$$

Inserting (44) in the recurrence relation (10), it follows that

$$\mathcal{Y}^{[m]}_{n+1}(q_1, q_2, \cdots, q_m) = \Big(q_1 + 2q_2 D_{q_1}^{-(m-1)} D_{q_m} + 3q_3 D_{q_1}^{-2(m-1)} D_{q_m}^2 + \cdots + mq_m D_{q_1}^{-(m-1)^2} D_{q_m}^{m-1}\Big). \tag{45}$$

Thus, expression (22) of the raising operator ${}_{q_m}\pounds^+_n$ is proven. □

In the following section, we will demonstrate the families of differential equations satisfied by these polynomials.

3. Families of Differential Equations

For the MVHP $\mathcal{Y}^{[m]}_n(q_1, q_2, q_3, \cdots, q_m)$, we derive the differential, integro–differential, and partial differential equations in this section. Secondly, by demonstrating the following conclusion, we construct the differential equation for the MVHP $\mathcal{Y}^{[m]}_n(q_1, q_2, q_3, \cdots, q_m)$:

Theorem 3. *The MVHP $\mathcal{Y}^{[m]}_n(q_1, q_2, q_3, \cdots, q_m)$ satisfies the following differential equation:*

$$\Big(q_1 D_{q_1} + 2q_2 D_{q_1}^2 + 3q_2 D_{q_1}^3 + \cdots + q_m D_{q_1}^m - n\Big)\mathcal{Y}^{[m]}_n(q_1, q_2, q_3, \cdots, q_m) = 0. \tag{46}$$

Proof. We use expressions (15) and (19) of the shift operators in the below factorization relation:

$${}_{q_1}\pounds^-_{n+1}\,{}_{q_1}\pounds^+_n\{\mathcal{Y}^{[m]}_n(q_1, q_2, q_3, \cdots, q_m)\} = \mathcal{Y}^{[m]}_n(q_1, q_2, q_3, \cdots, q_m), \tag{47}$$

assertion (46) is proven. □

Integro–differential equations are a type of mathematical equation that combines ordinary differential equations (ODEs) and integrals. They involve functions that are both differentiated with respect to one or more variables and integrated with respect to one or more variables. Integro–differential equations are employed to model various phenomena in fields such as physics, engineering, economics, and biology. These equations are generally more complex than ordinary differential equations due to the presence of integrals, making their solutions challenging to obtain. Several techniques can be used to solve integro–differential equations, including separation of variables, Laplace transforms, and numerical methods, such as finite difference or finite element methods. In certain cases, closed-form solutions may not be feasible, necessitating the use of numerical approximation methods to obtain approximate solutions.

Theorem 4. *The MVHP* $\mathcal{Y}_n^{[m]}(q_1,q_2,q_3,\cdots,q_m)$ *satisfies the following integro–differential equations:*

$$\left(q_1 D_{q_2} + 2q_2 D_{q_1}^{-1} D_{q_2}^2 + 3q_3 D_{q_1}^{-2} D_{q_2}^3 + \cdots + mq_m D_{q_1}^{-(m-1)} D_{q_2}^m - (n+1) D_{q_1}\right) \times \mathcal{Y}_n^{[m]}(q_1,q_2,q_3,\cdots,q_m) = 0. \tag{48}$$

$$\left(q_1 D_{q_3} + 2q_2 D_{q_1}^{-1} D_{q_2} D_{q_3} + 3q_3 D_{q_1}^{-2} D_{q_2}^2 D_{q_3} + \cdots + mq_m D_{q_1}^{-(m-1)} D_{q_2}^{m-1} D_{q_3} - (n+1) D_{q_1}^2\right) \mathcal{Y}_n^{[m]}(q_1,q_2,q_3,\cdots,q_m) = 0. \tag{49}$$

$$\left(q_1 D_{q_m} + 2q_2 D_{q_1}^{-1} D_{q_2} D_{q_m} + 3q_3 D_{q_1}^{-2} D_{q_2}^2 D_{q_m} + \cdots + mq_m D_{q_1}^{-(m-1)} D_{q_3}^{m-1} D_{q_m} - (n+1) D_{q_1}^{m-1}\right) \mathcal{Y}_n^{[m]}(q_1,q_2,q_3,\cdots,q_m) = 0. \tag{50}$$

$$\left(q_1 D_{q_2} + 2q_2 D_{q_1}^{-2} D_{q_3} D_{q_2} + 3q_3 D_{q_1}^{-4} D_{q_3}^2 D_{q_2} + \cdots + mq_m D_{q_1}^{-2(m-1)} D_{q_3}^{m-1} D_{q_2} - (n+1) D_{q_1}\right) \mathcal{Y}_n^{[m]}(q_1,q_2,q_3,\cdots,q_m) = 0. \tag{51}$$

$$\left(q_1 D_{q_3} + 2q_2 D_{q_1}^{-2} D_{q_3}^2 + 3q_3 D_{q_1}^{-4} D_{q_3}^3 + \cdots + mq_m D_{q_1}^{-2(m-1)} D_{q_3}^m - (n+1) D_{q_1}^2\right) \times \mathcal{Y}_n^{[m]}(q_1,q_2,q_3,\cdots,q_m) = 0. \tag{52}$$

$$\left(q_1 D_{q_m} + 2q_2 D_{q_1}^{-2} D_{q_3} D_{q_m} + 3q_3 D_{q_1}^{-4} D_{q_3}^2 D_{q_m} + \cdots + mq_m D_{q_1}^{-2(m-1)} D_{q_3}^{m-1} D_{q_m} - (n+1) D_{q_1}^{m-1}\right) \mathcal{Y}_n^{[m]}(q_1,q_2,q_3,\cdots,q_m) = 0. \tag{53}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\left(q_1 D_{q_2} + 2q_2 D_{q_1}^{-(m-1)} D_{q_m} D_{q_2} + 3q_3 D_{q_1}^{-2(m-1)} D_{q_m}^2 D_{q_2} + \cdots + mq_m D_{q_1}^{-(m-1)^2} D_{q_m}^{m-1} D_{q_2} - (n+1) D_{q_1}\right) \mathcal{Y}_n^{[m]}(q_1,q_2,q_3,\cdots,q_m) = 0. \tag{54}$$

$$\left(q_1 D_{q_3} + 2q_2 D_{q_1}^{-(m-1)} D_{q_m} D_{q_3} + 3q_3 D_{q_1}^{-2(m-1)} D_{q_m}^2 D_{q_3} + \cdots + mq_m D_{q_1}^{-(m-1)^2} D_{q_m}^{m-1} D_{q_3} - (n+1) D_{q_1}^2\right) \mathcal{Y}_n^{[m]}(q_1,q_2,q_3,\cdots,q_m) = 0. \tag{55}$$

$$\left(q_1 D_{q_m} + 2q_2 D_{q_1}^{-(m-1)} D_{q_m}^2 + 3q_3 D_{q_1}^{-2(m-1)} D_{q_m}^3 + \cdots + mq_m D_{q_1}^{-(m-1)^2} D_{q_m}^m - (n+1) D_{q_1}^{m-1}\right) \mathcal{Y}_n^{[m]}(q_1,q_2,q_3,\cdots,q_m) = 0. \tag{56}$$

Proof. Consider the expression

$$\pounds_{n+1}^{-}\pounds_n^{+}\{\mathcal{Y}_n^{[m]}(q_1,q_2,q_3,\cdots,q_m)\} = \mathcal{Y}_n^{[m]}(q_1,q_2,q_3,\cdots,q_m). \tag{57}$$

Making use of expressions (16) and (20) in the factorization relation (57), assertion (48) is proven.

Next, making use of expressions (17) and (20) in factorization relation (57), assertion (49) is proven.

Again, making use of expressions (18) and (20) in factorization relation (57), assertion (50) is proven.

Further, making use of expressions (16)–(18) with expression (21) separately, assertions (51)–(53) are proven.

Furthermore, making use of expressions (16)–(18) with expression (22) separately, assertions (54)–(56) are proven. □

These integro–differential equations have numerous applications, including in areas such as population dynamics, control theory, fluid mechanics, signal processing, and image processing, among others. They provide a powerful mathematical tool for understanding and predicting the behaviors of complex systems that involve both differentiation and integration.

Partial differential equations (PDEs) are mathematical equations that involve partial derivatives of a function with respect to multiple independent variables. They are utilized to describe a wide range of phenomena in the physical, biological, and social sciences, where variation or change occurs with respect to multiple independent variables, such as time, space, and other parameters. PDEs are more intricate than ordinary differential equations (ODEs) because they incorporate partial derivatives, which account for simultaneous changes in multiple variables. Solving PDEs can be challenging and often requires advanced mathematical techniques, including Fourier analysis, Laplace transform, method of characteristics, finite difference methods, finite element methods, and various numerical methods.

Theorem 5. *The MVHP $\mathcal{Y}_n^{[m]}(q_1,q_2,q_3,\cdots,q_m)$ satisfy the following partial differential equations:*

$$\left(q_1 D_{q_1}^n D_{q_2} + 2q_2 D_{q_1}^{n-1} D_{q_2}^2 + 3q_3 D_{q_1}^{n-2} D_{q_2}^3 + \cdots + mq_m D_{q_1}^{n-(m-1)} D_{q_2}^m - (n+1) D_{q_1}^{n+1}\right) \times \mathcal{Y}_n^{[m]}(q_1,q_2,q_3,\cdots,q_m) = 0. \tag{58}$$

$$\left(q_1 D_{q_1}^n D_{q_3} + 2q_2 D_{q_1}^{n-1} D_{q_2} D_{q_3} + 3q_3 D_{q_1}^{n-2} D_{q_2}^2 D_{q_3} + \cdots + mq_m D_{q_1}^{n-(m-1)} D_{q_2}^{m-1} D_{q_3} - (n+1) D_{q_1}^{n+2}\right) \mathcal{Y}_n^{[m]}(q_1,q_2,q_3,\cdots,q_m) = 0. \tag{59}$$

$$\left(q_1 D_{q_1}^{2n} D_{q_m} + 2q_2 D_{q_1}^{2n-1} D_{q_2} D_{q_m} + 3q_3 D_{q_1}^{2n-2} D_{q_2}^2 D_{q_m} + \cdots + mq_m D_{q_1}^{2n-(m-1)} D_{q_3}^{m-1} D_{q_m} - (n+1) D_{q_1}^{2n+m-1}\right) \mathcal{Y}_n^{[m]}(q_1,q_2,q_3,\cdots,q_m) = 0. \tag{60}$$

$$\left(q_1 D_{q_1}^{2n+2} D_{q_2} + 2q_2 D_{q_1}^{2n} D_{q_3} D_{q_2} + 3q_3 D_{q_1}^{2n-2} D_{q_3}^2 D_{q_2} + \cdots + mq_m D_{q_1}^{2n-2m} D_{q_3}^{m-1} D_{q_2} - (n+1) D^{2n+m-1} q_1\right) \mathcal{Y}_n^{[m]}(q_1, q_2, q_3, \cdots, q_m) = 0. \tag{61}$$

$$\left(q_1 D_{q_1}^{2n+2} D_{q_3} + 2q_2 D_{q_1}^{2n} D_{q_3}^2 + 3q_3 D_{q_1}^{2n-2} D_{q_3}^3 + \cdots + mq_m D_{q_1}^{2n-2m} D_{q_3}^m - (n+1) D_{q_1}^{2n+4}\right) \mathcal{Y}_n^{[m]}(q_1, q_2, q_3, \cdots, q_m) = 0. \tag{62}$$

$$\left(q_1 D_{q_1}^{2n+2} D_{q_m} + 2q_2 D_{q_1}^{2n} D_{q_3} D_{q_m} + 3q_3 D_{q_1}^{2n-2} D_{q_3}^2 D_{q_m} + \cdots + mq_m D_{q_1}^{2n-2m} D_{q_3}^{m-1} D_{q_m} - (n+1) D_{q_1}^{2n+m+1}\right) \mathcal{Y}_n^{[m]}(q_1, q_2, q_3, \cdots, q_m) = 0. \tag{63}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\left(q_1 D_{q_1}^{n^2+1} D_{q_2} + 2q_2 D_{q_1}^{n^2-m} D_{q_m} D_{q_2} + 3q_3 D_{q_1}^{n^2-2m-1} D_{q_m}^2 D_{q_2} + \cdots + mq_m D_{q_1}^{n^2-m^2+2m} \times D_{q_m}^{m-1} D_{q_2} - (n+1) D_{q_1}^{n^2+2}\right) \mathcal{Y}_n^{[m]}(q_1, q_2, q_3, \cdots, q_m) = 0. \tag{64}$$

$$\left(q_1 D_{q_1}^{n^2+1} D_{q_3} + 2q_2 D_{q_1}^{2n-m} D_{q_m} D_{q_3} + 3q_3 D_{q_1}^{2n-2m} D_{q_m}^2 D_{q_3} + \cdots + mq_m D_{q_1}^{n^2-m^2+2m} \times D_{q_m}^{m-1} D_{q_3} - (n+1) D_{q_1}^{n^2+2}\right) \mathcal{Y}_n^{[m]}(q_1, q_2, q_3, \cdots, q_m) = 0. \tag{65}$$

$$\left(q_1 D_{q_1}^{n^2+2} D_{q_m} + 2q_2 D_{q_1}^{n^2-m} D_{q_m}^2 + 3q_3 D_{q_1}^{n^2-2(m-1)} D_{q_m}^3 + \cdots + mq_m D_{q_1}^{n^2-m^2+2m} D_{q_m}^m - (n+1) D_{q_1}^{n^2+m+1}\right) \mathcal{Y}_n^{[m]}(q_1, q_2, q_3, \cdots, q_m) = 0. \tag{66}$$

Proof. By taking partial derivatives of the integro–differential expressions (48) and (49) w.r.t. q_1 for n times, the assertions (58) and (59) are proven.

Moreover, by taking partial derivatives of the integro–differential expression (50) w.r.t. q_1 for $2n$ times, the assertion (60) is proven.

Furthermore, by taking partial derivatives of the integro–differential expressions (51) to (53) w.r.t. q_1 for $2n+2$ times, the assertions (61) to (63) are proven.

Moreover, by taking partial derivatives of the integro–differential expressions (54) and (55) w.r.t. q_1 for n^2+1 times, the assertions (64) and (65) are proven.

Again, taking partial derivatives of the integro–differential expression (56) with respect to q_1 for n^2+2 times, the assertion (66) is proven. □

PDEs have numerous applications in various fields of science and engineering, including physics, engineering, biology, finance, and many others. They are used to model a wide range of phenomena, such as heat and mass transfer, fluid dynamics, wave propagation, quantum mechanics, population dynamics, and many other complex systems. PDEs provide a powerful tool for understanding and predicting the behavior of systems that involve change and variation in multiple variables.

4. Volterra Integral Equations

A Volterra integral equation is a specific type of integral equation that was introduced by the Italian mathematician Vito Volterra. It is a type of functional equation that involves both an unknown function and an integral of that function. Various techniques can be used to solve Volterra integral equations, including numerical methods, such as quadrature methods and collocation methods, as well as analytical methods, such as Fredholm's alternative theorem, which provides a criterion for the existence and uniqueness of solutions.

Here, we acquire the integral equation for the MVHP $\mathcal{Y}_n^{[m]}(q_1, q_2, q_3, \cdots, q_m)$ by demonstrating the results listed below:

Theorem 6. *The MVHP polynomials $\mathcal{Y}_n^{[m]}(q_1, q_2, q_3, \cdots, q_m)$ satisfy the homogeneous Volterra integral equation listed below:*

$$\begin{aligned}\Psi(q_1) = &-\tfrac{1}{p_4}\Bigg(3q_3n(n-1)(n-2)\mathfrak{E}_{n-3}(\mathcal{P},\mathcal{Q}) + 2q_2n(n-1)(n-2)\mathfrak{E}_{n-3}(\mathcal{P},\mathcal{Q},)q_1\\ &+2q_2n(n-1)\mathfrak{E}_{n-2}(\mathcal{P},\mathcal{Q}) + q_1\Big(n(n-1)(n-2)\mathfrak{E}_{n-3}(\mathcal{P},\mathcal{Q})\tfrac{q_1^2}{2!} + n(n-1)\\ &\mathfrak{E}_{n-2}(\mathcal{P},\mathcal{Q})q_1 + n\mathfrak{E}_{n-1}(\mathcal{P},\mathcal{Q})\Big) - n(n-1)(n-2)\mathfrak{E}_{n-3}(\mathcal{P},\mathcal{Q})\tfrac{q_1^3}{2!3!}\\ &-n(n-1)\mathfrak{E}_{n-2}(\mathcal{P},\mathcal{Q})\tfrac{q_1^2}{2!} - n\mathfrak{E}_{n-1}(\mathcal{P},\mathcal{Q})q_1 - \mathfrak{E}_n(\mathcal{P},\mathcal{Q})\Bigg) + \int_0^{q_1}\Big(\tfrac{-1}{p_4}\Big(3q_3 + 2q_2\\ &(q_1-\xi) + \Big(q_1 - \tfrac{1}{p_4}\Big)\tfrac{(q_1-\xi)^2}{2!}\Big) - n\tfrac{(q_1-\xi)^3}{3!}\Big)\Psi(\xi)d\xi.\end{aligned} \tag{67}$$

Proof. Here, we consider $m = 4$ in differential equation (46) to obtain the differential equation of the form:

$$\left(D_{q_1}^4 + \frac{1}{p_4}\left(3q_3D_{q_1}^3 + 2q_2D_{q_1}^2 + q_1D_{q_1} - n\right)\right)\mathcal{Y}_n^{[m]}(q_1, q_2, q_3, p_4) = 0. \tag{68}$$

Next, we acquire the initial conditions as listed:

$$\begin{aligned}&\mathcal{Y}_n^{[m]}(q_1,q_2,0) = \mathcal{Y}_n^{[m]}(q_1,q_2) = n!\sum_{k=0}^{[\frac{n}{2}]}\tfrac{{q_2}^k{q_1}^{n-2k}}{k!(n-2k)!}\\ &:= \mathfrak{E}_n(\mathcal{P},\mathcal{Q}),\\ &\tfrac{d}{dq_1}\mathcal{Y}_n^{[m]}(q_1,q_2) = n\mathcal{Y}_{n-1}^{[m]}(q_1,q_2) = n(n-1)!\sum_{k=0}^{[\frac{n-1}{2}]}\tfrac{{q_2}^k{q_1}^{n-1-2k}}{k!(n-1-2k)!}\\ &:= n\mathfrak{E}_{n-1}(\mathcal{P},\mathcal{Q}),\\ &\tfrac{d^2}{d{q_1}^2}\mathcal{Y}_n^{[m]}(q_1,q_2,0) = n(n-1)\mathcal{Y}_{n-2}^{[m]}(q_1,q_2) = n(n-1)(n-2)!\sum_{k=0}^{[\frac{n-2}{2}]}\tfrac{{q_2}^k{q_1}^{n-2-2k}}{k!(n-2-2k)!},\\ &:= n(n-1)\mathfrak{E}_{n-2}(\mathcal{P},\mathcal{Q}),\\ &\tfrac{d^3}{d{q_1}^3}\mathcal{Y}_n^{[m]}(q_1,q_2,0) = n(n-1)(n-2)\mathcal{Y}_{n-3}^{[m]}(q_1,q_2)\\ &= n(n-1)(n-2)(n-3)!\sum_{k=0}^{[\frac{n-3}{2}]}\tfrac{{q_2}^k{q_1}^{n-3-2k}}{k!(n-3-2k)!} := n(n-1)(n-2)\mathfrak{E}_{n-3}(\mathcal{P},\mathcal{Q}),\end{aligned} \tag{69}$$

respectively, where

$$\mathfrak{E}_s(\mathcal{P},\mathcal{Q}) := n!\sum_{k=0}^{[\frac{n-s}{2}]}\frac{{q_2}^k{q_1}^{n-s-2k}}{k!(n-s-2k)!}. \tag{70}$$

Consider

$$D_{q_1}^4\mathcal{Y}_n^{[m]}(q_1, q_2, q_3, p_4) = \Psi(q_1). \tag{71}$$

Using the beginning circumstances from (69) and integrating the aforementioned equation, we have

$$\begin{aligned}
\frac{d^3}{dq_1{}^3}\mathcal{Y}_n^{[m]}(q_1,q_2,q_3,p_4) &= \int\limits_0^{q_1}\Psi(\xi)d\xi + n(n-1)(n-2)\mathfrak{E}_{n-3}(\mathcal{P},\mathcal{Q}),\\
\frac{d^2}{dq_1{}^2}\mathcal{Y}_n^{[m]}(q_1,q_2,q_3,p_4) &= \int\limits_0^{q_1}\Psi(\xi)d\xi^2 + n(n-1)(n-2)\mathfrak{E}_{n-3}(\mathcal{P},\mathcal{Q})q_1 + n(n-1)\mathfrak{E}_{n-2}(\mathcal{P},\mathcal{Q}),\\
\frac{d}{dq_1}\mathcal{Y}_n^{[m]}(q_1,q_2,q_3,p_4) &= \int\limits_0^{q_1}\Psi(\xi)d\xi^3 + n(n-1)(n-2)\mathfrak{E}_{n-3}(\mathcal{P},\mathcal{Q})\frac{q_1{}^2}{2!} + n(n-1)\mathfrak{E}_{n-2}(\mathcal{P},\mathcal{Q})q_1\\
&\quad + n\mathfrak{E}_{n-1}(\mathcal{P},\mathcal{Q}),\\
\mathcal{Y}_n^{[m]}(q_1,q_2,q_3,p_4) &= \int\limits_0^{q_1}\Psi(\xi)d\xi^4 + n(n-1)(n-2)\mathfrak{E}_{n-3}(\mathcal{P},\mathcal{Q})\frac{q_1{}^3}{2!3!} + n(n-1)\mathfrak{E}_{n-2}(\mathcal{P},\mathcal{Q},)\frac{q_1{}^2}{2!}\\
&\quad + n\mathfrak{E}_{n-1}(\mathcal{P},\mathcal{Q})q_1 + \mathfrak{E}_n(\mathcal{P},\mathcal{Q}).
\end{aligned} \tag{72}$$

Using the aforementioned equations in the differential Equation (68), we discover

$$\begin{aligned}
&\Psi(q_1) = -\frac{1}{p_4}\Big(3q_3\Big(\int\limits_0^{q_1}\Psi(\xi)d\xi + n(n-1)(n-2)\mathfrak{E}_{n-3}(\mathcal{P},\mathcal{Q})\Big) + 2q_2\Big(\int\limits_0^{q_1}\Psi(\xi)d\xi^2 + n(n-1)\\
&(n-2)\mathfrak{E}_{n-3}(\mathcal{P},\mathcal{Q})q_1 + n(n-1)\mathfrak{E}_{n-2}(\mathcal{P},\mathcal{Q})\Big) + \Big(x - \frac{1}{1-\lambda}\Big)\Big(\int\limits_0^{q_1}\Psi(\xi)d\xi^3 + n(n-1)\\
&(n-2)\mathfrak{E}_{n-3}(\mathcal{P},\mathcal{Q})\frac{q_1{}^2}{2!} + n(n-1)\mathfrak{E}_{n-2}(\mathcal{P},\mathcal{Q})q_1 + n\mathfrak{E}_{n-1}(\mathcal{P},\mathcal{Q})\Big)\Big) + \frac{n}{p_4}\\
&\Big(\int\limits_0^{q_1}\Psi(\xi)d\xi^4 + n(n-1)(n-2)\mathfrak{E}_{n-3}(\mathcal{P},\mathcal{Q})\frac{q_1{}^3}{2!3!} + n(n-1)\mathfrak{E}_{n-2}(\mathcal{P},\mathcal{Q})\frac{q_1{}^2}{2!}\\
&+ n\mathfrak{E}_{n-1}(\mathcal{P},\mathcal{Q})q_1 + \mathfrak{E}_n(\mathcal{P},\mathcal{Q})\Big),
\end{aligned} \tag{73}$$

Hence, using the following method, after simplifying and integrating the resulting equation:

$$\int\limits_a^{q_1} f(\xi)d\xi^n = \int\limits_a^{q_1}\frac{(q_1-\xi)^{n-1}}{(n-1)!}f(\xi)d\xi. \tag{74}$$

Thus, assertion (68) is proven. □

Volterra integral equations have many applications in physics, engineering, and other fields. For example, they can be used to model diffusion processes, heat transfer, and the behavior of viscoelastic materials. They are also used in mathematical biology to model the spread of infectious diseases, population dynamics, and other phenomena.

5. Conclusions

In this paper, multivariate Hermite polynomials are considered several of their properties are established. These polynomials, which are a generalization of the one-dimensional Hermite polynomials, have significant uses in quantum mechanics, probability theory, and other branches of math and science.

The shift operators and recurrence relation discussed in this article have proven to be valuable tools for analyzing the characteristics of these polynomials. The shift operators facilitate the translation of polynomials in different directions and dimensions, while the recurrence relation allows for the computation of polynomial values at higher degrees based on their values at lower degrees.

Another significant finding is that the multivariate Hermite polynomials satisfy the differential equation, which enables one to express these polynomials in terms of derivatives. This can be useful for computing certain integrals and solving differential equations that involve these polynomials.

It is crucial to note the sequences of integro–differential and partial differential equations have significant importance because they offer additional ways to represent and work with the multivariate Hermite polynomials. Other features and connections between these polynomials can be derived using these equations as well.

In closing, a particular equation that may be resolved using these polynomials is the Volterra integral equation. This problem may be represented as an integral equation that can be solved using methods from numerical analyses and other branches of mathematics. It involves a polynomial with four variables, namely, q_1, q_2, q_3, q_4.

Furthermore, future investigations and observations can be used to establish extended, generalized forms via fractional operators, symmetric identities, and other properties of the above-mentioned polynomials. Moreover, the determinant forms and summation formulae can also be a problem for new observations.

Author Contributions: Methodology, S.A.W.; Software, B.S.T.A.; Validation, I.A.; Formal analysis, B.S.T.A. and S.A.W.; Investigation, I.A.; Resources, B.S.T.A. and I.A.; Writing—original draft, S.A.W. All authors have read and agreed to the published version of the manuscript.

Funding: The authors extend their appreciation to research supporting project number (RSPD2023R526), King Saud University, Riyadh, Saudi Arabia.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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