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Note on the Generalized Branching Random Walk on the Galton–Watson Tree

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Abstract: Let $\partial\mathsf{T}$ be a super-critical Galton–Watson tree. Recently, the first author computed almost surely and simultaneously the Hausdorff dimensions of the sets of infinite branches of the boundary of $\partial\mathsf{T}$ along which the sequence $S_n\mathsf{X}(t)/S_n\widetilde{\mathsf{X}}(t)$ has a given set of limit points, where $S_n\mathsf{X}(t)$ and $S_n\widetilde{\mathsf{X}}(t)$ are two branching random walks defined on $\partial\mathsf{T}$. In this study, we are interested in the study of the speed of convergence of this sequence. More precisely, for a given sequence $s = (s_n)$, we consider $E_{\alpha,s} = \Big\{t \in \partial\mathsf{T} : S_n\mathsf{X}(t) - \alpha S_n\widetilde{\mathsf{X}}(t) \sim s_n \text{ as } n \to +\infty\Big\}$. We will give a sufficient condition on (s_n) so that $E_{\alpha,s}$ has a maximal Hausdorff and packing dimension.

Keywords: random walk; Hausdorff and packing dimensions; Galton–Watson tree

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1. Introduction

Multifractal analysis is typically used to describe objects possessing some type of scale invariance. It was developed around 1980, following the work of B. Mandelbrot [1,2] and, since then, it has shown results of outstanding significative in theory and applications. Specifically, consider a signal $X : \mathbb{R}^{\mathrm{d}} \longrightarrow \mathbb{R}$, the multifractal analysis is a processing method that allows the examination of the signal X using the characteristics of its pointwise regularity, which are measured by using the exponent of pointwise regularity $\alpha_X(x)$. More precisely, consider the set

$$E_\alpha = \Big\{x \in \mathbb{R}^{\mathrm{d}},\ \ \alpha_X(x) = \alpha\Big\}. \tag{1}$$

The aim of the multifractal spectrum is to give a geometric and global account of the variations in X's regularity along x by computing the Hausdorff and packing dimensions of the set E_α, for each $\alpha \in \mathbb{R}$. Especially, the multifractal analysis is a powerful tool to study the time series since such series present complex statistical fluctuations that are associated with long-range correlations between the dynamical variables present in these series, and which obey the behavior usually described by the decay of the fractal power law.

Let $\partial\mathsf{T}$ be the boundary of the Galton–Watson tree T with defining element N. T is an elementary model for the genealogy of a branching population. Roughly speaking, for a given generation, each individual gives birth to a random number of children in the next generation independently of each other and all with the same distribution. For each $t \in \partial\mathsf{T}$, we may define the branching random walks $S_n\mathsf{X}(t)$ and $S_n\widetilde{\mathsf{X}}(t)$ defined as

$$S_n\mathsf{X}(t) = \sum_{k=1}^{n} \mathsf{X}_{t_1\cdots t_k} \quad \text{and} \quad S_n\widetilde{\mathsf{X}}(t) = \sum_{k=1}^{n} \widetilde{\mathsf{X}}_{t_1\cdots t_k},$$

(see definitions and notation in Section 2). Consider the level sets of the asymptotic behavior of the sequence $S_n\mathsf{X}(t)/S_n\widetilde{\mathsf{X}}(t)$, that is,

$$E_{\mathsf{X},\widetilde{\mathsf{X}}}(\alpha) = \Big\{ t \in \partial\mathsf{T} : \lim_{n\to\infty} \frac{S_n\mathsf{X}(t)}{S_n\widetilde{\mathsf{X}}(t)} = \alpha \Big\}, \tag{2}$$

where $\alpha \in \mathbb{R}$. It is natural to consider the multifractal analysis of $E_{\mathsf{X},\widetilde{\mathsf{X}}}(\alpha)$ and then compute the Hausdorff and packing dimensions of these sets [3].

we can show that there exists α_1 such that $E_{\mathsf{X},\widetilde{\mathsf{X}}}(\alpha_1)$ is of full Hausdorff and packing dimensions in the boundary of Galton–Watson tree [3,4] and then, it is natural to explore the other branches over which $S_n\mathsf{X}(t)/S_n\widetilde{\mathsf{X}}(t) \to \alpha$ for $\alpha \neq \alpha_1$ [5,6]. These level sets $E_{\mathsf{X},\widetilde{\mathsf{X}}}(\alpha)$ have been considered in many papers, see for instance [7–11] (the interested readers might consult [4,12] for a general case). The size of $E_{\mathsf{X},\widetilde{\mathsf{X}}}(\alpha)$ is related to the Legendre transform of some function, this principle is known as the multifractal formalism. In [13], the authors highlighted the link between the existence of auxiliary measures and multifractal formalism. In particular, almost all papers cited above are associated with the construction of Mandelbrot measures (see [14–16] for more details on these measures).

If $\widetilde{\mathsf{X}}_i = 1,\ 1 \le i \le N$. Then, the set $E_{\mathsf{X},\widetilde{\mathsf{X}}}(\alpha)$ will be denoted by $E_{\mathsf{X}}(\alpha)$ and it was treated in [4,12]. More precisely, we define the functions

$$S_{\mathsf{X}}(q) = \sum_{i=1}^{N} \exp(q\mathsf{X}_i) \quad \text{and} \quad \widetilde{P}(q) = \log S_{\mathsf{X}}(q) = \log \mathbb{E}\Big(\sum_{i=1}^{N} \exp(q\mathsf{X}_i) \Big)$$

and assume that $\widetilde{P}(q) < \infty$ for all $q \in \mathbb{R}$. In addition, assume that there exists $\gamma > 1$ such that $\mathbb{E}\big(S_{\mathsf{X}}(q)^{\gamma}\big) < \infty$, then the set $I_{\mathsf{X}} = \big\{\alpha \in \mathbb{R} : \widetilde{P}^*(\alpha) \ge 0\big\}$ is a non-empty convex compact set [4,12] and, almost surely (a.s.), for all $\alpha \in \mathbb{R}$, we have $E_{\mathsf{X}}(\alpha)$ is non-empty if and only if $\alpha \in I_{\mathsf{X}}$. Moreover, in this case, we have

$$\dim E_{\mathsf{X}}(\alpha) = \widetilde{P}^*(\alpha),$$

where dim stands for Hausdorff dimension and $\widetilde{P}^*$ is the Legendre transform of the $\widetilde{P}$ defined by $f^*(\alpha) = \inf_{q\in\mathbb{R}} f(q) - q\,\alpha$, for any function $f : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ and any $\alpha \in \mathbb{R}$ [4,12]. Let $\alpha \in int(I_{\mathsf{X}})$ and let $s = \{s_n\}$ be a positive sequence such that $s_n = o(n)$. We set

$$\widetilde{E}_{\alpha,s} = \Big\{ t \in \partial\mathsf{T} : S_n\mathsf{X}_n(t) - n\alpha \sim s_n \text{ as } n \to +\infty \Big\}, \tag{3}$$

where $S_n\mathsf{X}_n(t) - n\alpha \sim s_n$ means that $(S_n\mathsf{X}_n(t) - n\alpha)_n$ and $(s_n)_n$ are two equivalent sequences. Kahane and Fan in [17] computed almost surely, for given α, the Hausdorff dimension of $\widetilde{E}_{\alpha,s}$ when $\partial\mathsf{T} = \{0,1\}^{\mathbb{N}}$. They assume in addition that

$$\sqrt{n \ln\ln n} = \circ(s_n) \qquad \text{and} \qquad \eta_n = s_n - s_{n-1} = o(1).$$

This assumption is verified, in particular, when $s_n = n^{\beta}$ with $\beta \in (1/2, 1)$. Later, Attia in [18,19], generalize this result by computing that almost surely, for all $\alpha \in int(I_{\mathsf{X}})$, the Hausdorff dimensions of the sets $\widetilde{E}_{\alpha,s}$. In the present work, we are interested in the study of the set

$$E_{\alpha,s} = \Big\{ t \in \partial\mathsf{T} : S_n\mathsf{X}(t) - \alpha S_n\widetilde{\mathsf{X}}(t) \sim s_n \text{ as } n \to +\infty \Big\}, \tag{4}$$

for α belongs to the given set $\mathcal{K}$. We will give a sufficient condition on the sequence (s_n) so that the set $E_{\alpha,s}$ has a maximal Hausdorff and packing dimension. The motivation to introduce this kind of set comes from the idea of studying the dimension of the set $E_{\mathsf{X}}(\alpha)$ under the distance $d_{\widetilde{\mathsf{X}}}$ defined as

$$d_{\widetilde{\mathsf{X}}} : (s,t) \mapsto \exp(-S_{|s\wedge t|}\widetilde{\mathsf{X}}(s\wedge t)) \tag{5}$$

for all $s, t \in \partial\mathsf{T}$, where $s \wedge t$ stands for the longest common prefix of s and t, and with the convention that $\exp(-\infty) = 0$. This article is organized as follows: in the next section, we will recall the definitions of the various notation used in the paper and give some preliminary results. In Section 3, we will state and prove our main result concerning the study of the Hausdorff and packing dimension of the set $E_{\alpha,s}$. Finally, we mention that the method used here is not a direct extension of that used in [18]. Indeed, in this paper, we build simultaneously (on q and α) the Mandelbrot measures $\mu^s_{q,\alpha}$. This measure will be carried on the set $E_{\alpha,s}$ and approximate from below the Hausdorff dimension.

2. Notation and Preliminaries Results

2.1. Hausdorff and Packing Dimensions

Let $K \subseteq \mathbb{N}_+^{\mathbb{N}_+}$ and let d be a metric on K making it σ-compact. For $x \in K$, we denote by $B(x,r)$ the closed ball centered at x and with radius r. In the next, for $s > 0$, we recall the construction of the s-dimensional Hausdorff and packing measures denoted, respectively, $\mathcal{H}^s$ and P^s. We set, for $E \subseteq K$,

$$\mathcal{H}^s(E) = \lim_{\delta\to 0^+} \inf\Big\{\sum_{i\in\mathbb{N}}(\mathrm{diam}(U_i)^s\Big\} \text{ and } \overline{P}^s(E) = \lim_{\delta\to 0^+} \sup\Big\{\sum_{i\in\mathbb{N}}(\mathrm{diam}(B_i)^s\Big\},$$

where the infimum is taken over all the countable family $(U_i)_{i\in\mathbb{N}}$ such that $E \subseteq \bigcup_i U_i$ and $\mathrm{diam}(U_i) \le \delta$ and the supremum is taken over all the packings $\{B_i := B(x_i, r_i)\}_{i\in\mathbb{N}}$ with $x_i \in E$ and $\mathrm{diam}(B_i) \le \delta$. Then,

$$P^s(E) = \lim_{\delta\to 0^+} \inf\Big\{\sum_{i\in\mathbb{N}} \overline{P}^s(E_i)\Big\},$$

where the infimum being taken over all the countable family $(E_i)_{i\in\mathbb{N}}$ such that $E \subseteq \bigcup_i E_i$ and $\mathrm{diam}(E_i) \le \delta$. The Hausdorff and packing dimensions of E are defined, respectively, by

$$\dim E = \inf\{s > 0 : \mathcal{H}^s(E) = 0\} \text{ and } Dim E = \inf\{s > 0 : P^s(E) = 0\}$$

with the convention $\inf\emptyset = \infty$. [20,21].

Let μ be a positive and finite Borel measure supported on the set K, then the lower and upper Hausdorff dimensions of μ are defined, respectively, as follows [22]:

$$\underline{\dim}(\mu) = \text{ess inf}_\mu \liminf_{r\to 0^+} \frac{\log\mu(B(x,r))}{\log(r)}, \overline{\dim}(\mu) = \text{ess sup}_\mu \liminf_{r\to 0^+} \frac{\log\mu(B(x,r))}{\log(r)}$$

and the lower and upper packing dimensions of μ are defined, respectively, as follows

$$\underline{Dim}(\mu) = \text{ess inf}_\mu \limsup_{r\to 0^+} \frac{\log\mu(B(x,r))}{\log(r)}, \overline{Dim}(\mu) = \text{ess sup}_\mu \limsup_{r\to 0^+} \frac{\log\mu(B(x,r))}{\log(r)}.$$

In addition, if $\underline{\dim}(\mu) = \overline{\dim}(\mu)$ (resp. $\underline{Dim}(\mu) = \overline{Dim}(\mu)$), then the common value will denoted by $\dim\mu$ (resp. $Dim(\mu)$). One says that μ is exactly dimensional if $\dim\mu = Dim\mu$.

2.2. Branching Random Walk on the Boundary of Galton–Watson Tree

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and denote by $\mathbb{E}$ the expectation with respect to the probability $\mathbb{P}$. Let $\mathbb{N}$ denote the set of non-negative integers and $\mathcal{L} = (N, (\mathsf{X}_1, \widetilde{\mathsf{X}}_1), (\mathsf{X}_2, \widetilde{\mathsf{X}}_2), \dots)$

be a random vector with independent components taking values in $\mathbb{N} \times (\mathbb{R} \times (\mathbb{R}_+^*))^{\mathbb{N}_+}$. Consider a family of independent copies of $\mathcal{L}$:

$$\{(N_{u0}, (\mathsf{X}_{u1}, \widetilde{\mathsf{X}}_{u1}), (X_{u2}, \widetilde{\mathsf{X}}_{u2}), \ldots)\}_{u \in \bigcup_{n \geq 0} \mathbb{N}_+^n}$$

indexed by $u = u_1 \ldots u_n, n \geq 0, u_i \in \mathbb{N}_+^*$ ($n = 0$ corresponds to the empty sequence denoted $\emptyset$). Consider T to be the Galton–Watson tree with defining elements $\{N_u\}$ that is,

- $\emptyset \in \mathsf{T}$
- if $u \in \mathsf{T}$ and $i \in \mathbb{N}_+$ then $ui \in \mathsf{T}$ if and only if $1 \leq i \leq N_u$, where ui is the concatenation of u and i, and if $ui \in \mathsf{T}$ then $u \in \mathsf{T}$.

Let $u \in \bigcup_{n \geq 0} \mathbb{N}_+^n$, then we will denote by $\mathsf{T}(u)$ to be the Galton–Watson tree rooted at u with defining elements $\{N_{uv}\}$, $v \in \bigcup_{n \geq 0} \mathbb{N}_+^n$. We suppose that the Galton–Watson tree T is supercritical and the probability of extinction is equal to 0, that is, $\mathbb{E}(N) > 1$ and $\mathbb{P}(N \geq 1) = 1$. Let $t = t_1 t_2 \cdots \in \mathbb{N}_+^{\mathbb{N}_+}$ be an infinite word then, for $n \geq 0$, we set $t_{|n} = t_1 \cdots t_n \in \mathbb{N}_+^n$ with the convention $t_{|0} = \emptyset$. If $u \in \mathbb{N}_+^n$ for some integer $n \geq 0$, then the length of u is equal n and it is denoted by $|u|$. Hence, we denote by the cylinder $[u]$ the set of infinite words $t = t_1 t_2 \cdots$ such that $t_{||u|} = u$.

The space $\mathbb{N}_+^{\mathbb{N}_+}$ is endowed with the distance d defined as

$$d : (u, v) \mapsto e^{-\sup\{|w| : u \in [w], v \in [w]\}},$$

with the convention $\exp(-\infty) = 0$. Let $\mathsf{T}_n = \mathsf{T} \cap \mathbb{N}_+^n$ and define the boundary of T as the compact set

$$\partial \mathsf{T} = \bigcap_{n \geq 1} \bigcup_{u \in \mathsf{T}_n} [u].$$

For each $(q, \alpha, p) \in \mathbb{R}^3$, let us consider

$$S_\alpha(q) = \sum_{i=1}^{N} \exp\left(q(\mathsf{X}_i - \alpha \widetilde{\mathsf{X}}_i) - \widetilde{\mathcal{P}}_\alpha(q)\right)$$

and we suppose that for all $q \in \mathbb{R}$ there exists $\gamma > 1$ such that

$$\mathbb{E}(S_{\mathsf{X}}(q)^\gamma) + \mathbb{E}(S_{\widetilde{\mathsf{X}}}(q)^\gamma) < \infty. \tag{6}$$

In particular, for all $(q, \alpha) \in \mathbb{R}^2$, we have

$$\exists\, \gamma > 1, \mathbb{E}(S_\alpha(q)^\gamma) < \infty \tag{7}$$

and

$$\forall \gamma \in \mathbb{R}, \quad \mathbb{E}\Big(\sum_{i=1}^{N} e^{-\gamma \widetilde{\mathsf{X}}_i}\Big) < \infty. \tag{8}$$

In fact, Lemma 2.1 in [23], under (8), there exist $0 < \beta_1 < \beta_2 < 1$ such that, almost surely, for n large enough,

$$0 < \log(1/\beta_2) \leq \min\left\{\frac{S_n \widetilde{\mathsf{X}}(u)}{n} : u \in \mathsf{T}_n\right\} \leq \max\left\{\frac{S_n \widetilde{\mathsf{X}}(u)}{n} : u \in \mathsf{T}_n\right\} \leq \log(1/\beta_1).$$

Hence, $\Big\{t \in \partial \mathsf{T} : S_n \mathsf{X}(t)/S_n \widetilde{\mathsf{X}}(t) \to \alpha\Big\} = \Big\{t \in \partial \mathsf{T} : S_n(\mathsf{X} - \alpha \widetilde{\mathsf{X}})(t)/n \to 0\Big\}$. Therefore, for each $(q, \alpha, p) \in \mathbb{R}^3$, we introduce the functions

$$\widetilde{\mathcal{P}}_\alpha(q) = \log \mathbb{E}\Big(\sum_{i=1}^{N} \exp\left(q(\mathsf{X}_i - \alpha \widetilde{\mathsf{X}}_i)\right)\Big) \text{ and } \psi_\alpha(q, p) = \widetilde{\mathcal{P}}_\alpha(pq) - p \widetilde{\mathcal{P}}_\alpha(q).$$

Assume (6) then $\mathcal{K} = \{\alpha \in \mathbb{R} : \widetilde{\mathcal{P}}_\alpha^*(0) \geq 0\}$ is a non-empty compact interval. In addition, almost surely, for every $\alpha \in \mathbb{R}$, the set $E_{\mathsf{X},\widetilde{\mathsf{X}}}(\alpha) = \{t \in \partial\mathsf{T} : S_n\mathsf{X}(t)/S_n\widetilde{\mathsf{X}}(t) \to \alpha\} \neq \emptyset$ if and only if $\alpha \in \mathcal{K}$. In this case, we have

$$\dim E_{\mathsf{X},\widetilde{\mathsf{X}}}(\alpha) = Dim\, E_{\mathsf{X},\widetilde{\mathsf{X}}}(\alpha) = \widetilde{\mathcal{P}}_\alpha^*(0) = \inf_{q\in\mathbb{R}} \widetilde{\mathcal{P}}_\alpha(q).$$

It follows, if $s_n = \mathrm{o}(n)$ and since $E_{\alpha,s} \subset E_{\mathsf{X},\widetilde{\mathsf{X}}}(\alpha)$, that, almost surely, for all $\alpha \in \mathbb{R}$,

$$\dim E_{\alpha,s} \leq \dim E_\alpha \leq Dim E_\alpha \leq \widetilde{\mathcal{P}}_\alpha^*(0) \tag{9}$$

[3,12,23].

2.3. Mandelbrot Measures; Some Basic Properties

Let $\mathcal{S} = (N, (\mathsf{X}_1, \mathsf{X}'_1), (\mathsf{X}_2, \mathsf{X}'_2), \dots)$ be a random vector taking values in $\mathbb{N}_+ \times (\mathbb{R} \times \mathbb{R}_+^*)^{\mathbb{N}_+}$. Assume that the random variable N satisfies the assumptions above and suppose in addition

$$\begin{cases} \mathbb{E}\Big(\sum_{i=1}^N \mathsf{X}'_i \exp(\mathsf{X}'_i)\Big) < 0, \\ \mathbb{E}\Big(\sum_{i=1}^N \exp(\mathsf{X}'_i)\Big) = 1, \\ \quad \mathbb{E}\Big(\Big(\sum_{i=1}^N \exp(\mathsf{X}'_i)\Big) \log^+ \Big(\sum_{i=1}^N \exp(\mathsf{X}'_i)\Big)\Big) < \infty \end{cases}, \tag{10}$$

and

$$\mathbb{E}\Big(\sum_{i=1}^N |\mathsf{X}_i| \exp(\mathsf{X}'_i)\Big) < \infty. \tag{11}$$

Let $\{(N_u, (\mathsf{X}_{u1}, \mathsf{X}'_{u1}), (\mathsf{X}_{u2}, \mathsf{X}'_{u2}), \dots)\}_u$ be a family of independent copies of $\mathcal{S}$, defined on a probability space $(\Omega', \mathcal{A}', \mathbb{P}')$ and indexed by $u = u_1 \cdots u_n$, $n \geq 0$, $u_i \in \mathbb{N}_+$. Therefore, (10) implies that with almost surely, for all $n \geq 1$ and $u \in \mathbb{N}_+^n$,

$$Y'_p(u) = \sum_{v\in\mathsf{T}_p(u)} \exp(S_{n+p}\mathsf{X}'(uv) - S_n\mathsf{X}'(u)) \quad (p \geq 1)$$

converges to a positive limit $Y'(u)$, while, if the condition is violated then the limit exists and vanishes [14,24]. Therefore, using the family $\{(N_{u0}, (\mathsf{X}_{u1}, \mathsf{X}'_{u1}), (\mathsf{X}_{u2}, \mathsf{X}'_{u2}), \dots)\}_u$, we can associate the Mandelbrot measure defined on the σ-field $\mathcal{C}$ generated by the cylinders of $\mathbb{N}_+^{\mathbb{N}_+}$ by

$$\mu'([u]) = \begin{cases} \exp(S_n\mathsf{X}'(u))Y'(u) & \text{if } u \in \mathsf{T}_n \\ 0 & \text{otherwise} \end{cases},$$

and supported on $\partial\mathsf{T}$. Moreover, since $E(Y') < \infty$ [14,25], we have the following result.

Proposition 1. *Almost surely, for μ'-almost every (a.e.) $t \in \partial\mathsf{T}$, we have*

1. $\displaystyle\lim_{n\to\infty} \frac{S_n\mathsf{X}(t)}{n} = \mathbb{E}\Big(\sum_{i=1}^N \mathsf{X}_i \exp(\mathsf{X}'_i)\Big).$
2. $\displaystyle\limsup_{n\to\infty} \frac{\log Y'(t_{|n})}{-n} \leq 0.$
3. $\displaystyle\limsup_{n\to\infty} \frac{\log Y'(t_{|n})}{-n} \leq 0$ *and* $\displaystyle\limsup_{n\to\infty} \frac{\log \mu'([t_{|n}])}{-n} \leq -\mathbb{E}\Big(\sum_{i=1}^N \mathsf{X}'_i \exp(\mathsf{X}'_i)\Big).$

Assume that $\mathbb{E}(\big| \sum_{i=1}^N \exp(\mathsf{X}'_i)\big|^\gamma) < \infty$ for some $\gamma > 1$. Then, under the property $E(Y' \log^+ Y') < \infty$ (in particular when $E({Y'}^h) < \infty$ for some $h > 1$), we obtain the next result [14,25]. For more details on the multifractal analysis of Mandelbrot measures, the reader is referred to [10,11,16].

Proposition 2. *Almost surely, for μ'-almost every $t \in \partial\mathsf{T}$,*

$$\liminf_{n\to\infty} \frac{\log \mu'([t_{|n}])}{-n} \geq -\mathbb{E}\Big(\sum_{i=1}^{N} \mathsf{X}'_i \exp(\mathsf{X}'_i)\Big).$$

2.4. Preliminaries Results

In the following, we give a useful lemma that generalizes Lemma 2.11 in [4]. This result may be used, in particular, in the proof of Proposition 3.

Lemma 1. *Let, for $k \geq 1$, the function $f_k : \mathbb{C} \longrightarrow \mathbb{C}$ and $(N, W_1, W_2, \ldots)$ be a random vector taking values in $\mathbb{N}_+ \times \mathbb{C}^{\mathbb{N}_+}$ and such that $\sum_{i=1}^{N} f_k(W_i)$ is integrable and $\mathbb{E}\big(\sum_{i=1}^{N} f_k(W_i)\big) = 1$. Consider a sequence $\big\{(N_u, W_{u1}, W_{u2}, \ldots)\big\}_{u\in\bigcup_{n\geq 0}\mathbb{N}_+^n}$ of independent copies of $(N, W_1, \ldots, W_N)$. We define the sequence $(Z_n)_{n\geq 0}$ by $Z_0 = 1$ and for $n \geq 1$*

$$Z_n = \sum_{u\in\mathsf{T}_n} \prod_{k=1}^{n} f_k(W_{u_{|k}}).$$

Let $p \in (1,2]$, there exists a constant C_p which depend only on p such that for all $n \geq 1$

$$\mathbb{E}(|Z_n - Z_{n-1}|^p) \leq C_p \mathbb{E}\Big(|\sum_{i=1}^{N} f_n(W_i)|^p\Big) \prod_{k=1}^{n-1} \mathbb{E}\Big(\sum_{i=1}^{N} |f_k(W_i)|^p\Big).$$

Proof. For $n \geq 1$ we have

$$Z_n - Z_{n-1} = \sum_{u\in\mathsf{T}_{n-1}} \prod_{k=1}^{n-1} f_k(W_{u_{|k}}) \Big(\sum_{i=1}^{N_u} f_n(W_{ui}) - 1\Big). \tag{12}$$

Now, for $n \geq 1$, let $\mathcal{F}_n = \sigma\big\{(N_u, W_{u1}, \ldots) : |u| \leq n-1\big\}$ and let $\mathcal{F}_0$ be the trivial sigma-field. For $u \in \mathsf{T}_{n-1}$, we set $B_u(q) = \sum_{i=1}^{N_u} f_n(W_{ui})$. In fact, the random variables $(B_u(q) - 1), u \in \mathsf{T}_{n-1}$, are centered, independent, identically distributed (i.i.d.), and independent of $\mathcal{F}_{n-1}$. Hence, conditionally on $\mathcal{F}_{n-1}$, we can apply Lemma 2.10 in [4] to the family $\Big\{(B_u(q) - 1) \prod_{k=1}^{n-1} f_k(W_{u_{|k}})\Big\}$. Since $B_u(q), u \in \mathsf{T}_{n-1}$ has the same distribution, then

$$\begin{aligned} \mathbb{E}(|Z_n - Z_{n-1}|^p) &= \mathbb{E}\Big(\mathbb{E}(|Z_n - Z_{n-1}|^p \mid \mathcal{F}_{n-1})\Big) \\ &\leq 2^{p-1}\mathbb{E}(|B(q) - 1|^p)\mathbb{E}\Big(\sum_{u\in\mathsf{T}_{n-1}} \prod_{k=1}^{n-1} |f_k(W_{u_{|k}})|^p\Big), \end{aligned}$$

where $B(q)$ stands for any of the identically distributed variables $B_u(q)$. Using the independence of the random vectors $(N_u, W_{u1}, \ldots)$ and the branching property, we obtain

$$\begin{aligned} \mathbb{E}\Big(\sum_{u\in\mathsf{T}_{n-1}} \prod_{k=1}^{n-1} |f_k(W_{u_{|k}})|^p\Big) &= \mathbb{E}\Big[\mathbb{E}\Big(\sum_{u\in\mathsf{T}_{n-2}} \prod_{k=1}^{n-2} |f_k(W_{u_{|k}}))|^p \Big(\sum_{i=1}^{N_u} |f_{n-1}(W_{ui})|^p\Big) \Big| \mathcal{F}_{n-2}\Big)\Big] \\ &= \mathbb{E}\Big(\sum_{i=1}^{N} |f_{n-1}(W_i)|^p\Big)\mathbb{E}\Big(\sum_{u\in\mathsf{T}_{n-2}} \prod_{k=1}^{n-2} |f_k(W_{u_{|k}})|^p\Big) \end{aligned}$$

and then

$$\mathbb{E}\Big(\sum_{u\in\mathsf{T}_{n-1}} \prod_{k=1}^{n-1} |f_k(W_{u_{|k}})|^p\Big) = \prod_{k=1}^{n-1} \mathbb{E}\Big(\sum_{i=1}^{N} |f_k(W_i)|^p\Big).$$

Recall that, for $r > 1$, we have $|x+y|^r \le 2^{r-1}(|x|^r + |y|^r)$, which implies that

$$\mathbb{E}\Big(|\sum_{i=1}^{N_u} f_n(W_{ui}) - 1|^p\Big) \quad \le \quad 2^{p-1}\mathbb{E}\Big(\Big|\sum_{i=1}^{N_u} f_n(W_{ui})\Big|^p + 1\Big).$$

Since $1 = \Big(\mathbb{E}\big(\sum_{1=1}^{N_u} f_n(W_{ui})\big)\Big)^p \le \mathbb{E}\Big|\sum_{i=1}^{N_u} f_n(W_{ui})\Big|^p$ then, it follows from (Lemma 2.10) in [4] that

$$\mathbb{E}\big(\Big|\sum_{i=1}^{N_u} f_n(W_{ui}) - 1\Big|^p\big) \quad \le \quad 2^p\mathbb{E}\big(|\sum_{i=1}^{N_u} f_n(W_{ui})|^p\big) = 2^p\mathbb{E}\big(|\sum_{i=1}^{N} f_n(W_i)|^p\big).$$

Finally, we have

$$\mathbb{E}\Big(|Z_n - Z_{n-1}|^p\Big) \le 2^p\mathbb{E}\Big(|\sum_{i=1}^{N} f_n(W_i)|^p\Big)\prod_{k=1}^{n-1}\mathbb{E}\Big(\sum_{i=1}^{N}|f_k(W_i)|^p\Big).$$

□

We end this section with the Cauchy formula for holomorphic functions, which will be useful in Propositions 5 and 6.

Definition 1. *Let $D \subseteq \mathbb{C}^{\mathsf{d}}$ is an open polydisc that is $D = D_1 \times \cdots \times D_{\mathsf{d}}$, where D_i is an open disc of $\mathbb{C}$ for all $i = 1, \ldots, d$. We denote $D(\zeta, r)$, the polydisc with center $\zeta = (\zeta_1, \ldots, \zeta_{\mathsf{d}})$, and radius $r = (r_1, \ldots, r_{\mathsf{d}})$. The set $\partial D = \partial D_1 \times \cdots \times \partial D_{\mathsf{d}}$ is the distinguished boundary of the polydisc D.*

Let f be a continuous function on ∂D, the boundary of the polydisc $D = D(\zeta, r)$ in $\mathbb{C}^{\mathsf{d}}$. The integral of the function f on ∂D is defined as

$$\int_{\partial D} f(\zeta)d\zeta_1 \ldots d\zeta_{\mathsf{d}} = (2i\pi)^{\mathsf{d}} r_1 \ldots r_{\mathsf{d}} \int_{[0,1]^{\mathsf{d}}} f(\zeta(\theta))e^{i2\pi\theta_1} \ldots e^{i2\pi\theta_d} d\theta_1 \ldots d\theta_{\mathsf{d}},$$

where the function $\zeta(\theta) = (\zeta_1(\theta), \ldots, \zeta_d(\theta))$ and, for $j = 1, \ldots, \mathsf{d}$, one has $\zeta_j(\theta) = \zeta_j + r_j e^{i2\pi\theta_j}$.

Theorem 1. *Let $D = D(a, r)$ be a polydisc in $\mathbb{C}^{\mathsf{d}}$ and f be a holomorphic function in a neighborhood of D. Then, for $z \in D$, one has*

$$f(z) = \frac{1}{(2i\pi)^{\mathsf{d}}}\int_{\partial D}\frac{f(\zeta)d\zeta_1 \ldots d\zeta_{\mathsf{d}}}{(\zeta_1 - z_1)\ldots(\zeta_{\mathsf{d}} - z_{\mathsf{d}})}.$$

It follows that

$$\sup_{z \in D(a, r/2)} |f(z)| \le 2^{\mathsf{d}} \int_{[0,1]^d} |f(\zeta(\theta))| d\theta_1 \ldots d\theta_{\mathsf{d}}. \tag{13}$$

3. Main Result

In this section, we give our main result concerning the study of the size of the set $E_{\alpha,s}$ (Theorem 2). Let us mention that the method used in [18] to compute the Hausdorff and the packing dimension of the set $\widetilde{E}_{\alpha,s}$ does not give results on dim $E_{\alpha,s}$. Let $s = (s_n)_{n\ge 0}$ be a positive sequence and for $n \ge 1$, $\eta_n = s_n - s_{n-1}$. Assume

$$s_n = \mathrm{o}(n), \quad \eta_n = \mathrm{o}(1) \tag{14}$$

and there exist $\epsilon_n \to 0$ such that

$$\sum_{n\geq 1} \exp\Big(-\epsilon \sum_{k=1}^{n} \epsilon_k \, \eta_k^2 \Big) < +\infty, \qquad \forall \epsilon > 0. \tag{15}$$

In particular, we can choose for $n \geq 1$,

$$s_n = \sum_{k=1}^{n} \frac{1}{k^\alpha} \quad \text{and} \quad \epsilon_n = n^{-\xi} \tag{16}$$

such that $\alpha \in (0, \frac{1}{2})$ and $\xi > 0$ such that $1 - 2\alpha - \xi > 0$. We are now able to state our main result.

Theorem 2. *Let $s = (s_n)_{n\geq 1}$ be a positive sequence such that (14) and (15) are satisfied. Then, a.s., for all $\alpha \in int(\mathcal{K})$,*

$$\dim E_{\alpha,s} = Dim\, E_{\alpha,s} = \widetilde{\mathcal{P}}_\alpha^*(0) = \inf_{q\in\mathbb{R}} \widetilde{\mathcal{P}}_\alpha(q).$$

In fact, we have $\dim E_{\alpha,s} \leq \dim E_\alpha \leq \widetilde{\mathcal{P}}_\alpha^*(0)$, this result also yields the packing dimensions simultaneously (9). Therefore, we need to prove Theorem 2, a simultaneous building, for ϖ belonging to a suitable set $\mathcal{J}$ of Mandelbrot measures μ_ϖ^s and computing their Hausdorff and packing dimensions; it uses extensive techniques combining analytic functions theory and large deviations estimates. However, our approach covers only levels $\alpha \in int(\mathcal{K})$ and cannot be applied to cover the set $E_{\alpha,s}$ with $\alpha \in \partial\mathcal{K}$ (see Section 4). In the following, we will prove that $\mu_\varpi^s(E_{\alpha,s}) = 1$ (Proposition 5). Moreover, almost surely, for all $\varpi \in \mathcal{J}$, for μ_ϖ^s-almost every $t \in E_{\alpha,s}$, we have (Propositions 5 and 6)

$$\dim \mu_\varpi^s := \lim_{n\to\infty} \frac{\log(\mu_\varpi^s[t_{|n}])}{\log(\mathrm{diam}([t_{|n}]))} = \widetilde{\mathcal{P}}_\alpha^*(0)$$

then, using (Theorem 4.2 in [20]), we get

$$\dim E_{\alpha,s} \geq \widetilde{\mathcal{P}}_\alpha^*(0) = \inf_{q\in\mathbb{R}} \widetilde{\mathcal{P}}_\alpha(q). \tag{17}$$

which gives the desired result.

3.1. Construction of Inhomogeneous Mandelbrot Measures

We consider the set $\mathcal{J} = \Big\{ (q,\alpha) \in \mathbb{R} \times int(\mathcal{K}) : \ \widetilde{\mathcal{P}}_\alpha^*(\widetilde{\mathcal{P}}_\alpha'(q)) > 0 \Big\}$. The same lines as in (Proposition 3.2) in [23] show, for each $\alpha \in int(\mathcal{K})$, the existence of unique $q := q_\alpha$ such that $\widetilde{\mathcal{P}}_\alpha'(q) = 0$. Moreover, $\alpha \in int(\mathcal{K}) \longmapsto q_\alpha$ is analytic. This fact will be used in the construction of the inhomogeneous Mandelbrot measures.

Lemma 2. *Let K be a nontrivial compact set of $\mathcal{J}$. Then, there exists a real number*

1. $1 < p_K < 2$ *such that for all* $p \in (1, p_K]$ *we have*

$$\sup_{(q,\alpha)\in K} \psi_\alpha(q, p_K) < 0.$$

2. $\widetilde{p}_K > 1$, *for which*

$$\sup_{(q,\alpha)\in K} \mathbb{E}\Big(\Big| \sum_{i=1}^{N} e^{q(\mathsf{X}_i - \alpha \widetilde{\mathsf{X}}_i)} \Big|^{\widetilde{p}_K} \Big) < \infty.$$

Proof.

1. Let $\varpi = (q,\alpha) \in \mathcal{J}$. One has $\dfrac{\partial\psi_\alpha}{\partial p}(q,1^+) < 0$. Therefore, $\exists\ p_\varpi > 1$, such that $\psi_\alpha(q,p_\varpi) < 0$ and, in a neighborhood V_ϖ of ϖ, one has
$$\psi_{\alpha'}(q',p_\varpi) < 0, \quad \text{for all} \quad (q',\alpha') \in V_\varpi.$$
If K is a nontrivial compact of $\mathcal{J}$, it is covered by a finite number of such V_{ϖ_i}. Finally, we may take $p_K = \inf_i p_{\varpi_i}$. If $1 < p \leq p_K$ and $\sup_{\varpi\in K}\psi_\alpha(q,p) \geq 0$, there exists $(q_0,\alpha_0) \in K$ such that
$$\psi_{\alpha_0}(q_0,p) \geq 0, \quad \text{and} \quad (q_0,\alpha_0) \in V_{\varpi_i}, \qquad \text{for some } i.$$
Now, the function $p \mapsto \psi_\alpha(q,p)$ is convex and $\psi_\alpha(q,1) = 0$. Since $1 < p \leq p_{\varpi_i}$, we have $\psi_{\alpha_0}(q_0,p) < 0$, which is a contradiction.
2. Since the mapping $\varpi = (q,\alpha) \longmapsto \mathbb{E}\Big(\big(\sum_{i=1}^N e^{q(\mathsf{X}_i-\alpha\widetilde{\mathsf{X}}_i)}\big)^{\widetilde{p}_K}\Big)$ is continuous over $\mathcal{J}$ and K is a compact subset of $\mathcal{J}$ then, using (7), there exists $\gamma := \widetilde{p}_K \in (1,2]$ such that
$$\sup_{\varpi\in K}\mathbb{E}\Big(\Big|\sum_{i=1}^N e^{q(\mathsf{X}_i-\alpha\widetilde{\mathsf{X}}_i)}\Big|^{\widetilde{p}_K}\Big) < \infty.$$

□

In the following, for $\varpi = (q,\alpha) \in \mathcal{J}$, we will construct an auxiliary measure μ_ϖ. We define, for $k \geq 1$, $\psi_k(\varpi)$ as the unique real t, such that
$$\widetilde{\mathcal{P}}'_\alpha(t) = \eta_k. \tag{18}$$

For $u \in \bigcup_{n\geq 0}\mathbb{N}_+^n$ and $\varpi \in \mathcal{J}$, we set for $1 \leq i \leq N_u$,
$$V(ui,\varpi) = \frac{\exp\big(q\mathsf{X}_{ui} - q\alpha\widetilde{\mathsf{X}}_{ui}\big)}{\mathbb{E}\Big(\sum_{i=1}^N \exp\big(q\mathsf{X}_i - q\alpha\widetilde{\mathsf{X}}_i\big)\Big)} = \exp\big(q\mathsf{X}_{ui} - q\alpha\widetilde{\mathsf{X}}_{ui} - \widetilde{\mathcal{P}}_\alpha(q)\big)$$
and, for all $n \geq 0$,
$$Y_n^s(\varpi,u) = \sum_{v_1\cdots v_n\in\mathsf{T}_n(u)}\prod_{k=1}^n V\big(u\cdot v_1\cdots v_k,\psi_{|u|+k}(\varpi)\big).$$

In addition, $Y_n^s(\varpi,\emptyset)$ will be denoted by $Y_n^s(\varpi)$ and $Y_0^s(\varpi,u) = 1$.

It is not difficult to observe that $\big(Y_n^s(\varpi,u)\big)_{n\geq1}$ is a positive martingale such that $\mathbb{E}\big(Y_n^s(\varpi,u)\big) = 1$. Therefore, it converges almost surely and in L^1 norm to a positive random variable $Y^s(\varpi,u)$ (see for instance [3,4,14,24,26] for a study of a similar sequence). In this paper, we need the almost surely simultaneous convergence of $\big(Y_n^s(\varpi,u)\big)_{n\geq1}$ to positive limits. This fact will be proven in the next proposition which generalizes Proposition 2.3 in [4] and Proposition 2 in [18]. The proof is almost the same lines as Proposition 2 in [18], the difference is that in the next proposition, we will prove the convergence of $\big(Y_n^s(\varpi,u)\big)_{n\geq1}$ almost surely and simultaneously on $(q,\alpha) \in \mathcal{J}$ and not only on q. However, this idea will be considered during the hold of the paper (see the proof of Propositions 5 and 6) so we keep the proof of Proposition 3 to the reader.

Proposition 3. *Let $K \subset \mathcal{J}$ be a compact set and consider the continuous functions $g_n : \varpi \in K \mapsto Y_n^s(\varpi,u)$. We can find a real number $p_K \in (1,2]$ such that g converge uniformly, a.s. and in L_{p_K} norm, to a limit $\varpi \in K \mapsto Y^s(\varpi,u)$. In particular, $\mathbb{E}(\sup_{\varpi\in K} Y^s(\varpi,u)^{p_K}) < \infty$. Furthermore, $Y^s(\cdot,u)$ is positive a.s.*

In addition, for all $n \geq 0$, $\sigma\big(\{(\mathsf{X}_{u1}, \widetilde{\mathsf{X}}_{u1}), \ldots, (\mathsf{X}_{uN_u}, \widetilde{\mathsf{X}}_{uN_u}), u \in \mathsf{T}_n\}\big)$ *and* $\sigma\big(\{Y^s(\cdot, u), u \in \mathsf{T}_{n+1}\}\big)$ *are independent, and the random functions* $Y^s(\cdot, u), u \in \mathsf{T}_{n+1}$*, are independent copies of* $Y^s(\cdot) := Y^s(\cdot, \emptyset)$.

It follows, using the branching property

$$Y_n^s(\varpi, u) = \sum_{i=1}^{N} \exp\left(q\mathsf{X}_{ui} - q\alpha\widetilde{\mathsf{X}}_{ui} - \widetilde{\mathcal{P}}_\alpha(q)\right) Y_n^s(\varpi, ui)$$

that we can construct the inhomogeneous Mandelbrot measures μ_ϖ^s.

Proposition 4. *Almost surely, for all* $\varpi \in \mathcal{J}$*, we have*

$$\mu_\varpi^s([u]) = \left[\prod_{k=1}^{n} \exp\Big(\psi_k(q)(\mathsf{X}_{u_1\ldots u_k} - \alpha\widetilde{\mathsf{X}}_{u_1\ldots u_k}) - \widetilde{\mathcal{P}}_\alpha(\psi_k(q))\Big)\right] Y^s(\varpi, u)$$

define a positive measure on the boundary of the Galton–Watson tree, where ψ_k *is defined in* (18).

The measure μ_ϖ^s will be useful to estimate below the dimension of $E_{\alpha,s}$.

3.2. Proof of Theorem 2

Theorem 2 is a direct consequence of the following two propositions. Their proofs are developed in the next subsections.

Proposition 5. *Almost surely, for all* $\varpi := (q_\alpha, \alpha) \in \mathcal{J}$,

$$S_n\mathsf{X}(t) - \alpha S_n\widetilde{\mathsf{X}}(t) \sim s_n \quad \text{for } \mu_\varpi^s\text{-a.e. } t \in \partial\mathsf{T}.$$

Proposition 6. *Almost surely, for all* $\varpi \in \mathcal{J}$*, for* μ_ϖ^s*-a.e.* $t \in \partial\mathsf{T}$,

$$\lim_{n\to\infty} \frac{\log Y^s(\varpi, t_{|n})}{n} = 0.$$

Using Proposition 5, we deduce that a.s., for all $\varpi := (q_\alpha, \alpha) \in \mathcal{J}$, $\mu_\varpi^s(E_{\alpha,s}) = 1$. Furthermore, a.s., for all $\varpi := (q_\alpha, \alpha) \in \mathcal{J}$, for μ_ϖ^s-a.e. $t \in E_{\alpha,s}$, we have (Proposition 5 and 6)

$$\begin{aligned}
\dim \mu_\varpi^s &:= \lim_{n\to\infty} \frac{\log(\mu_\varpi^s[t_{|n}])}{\log(\operatorname{diam}([t_{|n}]))} \\
&= \lim_{n\to\infty} -\frac{1}{n} \log\left[\prod_{k=1}^{n} \exp\Big(\psi_k(q)(\mathsf{X}_{t_1\ldots t_k} - \alpha\widetilde{\mathsf{X}}_{t_1\ldots t_k}) - \widetilde{\mathcal{P}}_\alpha(\psi_k(q))\Big)\right] Y^s(\varpi, t_{|n}) \\
&= \lim_{n\to\infty} -\frac{1}{n} \sum_{k=1}^{n} \psi_k(q)(\mathsf{X}_{t_1\ldots t_k} - \alpha\widetilde{\mathsf{X}}_{t_1\ldots t_k}) + \widetilde{\mathcal{P}}_\alpha(\psi_k(q)) - \frac{\log Y^s(\varpi, t_{|n})}{n} \\
&= \widetilde{\mathcal{P}}_\alpha(q) = \widetilde{\mathcal{P}}_\alpha^*(0).
\end{aligned}$$

We deduce the result from (Theorem 4.2 in [20]) and (9).

Example 1. *Let* $p \in (0,1)$*. In this example, we suppose that* X *is random variable with Bernoulli distribution, that is,*

$$\mathbb{P}(\mathsf{X} = 1) = p = 1 - \mathbb{P}(\mathsf{X} = 0).$$

Therefore, for $t \in \partial\mathsf{T}$*, the random walk* $S_n\mathsf{X}(t)$ *should be interpreted as the covering number of* t *by the family of cylinder* $[u]$ *of generation* $k \leq n$ *with* $\mathsf{X}_u = 1$*. Therefore, the result proven in* *this paper improves and covers the result in [17] which only proves the multifractal analysis for each α a.s.*

Example 2. *In this example, we consider the branching random walk $S_n\mathsf{X}(t)$ to be the branching process itself, that is, X is the branching numbers N defined above assuming it is not constant. Therefore, the natural branching random walk is denoted by*

$$S_n N(t) = N_{t_1} + N_{t_1 t_2} + \cdots + N_{t_1\cdots t_n}.$$

The result in this paper provides a geometric and large deviation description of the heterogeneity of the birth process along different infinite branches.

3.3. Proof of Proposition 5

Let $K \subseteq int(\mathcal{K})$ be a compact set and consider $\mathsf{K} = \{(q_\alpha, \alpha) : \alpha \in K\}$, where q_α is a number such that $\widetilde{\mathcal{P}}'_\alpha(q_\alpha) = 0$. For $n \geq 1, \epsilon > 0, \varpi = (q_\alpha, \alpha) \in \mathsf{K}$, and $s = (s_n)_{n\geq 1}$, we set

$$E^1_{\varpi,s,n,\epsilon} = \Big\{ t \in \partial\mathsf{T} : \sum_{k=1}^n \mathsf{X}_{t_1\cdots t_k}(t) - \alpha\widetilde{\mathsf{X}}_{t_1\cdots t_k}(t) - \eta_k \geq \epsilon \sum_{k=1}^n \eta_k \Big\}$$
$$E^{-1}_{\varpi,n,s,\epsilon} = \Big\{ t \in \partial\mathsf{T} : \sum_{k=1}^n \mathsf{X}_{t_1\cdots t_k}(t) - \alpha\widetilde{\mathsf{X}}_{t_1\cdots t_k}(t) - \eta_k \leq -\epsilon \sum_{k=1}^n \eta_k \Big\}.$$

For $\lambda \in \{-1, 1\}$, suppose that we have shown

$$\mathbb{E}\Big(\sup_{\varpi\in\mathsf{K}} \sum_{n\geq 1} \mu^s_\varpi(E^\lambda_{\varpi,n,s,\epsilon}) \Big) < \infty. \tag{19}$$

Then, almost surely, for all $\alpha \in int(\mathcal{K})$, $\epsilon \in \mathbb{Q}^*_+$ and $\lambda \in \{-1, 1\}$, we have $\sum_{n\geq 1} \mu^s_\varpi(E^\lambda_{\varpi,n,s,\epsilon}) < \infty$. Whence, we obtain the desired result using the Borel–Cantelli lemma. In the following, we will prove (19) for $\lambda = 1$ (the case $\lambda = -1$ is similar). Consider a positive sequence $\theta = (\theta_n)$ and $\varpi \in \mathsf{K}$ one has

$$\sup_{\varpi\in\mathsf{K}} \mu^s_\alpha\Big(E^1_{\varpi,n,s,\epsilon}\Big) \leq \sup_{\varpi\in\mathsf{K}} \sum_{u\in\mathsf{T}_n} \mu^s_\varpi([u])\, \mathbf{1}_{\{E^1_{\varpi,n,s,\epsilon}\}}(t_u)$$

where t_u is any point in the cylinder $[u]$. For simplicity, we will denote t_u by t, then

$$\begin{aligned}
&\sup_{\varpi\in\mathsf{K}} \mu^s_\varpi\Big(E^1_{\varpi,n,s,\epsilon}\Big) \\
\leq\; &\sup_{\varpi\in\mathsf{K}} \sum_{u\in\mathsf{T}_n} \mu^s_\varpi[u] \prod_{k=1}^n \exp\Big(\theta_k \mathsf{X}_{t_1\cdots t_k} - \theta_k\alpha\widetilde{\mathsf{X}}_{t_1\cdots t_k} - \theta_k\eta_k(1+\epsilon)\Big) \\
\leq\; &\sup_{\varpi\in\mathsf{K}} \sum_{u\in\mathsf{T}_n} \prod_{k=1}^n A_k(\alpha,\omega,\theta,\epsilon)\; Y^s(\varpi, u),
\end{aligned}$$

where
$A_k(\alpha,\omega,\theta,\epsilon) := \exp\Big((\psi_k(\varpi) + \theta_k)\mathsf{X}_{t_1\cdots t_k} - \widetilde{\mathcal{P}}_\alpha(\psi_k(\varpi)) - (\theta_k + \psi_k(\varpi))\alpha\widetilde{\mathsf{X}}_{t_1\cdots t_k} - \theta_k\eta_k(1+\epsilon)\Big)$.
For $\varpi \in \mathsf{K}$, $\theta = (\theta_n)$ and $n \geq 1$, we set

$$F^s_n(\varpi,\theta) \;=\; \sum_{u\in\mathsf{T}_n} \prod_{k=1}^n A_k(\alpha,\omega,\theta,\epsilon)\mathsf{Y}^s(u),$$

where

$$\mathsf{Y}^s(u) = \sup_{\alpha\in K} Y^s(\varpi, u).$$

There exists a neighborhood $V_{\mathsf{K}} \subset \mathbb{C}^2$ of K such that,

$$\Gamma(z,z') = \frac{\mathbb{E}\Big(\sum_{i=1}^{N}(\mathsf{X}_i - z'\widetilde{\mathsf{X}}_i)\exp(z\mathsf{X}_i - zz'\widetilde{\mathsf{X}}_i)\Big)}{\mathbb{E}\Big(\sum_{i=1}^{N}\exp(z\mathsf{X}_i - zz'\widetilde{\mathsf{X}}_i)\Big)} \quad \text{and} \quad \psi_k(\overline{\omega}),$$

$(k \geq 1)$, are well defined for all $\overline{\omega} = (z,z') \in V_{\mathsf{K}}$. For $\epsilon > 0$ and $n \geq 1$, we define

$$\begin{aligned} F_n^s(\overline{\omega},\theta) \quad &= \quad \sum_{u\in\mathsf{T}_n}\prod_{k=1}^{n}\exp\Big((\psi_k(\overline{\omega}) + \theta_k)\mathsf{X}_{u_{|k}} - \theta_k\Gamma(z,z') - \theta_k\eta_k(1+\epsilon)\Big) \times \\ & \quad\quad \mathbb{E}\Big(\sum_{i=1}^{N}\exp\left(\psi_k(\overline{\omega})\mathsf{X}_i\right)\Big)^{-1}\mathsf{Y}^s(u). \end{aligned}$$

Proposition 7. *There exist a positive constant $\mathcal{C}_{\mathsf{K}}$, a positive sequence θ, and a neighborhood $V \subset V_{\mathsf{K}}$ of K, such that for all $\overline{\omega} = (z,z') \in V$, for all $n \in \mathbb{N}^*$,*

$$\mathbb{E}(|F_n^s(\overline{\omega},\theta)|) \leq \mathcal{C}_{\mathsf{K}}e^{-b_n/2}$$

where $b_n = \frac{\epsilon}{4}\sum_{k=1}^{n}\epsilon_k\eta_k^2$ and $(\epsilon_n)_n$ is the sequence defined in (15).

Proof. Assume, we have proved for all $\varpi \in \mathsf{K}$, that

$$\mathbb{E}\big(F_n^s(\varpi,\theta)\big) \leq \mathcal{C}_{\mathsf{K}}e^{-b_n}, \tag{20}$$

where $\theta = (\theta_n)$ is a positive sequence and $\mathcal{C}_{\mathsf{K}}$ is a positive constant. Then, we can find a neighborhood $V_\varpi \subset V_{\mathsf{K}}$ of ϖ such that $\mathbb{E}(|F_n^s(\overline{\omega},\theta)|) \leq \mathcal{C}_{\mathsf{K}}e^{-b_n/2}$, for all $\overline{\omega} = (z,z') \in V_\varpi$. By extracting, from $\bigcup_{\varpi\in\mathsf{K}} V_\varpi$, a finite covering of K, we construct a neighborhood $V \subset V_{\mathsf{K}}$ of K such that

$$\mathbb{E}(|F_n^s(\overline{\omega},\theta)|) \leq \mathcal{C}_{\mathsf{K}}e^{-b_n/2}.$$

Now, we will prove (20). First, remark, for any positive sequence $\theta = (\theta_n)$, we have

$$\begin{aligned} \mathbb{E}\big(F_n^s(\varpi,\theta)\big) \quad &= \quad \prod_{k=1}^{n}\mathbb{E}\Big(\sum_{i=1}^{N}\exp\Big((\psi_k(\varpi) + \theta_k)\mathsf{X}_i - (\theta_k + \psi_k(\varpi))\alpha\widetilde{\mathsf{X}}_i\Big) \times \\ & \quad\quad \exp\Big(-\widetilde{\mathcal{P}}_\alpha(\psi_k(\varpi)) - \theta_k\eta_k(1+\epsilon)\Big)\mathbb{E}(\mathsf{Y}^s(u)) \\ &\leq \quad \mathcal{C}'_{\mathsf{K}}\prod_{k=1}^{n}\exp\Big(\widetilde{\mathcal{P}}_\alpha(\psi_k(\varpi) + \theta_k) - \widetilde{\mathcal{P}}_\alpha(\psi_k(\varpi)) - \theta_k\eta_k(1+\epsilon)\Big), \end{aligned}$$

where, using Proposition (3), we have

$$\mathcal{C}'_{\mathsf{K}} = \mathbb{E}\big(\mathsf{Y}^s(u)\big) = \mathbb{E}\big(\mathsf{Y}^s(\emptyset)\big) < \infty,$$

$\forall u \in \bigcup_{n\geq 0}\mathbb{N}_+^n$. Notice that $\eta_k = \mathrm{o}(1)$, therefore, we can find a compact neighborhood K' of K such that $\psi_k(\varpi) \in \mathsf{K}'$, for all $\forall\, k \geq 1$ and $\forall\, \varpi \in \mathsf{K}$. Now, consider the function $h : \theta \mapsto \widetilde{\mathcal{P}}_\alpha(\psi_k(\varpi) + \theta)$, then a direct application of the Taylor expansion with integral rest of order 2 of h at 0, we obtain

$$h(\theta) = h(0) + \theta h'(0) + \theta^2\int_0^1 (1-t)h''(t\theta)dt,$$

where $h''(t\theta) \leq m_{\mathsf{K}} = \sup_{t\in[0,1]}\sup_{\varpi\in\mathsf{K}} h''(t\theta)$. Therefore,

$$\widetilde{\mathcal{P}}_\alpha(\psi_k(\varpi) + \theta_k) - \widetilde{\mathcal{P}}_\alpha(\psi_k(\varpi)) - \theta_k\widetilde{\mathcal{P}}'_\alpha(\psi_k(\varpi)) \leq \theta_k^2 m_{\mathsf{K}}, \qquad (k \geq 1).$$

Recall that $\widetilde{\mathcal{P}}'_\alpha(\psi_k(\varpi)) = \eta_k$. Then

$$\begin{aligned}\mathbb{E}\big(F_n^s(\varpi,\theta)\big) &\leq \mathcal{C'}_{\mathsf{K}} \prod_{k=1}^{n} \exp\Big(\widetilde{\mathcal{P}}_\alpha(\psi_k(\varpi)+\theta_k) - \widetilde{\mathcal{P}}_\alpha(\psi_k(\varpi)) - \theta_k\eta_k(1+\epsilon)\Big) \\ &\leq \mathcal{C'}_{\mathsf{K}} \prod_{k=1}^{n} \exp\Big(-\theta_k\eta_k\epsilon + \theta_k^2 m_{\mathsf{K}}\Big).\end{aligned}$$

since θ is an arbitrarily positive sequence, we may consider $\theta_k = \epsilon_k\eta_k$. Hence, we get

$$\mathbb{E}\big(F_n^s(\varpi,\theta)\big) \leq \mathcal{C'}_{\mathsf{K}} \prod_{k=1}^{n} \exp\Big(-\epsilon_k\eta_k^2(\epsilon - \epsilon_k m_{\mathsf{K}})\Big).$$

Since the sequence (ϵ_k) tends to zero, we have $\epsilon - \epsilon_k m_{\mathsf{K}} > \dfrac{\epsilon}{2}$, for k large enough. Then, we obtain (20) with $b_n = \frac{\epsilon}{2}\sum_{k=1}^n \epsilon_k\eta_k^2$. □

With probability 1, the mapping $\overline{\omega} \in V \longmapsto F_n^s(\overline{\omega},\theta)$ is analytic. Fix $\rho > 0$ and a closed polydisc $D(\overline{\omega}_0, 2\rho) \subset V$, $\rho > 0$. Using Theorem 1, we obtain

$$\sup_{\overline{\omega}\in D(\overline{\omega}_0,\rho)} \big|F_n^s(\overline{\omega},\theta)\big| \leq 4\int_{[0,1]^2} \big|F_n(\zeta(t),\theta)\big| dt,$$

where, for $t = (t_1,t_2) \in [0,1]^2$, $\zeta(t) = \overline{\omega}_0 + 2\rho(e^{i2\pi t_1}, e^{i2\pi t_2})$. Furthermore, Fubini's Theorem gives

$$\begin{aligned}\mathbb{E}\big(\sup_{\overline{\omega}\in D(\overline{\omega}_0,\rho)} |F_n^s(\overline{\omega},\theta)|\big) &\leq \mathbb{E}\big(4\int_{[0,1]^2}|F_n^s(\zeta(t),\theta)|dt\big) \leq 4\int_{[0,1]^2}\mathbb{E}|F_n^s(\zeta(t),\theta)|dt \\ &\leq 4C_{\mathsf{K}}e^{-b_n/2}.\end{aligned}$$

Finally, we get

$$\mathbb{E}\Big(\sup_{\varpi\in\mathsf{K}} \mu_\varpi^s(E^1_{\alpha,n,s,\epsilon})\Big) \leq 4C_{\mathsf{K}}e^{-b_n/2}$$

and, then, under (15), we get (19) as required.

3.4. *Proof of Propostion 6*

Let $K \subset \mathcal{J}$ be a compact set and $a > 1$. We define the following set

$$E^+_{n,a} = \big\{t \in \partial\mathsf{T} : Y^s(\varpi, t_{|n}) > a^n\big\} \text{ and } E^-_{n,a} = \big\{t \in \partial\mathsf{T} : Y^s(\varpi, t_{|n}) < a^{-n}\big\},$$

where $\varpi := (q,\alpha) \in K$ and $n \geq 1$. We suppose, for some $\widetilde{\nu} > 0$ and $C > 0$, that

$$\sup_{\varpi\in K} \mu_\varpi^s(E) < Ce^{-n\widetilde{\nu}/2}, \tag{21}$$

for all $E \in \{E^+_{n,a}, E^-_{n,a}\}$. This implies that $\mathbb{E}\big(\sup_{\varpi\in K}\sum_{n\geq1}\mu_\varpi^s(E)\big) < \infty$ and then a.s., for each $q \in K$ and $E \in \{E^+_{n,a}, E^-_{n,a}\}$ we have $\sum_{n\geq1}\mu_\varpi^s(E) < \infty$. Therefore, using the Borel–Cantelli lemma, we get, for μ_ϖ^s-a.e. $t \in \partial\mathsf{T}$ and n, which is large enough,

$$-\log a \leq \liminf_{n\to\infty}\frac{1}{n}\log Y^s(\varpi, t_{|n}) \leq \limsup_{n\to\infty}\frac{1}{n}\log Y^s(\varpi, t_{|n}) \leq \log a,$$

which gives the desired result by letting a tend to 1.

In the next, we will only prove (21) for $E = E^+_{n,a}$ (the case $E = E^-_{n,a}$ is similar). First we have,

$$
\begin{aligned}
\sup_{\varpi\in K}\mu^s_\varpi(E^+_{n,a}) &= \sup_{\varpi\in K}\sum_{u\in\mathsf{T}_n}\mu^s_\varpi([u])\mathbf{1}_{\{Y^s(\varpi,u)>a^n\}}\\
&= \sup_{\varpi\in K}\sum_{u\in\mathsf{T}_n}Y^s(\varpi,u)\prod_{k=1}^n\exp\Big(\psi_k(\varpi)(\mathsf{X}_{u_{|k}}-\alpha\widetilde{\mathsf{X}}_{u_{|k}})-\widetilde{\mathcal{P}}_\alpha(\psi_k(\varpi))\Big)\mathbf{1}_{\{Y^s(\varpi,u)>a^n\}}\\
&\le \sup_{\varpi\in K}\sum_{u\in\mathsf{T}_n}(Y^s(\varpi,u))^{1+\nu}\prod_{k=1}^n\exp\Big(\psi_k(\varpi)(\mathsf{X}_{u_{|k}}-\alpha\widetilde{\mathsf{X}}_{u_{|k}})-\widetilde{\mathcal{P}}_\alpha(\psi_k(\varpi))\Big)a^{-\nu}\\
&\le \sup_{\varpi\in K}\sum_{u\in\mathsf{T}_n}\mathsf{Y}^s(u)^{1+\nu}\prod_{k=1}^n\exp\Big(\psi_k(\varpi)(\mathsf{X}_{u_{|k}}-\alpha\widetilde{\mathsf{X}}_{u_{|k}})-\widetilde{\mathcal{P}}_\alpha(\psi_k(\varpi))\Big)a^{-\nu},
\end{aligned}
$$

where $\mathsf{Y}^s(u)=\sup_{\varpi\in K}Y^s(\varpi,u)$ and $\nu>0$. For $\varpi\in K$ and $\nu>0$, we set

$$
H_n(\varpi,\nu)=\sum_{u\in\mathsf{T}_n}\mathsf{Y}^s(u)^{1+\nu}\prod_{k=1}^n\exp\Big(\psi_k(\varpi)(\mathsf{X}_{u_{|k}}-\alpha\widetilde{\mathsf{X}}_{u_{|k}})-\widetilde{\mathcal{P}}_\alpha(\psi_k(\varpi))\Big)a^{-\nu}.
$$

We can find a neighborhood $U_K\subset\mathbb{C}^2$ of K such that for all $\overline{\omega}=(z,z')\in U_K$, and $k\ge 1$

$$
\psi_k(\overline{\omega})\quad\text{is defined and}\quad \mathbb{E}\Big(\sum_{i=1}^N e^{\psi_k(\overline{\omega})(\mathsf{X}_i-z'\widetilde{\mathsf{X}}_i)}\Big)\neq 0,
$$

so that, we may define, for $\overline{\omega}=(z,z')\in U_K$, the mapping

$$
\begin{aligned}
H_n(\overline{\omega},\nu) &= \Big[\prod_{k=1}^n\mathbb{E}\Big(\sum_{i=1}^N\exp\big(\psi_k(z)(\mathsf{X}_i-z'\widetilde{\mathsf{X}}_i)\big)\Big)^{-1}\Big]\times\\
&\qquad\sum_{u\in\mathsf{T}_n}\mathsf{Y}^s(u)^{1+\nu}\prod_{k=1}^n\exp\Big(\psi_k(z)(\mathsf{X}_{u_{|k}}-z'\widetilde{\mathsf{X}}_{u_{|k}})\Big)a^{-\nu}.
\end{aligned}
$$

Moreover, we can find a neighborhood $V\subset\mathbb{C}^2$ of K and a positive constant C_K such that, for all $\overline{\omega}\in V$, for all $n\ge 1$,

$$
\mathbb{E}\Big(\Big|H_n(\overline{\omega},p_K-1)\Big|\Big)\le C_K a^{-n(p_K-1)/2}, \tag{22}
$$

where p_K is the real defined in Proposition (3).

Now, almost surely, the mapping $\overline{\omega}\in V\longmapsto H_n(\overline{\omega},\nu)$ is analytic. Fix $\rho>0$ and $D(\overline{\omega}_0,2\rho)\subset V$. It follows, using Theorem 1, that

$$
\sup_{\overline{\omega}\in D(\overline{\omega}_0,\rho)}\big|H_n(\overline{\omega},p_K-1)\big|\le 4\int_{[0,1]^2}\big|H_n(\zeta(t),p_K-1)\big|dt,
$$

where $\zeta(t)=\overline{\omega}_0+2\rho(e^{i2\pi t_1},e^{i2\pi t_2})$, for $t=(t_1,t_2)\in[0,1]^2$. Therefore, by Fubini's Theorem, we obtain

$$
\begin{aligned}
\mathbb{E}\big(\sup_{\overline{\omega}\in D(\overline{\omega}_0,\rho)}|H_n(\overline{\omega},p_K-1)|\big) &\le \mathbb{E}\Big(4\int_{[0,1]^2}|H_n(\zeta(t),p_K-1)|dt\Big)\\
&\le 4\int_{[0,1]^2}\mathbb{E}|H_n(\zeta(t),p_K-1)|dt\\
&\le 4C_K a^{-n(p_K-1)/2}.
\end{aligned}
$$

Since $a>1$ and $p_K-1>0$, we get (21).

Now, turn back to prove the Equation (22). For $\overline{\omega} \in U_K$ and $\nu > 0$, we set

$$\widetilde{H}_1(\overline{\omega},\nu) = \Big|\mathbb{E}\big(\sum_{i=1}^{N} \exp\big(z(\mathsf{X}_i - z'\widetilde{\mathsf{X}}_i)\big)\big)\Big|^{-1} \mathbb{E}\Big(\sum_{i=1}^{N}\Big|\exp\big(z(\mathsf{X}_i - z'\widetilde{\mathsf{X}}_i)\big)\Big|\Big)\, a^{-\nu}.$$

Let $\varpi = (q,\alpha) \in K$. Since $\mathbb{E}(\widetilde{H}_1(\varpi,\nu)) = a^{-\nu}$, there exists a neighborhood $V_\varpi \subset U_K$ of ϖ such that

$$\mathbb{E}\Big(\big|\widetilde{H}_1(z,\nu)\big|\Big) \le a^{-\nu/2},$$

for all $z \in V_q$. Therefore, from $\bigcup_{\varpi\in K} V_\varpi$, we can extract a finite covering of K and then find a neighborhood $V \subset U_K$ of K such that $\mathbb{E}\Big(\big|\widetilde{H}_1(\overline{\omega},\nu)\big|\Big) \le a^{-\nu/2}$, for all $\overline{\omega} = (z,z') \in V$. Without loss of generality, since $\eta_k = \mathrm{o}(1)$, we can assume that

$$\mathbb{E}\Big(\big|\widetilde{H}_1(\overline{\omega}_k,\nu)\big|\Big) \le a^{-\nu/2} \qquad (k \ge 1),$$

where $\overline{\omega}_k = (\psi_k(\overline{\omega}), z')$. Therefore,

$$\begin{aligned}
\mathbb{E}\big(\big|H_n(\overline{\omega},\nu)\big|\big) &= \Big[\prod_{k=1}^{n}\Big|\mathbb{E}\big(\sum_{i=1}^{N}\exp\big(\psi_k(z)(\mathsf{X}_i - z'\widetilde{\mathsf{X}}_i)\big)\Big|^{-1}\Big] \times \\
&\qquad \mathbb{E}\Big(\Big|\sum_{u\in\mathsf{T}_n}\mathsf{Y}^s(u)^{1+\nu}\prod_{k=1}^{n}\exp\Big(\psi_k(z)(\mathsf{X}_{u_{|k}} - z'\widetilde{\mathsf{X}}_{u_{|k}})\Big)\Big|\Big)\, a^{-n\nu} \\
&\le \Big[\prod_{k=1}^{n}\big|\mathbb{E}\Big(\sum_{i=1}^{N}\exp\big(\psi_k(z)(\mathsf{X}_i - z'\widetilde{\mathsf{X}}_i)\big)\Big)\big|^{-1}\Big] \times \\
&\qquad \mathbb{E}\Big(\sum_{u\in\mathsf{T}_n}\mathsf{Y}^s(u)^{1+\nu}\prod_{k=1}^{n}\Big|\exp\Big(\psi_k(z)(\mathsf{X}_{u_{|k}} - z'\widetilde{\mathsf{X}}_{u_{|k}})\Big)\Big|\Big)\, a^{-n\nu}.
\end{aligned}$$

According to Proposition 3, we can find a real $1 < p_K \le 2$ such that

$$\mathbb{E}\big(\mathsf{Y}^s(u)^{p_K}\big) = \mathbb{E}\big(\mathsf{Y}^s(\varnothing)^{p_K}\big) = C_K < \infty,$$

for all $u \in \bigcup_{n\ge0}\mathbb{N}_+^n$. Since $\sigma\big(\{(\mathsf{X}_{u1},\widetilde{\mathsf{X}}_{u1}),\ldots,(\mathsf{X}_{uN_u}\widetilde{\mathsf{X}}_{uN_u})\}, u \in \mathsf{T}_{n-1}\}\big)$ and $\sigma\big(\{Y^s(\cdot,u), u \in \mathsf{T}_n\}\big)$ are independent for all $n \ge 1$, then, for $\nu = p_K - 1$, we obtain

$$\begin{aligned}
\mathbb{E}\Big(\Big|H_n(\overline{\omega},p_K-1)\Big|\Big) &\le \Big[\prod_{k=1}^{n}\big|\mathbb{E}\big(\sum_{i=1}^{N}\exp\Big(\psi_k(z)(\mathsf{X}_i - z'\widetilde{\mathsf{X}}_i)\Big)\big)\big|^{-1}\Big] \cdot \\
&\qquad \prod_{k=1}^{n}\mathbb{E}\Big(\sum_{i=1}^{N}\Big|\exp\Big(\psi_k(z)(\mathsf{X}_i - z'\widetilde{\mathsf{X}}_i)\Big)\Big|\Big)^{n} C_K a^{-n(p_K-1)} \\
&= C_K\prod_{k=1}^{n}\mathbb{E}\Big(\big|\widetilde{H}_1(\overline{\omega}_k,p_K-1)\big|\Big) \le C_K a^{-n(p_K-1)/2},
\end{aligned}$$

which gives the desired result.

4. Perspective and Concluding Remarks

1. As mentioned in (16), we can choose the sequence (s_n) as follows:

$$s_n := s_{n,\beta} = \sum_{k=1}^{n}\frac{1}{k^\beta} \qquad \text{with} \qquad \beta \in (0,1/2).$$

Therefore, using Theorem 2 for each $\beta \in (0, 1/2)$ such that (14) and (15) are satisfied, we have almost surely for all $\alpha \in int(\mathcal{K})$,

$$\dim E_{\alpha,s} = Dim\, E_{\alpha,s} = \widetilde{\mathcal{P}}^*_\alpha(0) = \inf_{q \in \mathbb{R}} \widetilde{\mathcal{P}}_\alpha(q). \tag{23}$$

That is, $E_{\alpha,s}$ has a maximal Hausdorff and packing dimension. It is natural to ask whether it is possible to have the dimension uniformly on β. For this, we first define $\eta_n(\beta) = s_{n,\beta} - s_{n-1,\beta}$ and let us consider the set Λ_s such that

$$\Lambda_s \subseteq \Big\{\beta \in \mathbb{R}, \text{ such that } (s_{n,\beta}) \text{ satisfies } (14) \text{ and } (15)\Big\}.$$

Assume, for $k \geq 1$, that

$$\widetilde{\eta}_k = \inf_{\gamma \in \Lambda_s} \eta_k(\beta) > 0$$

and we suppose that there exists a sequence $\epsilon_n \to 0$ such that

$$\sum_{n \geq 1} \exp\Big(-\sum_{k=1}^{n} \epsilon_k \widetilde{\eta}_k^2\Big) < +\infty.$$

our approach gives the result in this context and we can prove that, under the previous assumptions, almost surely, for all $\alpha \in int(\mathcal{K})$ and all $\beta \in \Lambda_s$, we have the result mentioned in (23). This result generalizes Theorem 1.3 in [23].

2. Our approach gives results for the sequences (s_n) satisfying (14) and (15). It is natural to ask, for a given sequence $s_n = \circ(n)$, what is the size of the set $E_{\alpha,s}$. In particular, it is possible to obtain
   (a) $DimE_{\alpha,s} = 0$ with $E_{\alpha,s} \neq \emptyset$.
   (b) $\dim E_{\alpha,s} \neq DimE_{\alpha,s}$.
3. As mentioned in the introduction, the set $E_{\mathsf{X},\widetilde{\mathsf{X}}}(\alpha) \neq \emptyset$ if and only if $\alpha \in \mathcal{K}$ [3]. It remains then the nontrivial question of whether the approach introduced in this paper can be used to study the sets $E_{\alpha,s}$ for $\alpha \in \partial\mathcal{K}$. Since, for $\alpha \in \partial\mathcal{K}$, there is no q_α in general, such that $\widetilde{\mathcal{P}}'_\alpha(q_\alpha) = 0$, then the method used in this paper cannot be used to compute the Hausdorff and packing dimension of the set $E_{\alpha,s}$ in this case. However, it would be possible to use a concatenation method used in [12] to construct a Mandelbrot measure carried by $E_{\alpha,s}$ and with dimension $\widetilde{\mathcal{P}}^*_\alpha(0)$. In this case, it is possible to obtain $\widetilde{\mathcal{P}}^*_\alpha(0)$ and then $\dim E_{\alpha,s} = DimE_{\alpha,s} = 0$ but the set $E_{\alpha,s} \neq \emptyset$.

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