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A Space-Time Finite Element Method for the Fractional Ginzburg–Landau Equation

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Abstract: A fully discrete space-time finite element method for the fractional Ginzburg–Landau equation is developed, in which the discontinuous Galerkin finite element scheme is adopted in the temporal direction and the Galerkin finite element scheme is used in the spatial orientation. By taking advantage of the valuable properties of Radau numerical integration and Lagrange interpolation polynomials at the Radau points of each time subinterval $\mathcal{I}_n$, the well-posedness of the discrete solution is proven. Moreover, the optimal order error estimate in $L^\infty(L^2)$ is also considered in detail. Some numerical examples are provided to evaluate the validity and effectiveness of the theoretical analysis.

Keywords: fractional Ginzburg–Landau equation; discontinuous Galerkin finite element method; error estimate; numerical test

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1. Introduction

In this paper, we consider the following fractional Ginzburg–Landau equation with the Riesz fractional derivative:

$$\begin{cases} u_t + (\nu + i\eta)(-\triangle)^{\frac{\alpha}{2}} u + (\kappa + i\zeta)|u|^2 u - \gamma u = 0, & x \in \mathbb{R}, t \in (0, T], \\ u(x, 0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1}$$

where $u_t = \frac{\partial u(x,t)}{\partial t}$, $1 < \alpha \leq 2$, $i^2 = -1$, and $\nu > 0$, $\kappa > 0$, η, ζ, γ are real parameters. Here, $u(x,t)$ is a complex-valued function, and $u_0(x)$ is a given function. The Riesz fractional derivative is defined as follows [1]:

$$(-\triangle)^{\frac{\alpha}{2}} u = -\frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} = \frac{1}{2\cos\frac{\pi\alpha}{2}} [{}_{-\infty}D_x^\alpha u(x,t) + {}_x D_\infty^\alpha u(x,t)], \quad 1 < \alpha < 2.$$

Here, ${}_{-\infty}D_x^\alpha$ is the left Riemann–Liouville fractional derivative [2]

$${}_{-\infty}D_x^\alpha u(x,t) = \frac{1}{\Gamma(2-\alpha)} \frac{\mathrm{d}^2}{\mathrm{d}x^2} \int_{-\infty}^x \frac{u(s,t)}{(x-s)^{\alpha-1}} \mathrm{d}s,$$

and ${}_xD_{+\infty}^\alpha u(x,t)$ is the right Riemann–Liouville fractional derivative

$${}_xD_{+\infty}^\alpha u(x,t) = \frac{1}{\Gamma(2-\alpha)} \frac{\mathrm{d}^2}{\mathrm{d}x^2} \int_x^\infty \frac{u(s,t)}{(s-x)^{\alpha-1}} \mathrm{d}s,$$

where $\Gamma(\cdot)$ denotes the Gamma function. When $\alpha = 2$, $-(-\triangle)^{\frac{\alpha}{2}}$ coincides with the standard Laplace operator.

The fractional Ginzburg–Landau equation was suggested in [3,4]. It can be used to describe the dynamical process in a medium with fractal dispersion [4] and media with a fractal mass

dimension [5]. The analytical and closed solutions of the fractional Ginzburg–Landau equation cannot be obtained in general. Recently, some numerical methods for the fractional Ginzburg–Landau equation have been proposed to analyze the behavior of the solution of the equation. For solving this model, some efficient numerical methods, including finite difference methods [6–16] and finite element methods [17,18], were developed. Since the Galerkin finite element method can solve differential equations with more complex geometries and has high-order accuracy, numerous researchers considered the finite element method along the spacial direction to solve space fractional differential equations (see [19,20] and the references therein).

The space-time finite element method (STFEM) is a more attractive tool for solving partial differential equations. Its idea was first put forward by scholars such as Nickell, Sackman, and Oden (see [21,22]), and then it was constantly developed and improved. This method treats the time and space variables with a unified Galerkin finite-element framework. It generalizes the finite element scheme of layer-by-layer iteration, which is more flexible in dealing with discontinuous problems on unstructured meshes. It has been successfully applied to solve some partial differential equations with the integer order (see [23–31] for more details). Recently, Mustapha [32], Zheng and Zhao [33], Liu et al. [34], Liu et al. [35], Bu et al. [36], Yue et al. [37], and Li et al. [38] developed a finite element scheme for the fractional diffusion equation, fractional diffusion wave equation, linear space fractional PDE, nonlinear fractional reaction diffusion system, multi-term time-space fractional diffusion equation, multi-term time fractional advection–diffusion equations, and fractional wave problems, for which the space-time finite element method was adopted.

To our knowledge, few papers have been published concerning the STFEM for fractional equations. Therefore, this paper aims to generalize the STFEM to the fractional Ginzburg–Landau equation (Equation (1)). The existence, uniqueness, and stability of discrete solutions are analyzed based on the Radau numerical integration formula and the advantage of the useful properties of Lagrange interpolating polynomials at the Radau points of each time slab. The optimal order error estimate in $L^\infty(L^2)$ is provided under weak restrictions on the space-time meshes. In the future, we will propose an adaptive algorithm based on the present work.

The modulus of the initial value $u_0(x)$ decays to zero as the spatial variable x moves away from the origin in general (e.g., see (57) and (58) in Section 6). Also, for the needs of the error analysis, the infinite interval problem is usually truncated on a finite interval [17,39]. Based on this, we consider the following extended Dirichlet boundary problem:

$$\begin{cases} u_t + (\nu + i\eta)(-\triangle)^{\frac{\alpha}{2}} u + (\kappa + i\zeta)|u|^2 u - \gamma u = 0, & x \in \Omega, t \in (0, T], \\ u(x,t) = 0, & x \in \mathbb{R}\backslash\Omega, t \in (0, T], \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{2}$$

where $\Omega = [a, b]$ is a finite interval, $u_t = \frac{\partial u(x,t)}{\partial t}$, and $u_0(x)$ is a given function. The Riesz fractional derivative is then given by

$$(-\triangle)^{\frac{\alpha}{2}} u = -\frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} = \frac{1}{2\cos\frac{\pi\alpha}{2}} [{}_aD_x^\alpha u(x,t) + {}_xD_b^\alpha u(x,t)],$$

where ${}_aD_x^\alpha u(x,t)$ and ${}_xD_b^\alpha u(x,t)$ denote the left Riemann–Liouville and right Riemann–Liouville fractional derivatives of the order $1 < \alpha < 2$, respectively:

$${}_aD_x^\alpha u(x,t) = D_x^2 J_{a+}^{2-\alpha} u(x,t), \quad {}_xD_b^\alpha u(x,t) = D_x^2 J_{b-}^{2-\alpha} u(x,t).$$

Here, $D_x^2 = \partial^2/\partial x^2$ and $J_{a+}^{\mu}u(x,t)$, $J_{b-}^{\mu}u(x,t)$ are the Riemann–Liouville fractional integrals of the order μ of the following respective forms [1]:

$$J_{a+}^{\mu}u(x,t) = \frac{1}{\Gamma(\mu)}\int_a^x (s-x)^{\mu-1}u(s,t)\mathrm{d}s \ (x > a, \mu > 0),$$
$$J_{b-}^{\mu}u(x,t) = \frac{1}{\Gamma(\mu)}\int_x^b (x-s)^{\mu-1}u(s,t)\mathrm{d}s \ (x < b, \mu > 0).$$

2. Preliminaries

We begin with some notations. As usual, $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ are the inner product and norm in the Hilbert space $L^2(\Omega)$, respectively, while $\langle\cdot,\cdot\rangle_\alpha$ and $\|\cdot\|_\alpha$ will denote the inner product and norm equipped in the fractional Sobolev space $H^\alpha(\Omega)$, respectively. For convenience, $H_0^\alpha(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ with respect to $\|\cdot\|_\alpha$. We also introduce the space $L^\infty(\Omega)$ and the L^∞ norm as $\|\cdot\|_\infty$. Throughout this paper, C will denote the constants, which may differ in different places.

The following lemmas are very useful for us to establish the numerical theories of the space-time finite element method for Equation (2):

Lemma 1 ([40,41])**.** *Let $0 < \alpha < 1$, $\alpha \neq \frac{1}{2}$, and $\Omega = [a,b]$ be a finite interval, where $a, b \in \mathbb{R}$. If $u(x), v(x) \in H_0^\alpha(\Omega)$, and the first derivative of $v(x)$ is integrable in the open interval (a,b), then we have*

$$\langle {}_aD_x^{2\alpha}u, v\rangle = \langle {}_aD_x^{\alpha}u, {}_xD_b^{\alpha}v\rangle, \quad \langle {}_xD_b^{2\alpha}u, v\rangle = \langle {}_xD_b^{\alpha}u, {}_aD_x^{\alpha}v\rangle.$$

Lemma 2 ([40,41])**.** *If $\alpha > 0$, $u(x) \in H_0^\alpha(\Omega)$, then*

$$\langle {}_aD_x^{\alpha}u, {}_xD_b^{\alpha}u\rangle = \cos(\alpha\pi)\ \|\ {}_aD_x^{\alpha}u\ \|^2 .$$

Lemma 3. *If $1 < \alpha < 2$, $u(x), v(x) \in H_0^\alpha(\Omega)$, then it holds that*

$$\langle D_x^2\, {}_aD_x^{\alpha}u, v\rangle = \langle {}_aD_x^{\frac{\alpha}{2}+1}u, {}_xD_b^{\frac{\alpha}{2}+1}v\rangle, \tag{3a}$$
$$\langle D_x^2\, {}_xD_b^{\alpha}u, v\rangle = \langle {}_xD_b^{\frac{\alpha}{2}+1}u, {}_aD_x^{\frac{\alpha}{2}+1}v\rangle. \tag{3b}$$

Proof. It is obvious that $1 - \frac{\alpha}{2} \in (0,1)$ because of $1 < \alpha < 2$. From ([1], pages 74 and 92), we have

$$\begin{aligned} &D_x({}_aD_x^{\frac{\alpha}{2}}u) = {}_aD_x^{\frac{\alpha}{2}+1}u, \\ &{}_xD_b^{\frac{\alpha}{2}}D_x^2v = -D_x\, J_{b-}^{1-\frac{\alpha}{2}}D_x^2v = -D_x^3J_{b-}^{1-\frac{\alpha}{2}}v = -D_x({}_xD_b^{\frac{\alpha}{2}+1}v). \end{aligned} \tag{4}$$

Using integration by parts, Lemma 1, and Equation (4), we have

$$\begin{aligned} \langle D_x^2{}_aD_x^{\alpha}u, v\rangle &= \langle {}_aD_x^{\alpha}u, D_x^2v\rangle = \langle {}_aD_x^{\frac{\alpha}{2}}u, {}_xD_b^{\frac{\alpha}{2}}D_x^2v\rangle \\ &= \langle {}_aD_x^{\frac{\alpha}{2}}u, -D_x({}_xD_b^{\frac{\alpha}{2}+1}v)\rangle = \langle D_x({}_aD_x^{\frac{\alpha}{2}}u), {}_xD_b^{\frac{\alpha}{2}+1}v\rangle \\ &= \langle {}_aD_x^{\frac{\alpha}{2}+1}u, {}_xD_b^{\frac{\alpha}{2}+1}v\rangle. \end{aligned} \tag{5}$$

Therefore, the identity in Equation (3a) is verified. The identity in Equation (3b) can be proven similarly. □

Lemma 4 (Brouwder fixed-point theorem [42])**.** *Let $(H, \langle\cdot,\cdot\rangle)$ be a finite dimensional Hilbert space endowed with the norm $\|\cdot\|$ and $f : H \to H$ be continuous. Assume that there is $\delta > 0$ such that*

$$Re\langle f(z), z\rangle \geq 0, \text{ for every } z \in H, \ \|\ z\ \| = \delta.$$

Then, there exists an element $z^* \in H$ such that $f(z^*) = 0$ and $\|\ z^*\ \| \leq \delta$.

Now, we introduce the formula of Radau numerical integrals, which will be used in the following theoretical analysis of the space-time finite element scheme:

Lemma 5 ([43]). *For each integer $q \geq 1$, assume $g(\tau) \in C^{2q+1}[0,1]$. Then, there exist weights w_j $(j = 1, 2, \cdots, q)$ such that*

$$\int_0^1 g(\tau)\mathrm{d}\tau \cong \sum_{j=1}^{q} w_j g(\tau_j), \quad 0 < \tau_1 < \tau_2 < \cdots < \tau_q = 1 \tag{6}$$

holds. This is called the Radau quadrature rule. The Radau method is exact for all polynomials of degrees no more than $2q-2$. The nodes τ_j $(j = 1, 2, \ldots, q)$ are called the Radau points over $(0,1]$. Moreover, $\tau_j = \frac{1}{2} + \frac{1}{2}x_j$, x_j is the jth zero of $\dfrac{P_q(x) + P_{q+1}(x)}{x-1}$, where $P_q(x)$ is the Legendre polynomial of a degree q, $x \in [-1,1]$, and $w_j = \dfrac{1}{2(1-x_j)} \dfrac{1}{[P_q'(x_j)]^2}$, $j = 1, 2, \ldots, q$.

3. The Fully Discrete Space-Time Finite Element Scheme

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a partition of the time domain $[0,T]$ and $\mathcal{I}_n = (t_n, t_{n+1}]$, $\lambda_n = t_{n+1} - t_n$, $n = 0, 1, \ldots, N-1$. Let $S_h \subset H_0^{\frac{\alpha}{2}}(\Omega)$ denote the space of continuous and piecewise polynomials of a total degree $r-1$ (where r is a positive integer) concerning a partition $\mathcal{T}_h^n = \{\mathcal{K}\}$ of $\Omega = [a,b]$. We associate a partition $\mathcal{T}_h^n$ of Ω and a finite element space $\mathcal{S}_h^n$ to each interval $\mathcal{I}_n$; that is, we have

$$\mathcal{S}_h^n = \{v(x) \in H_0^{\frac{\alpha}{2}}(\Omega) : v(x)|_{\mathcal{K}} \in P_{r-1}(\mathcal{K}),\ \mathcal{K} \in \mathcal{T}_h^n\},$$

where $P_{r-1}(\mathcal{K})$ is a set of polynomials of a degree $r-1$ on a given spacial domain $\mathcal{K}$. We also associate a space S_h^{-1} with $\{t_0\}$. For simplicity, we take $S_h^{-1} = S_h^0$. Here, we denote with $\mathcal{K}$ the element of the partition $\mathcal{T}_h^n$, $h_{\mathcal{K}}$ the diameter of $\mathcal{K}$, and $h_n = \max\limits_{\mathcal{K} \in \mathcal{T}_h^n} h_{\mathcal{K}}$, $n = 0, 1, \ldots, N-1$.

Now, with a given positive integer q, we introduce the space-time finite element space over the whole region $\Omega \times (0,T]$:

$$\mathcal{V}_{h\lambda} = \{\phi : \Omega \times (0,T] \to \mathbb{C},\ \phi|_{\Omega \times \mathcal{I}_n} = \sum_{j=0}^{q-1} t^j v_j(x),\ v_j(x) \in \mathcal{S}_h^n\}.$$

The functions of $\mathcal{V}_{h\lambda}$ are, for fixed $t \in \mathcal{I}_n$, elements of $\mathcal{S}_h^n$, and for each $x \in \Omega$, polynomial functions are of a degree of at most $q-1$ in t on each subinterval $\mathcal{I}_n$. Note that the functions in $\mathcal{V}_{h\lambda}$ are allowed to be discontinuous at the nodes t_n, $n = 0, 1, \ldots, N-1$. Also, let $\mathcal{V}_{h\lambda}^n = \{\phi|_{\Omega \times \mathcal{I}_n} : \phi \in \mathcal{V}_{h\lambda}\}$.

Let $\langle Bu, v\rangle = \langle (-\triangle)^{\frac{\alpha}{2}} u, v\rangle$ be a bilinear mapping. Then, from [40], there exists a constant C such that

$$|\langle Bu, v\rangle| \leq C \parallel u \parallel_{\frac{\alpha}{2}} \parallel v \parallel_{\frac{\alpha}{2}}, \quad \forall\, u, v \in L^2(\Omega), \tag{7}$$

and

$$\langle Bu, u\rangle = \parallel {}_aD_x^{\frac{\alpha}{2}} u \parallel^2 + \parallel {}_xD_b^{\frac{\alpha}{2}} u \parallel^2 \geq 0, \quad \forall u \in L^2(\Omega) \tag{8}$$

according to Lemmas 1 and 2.

Let $f : \mathbb{C} \to \mathbb{C}$ be a function with the constants $C_1 > 0$, $C_2 > 0$ such that

$$\begin{aligned} &|f(z)| \leq C_1|z|, \quad \forall z \in \mathbb{C}, \\ &|f(z) - f(w)| \leq C_2|z - w|, \quad \forall z, w \in \mathbb{C}. \end{aligned} \tag{9}$$

We assume that Equation (2) admits a unique smooth solution on $[0, T]$. We define the approximation $U \in \mathcal{V}_{h\lambda}$ to the solution u of Equation (2) as follows.

Find $U \in \mathcal{V}_{h\lambda}$ such that it satisfies

$$\begin{aligned} &\int_{\mathcal{I}_n} \langle U_t, \Phi \rangle \mathrm{d}t + (\nu + i\eta) \int_{\mathcal{I}_n} \langle BU, \Phi \rangle \mathrm{d}t + (\kappa + i\zeta) \int_{\mathcal{I}_n} \langle f(U), \Phi \rangle \mathrm{d}t \\ &- \gamma \int_{\mathcal{I}_n} \langle U, \Phi \rangle \mathrm{d}t + \langle [U^n], \Phi^n_+ \rangle = 0, \quad \forall \Phi \in \mathcal{V}^n_{h\lambda},\ n = 0, 1, \ldots, N-1. \end{aligned} \tag{10}$$

Here, $[U^n] = U^n_+ - U^n_-$, $U^n_\pm = \lim\limits_{t \to t_n^\pm} U(x,t)$, and $\Phi^n_+ = \lim\limits_{t \to t_n^+} \Phi(x,t)$. We have set $U^0_- = u_0(x)$. The inner product $\langle [U^n], \Phi^n_+ \rangle$ denotes the data transport process during different time-space slabs, and it reflects the discontinuity of the scheme.

By applying the partial integration, we have

$$\int_{\mathcal{I}_n} \langle U_t, \Phi \rangle \mathrm{d}t + \langle [U^n], \Phi^n_+ \rangle = \langle U^{n+1}, \Phi^{n+1} \rangle - \int_{\mathcal{I}_n} \langle U, \Phi_t \rangle \mathrm{d}t - \langle U^n, \Phi^n_+ \rangle,$$

where $U^{n+1} = U^{n+1}_-$, $U^n = U^n_-$, and $\Phi^{n+1} = \Phi^{n+1}_-$. Then, the scheme in Equation (10) can be modified as follows:

$$\begin{aligned} &\langle U^{n+1}, \Phi^{n+1} \rangle - \int_{\mathcal{I}_n} \langle U, \Phi_t \rangle \mathrm{d}t + (\nu + i\eta) \int_{\mathcal{I}_n} \langle BU, \Phi \rangle \mathrm{d}t + (\kappa + i\zeta) \int_{\mathcal{I}_n} \langle f(U), \Phi \rangle \mathrm{d}t \\ &- \gamma \int_{\mathcal{I}_n} \langle U, \Phi \rangle \mathrm{d}t = \langle U^n, \Phi^n_+ \rangle, \quad \forall \Phi \in \mathcal{V}^n_{h\lambda}, \quad n = 0, 1, \ldots, N-1, \end{aligned} \tag{11}$$

where $U^0 = u_0(x)$ and f is endowed with the properties of Equation (9).

Next, we prove the well-posedness of the numerical scheme in Equation (11).

4. Well-Posedness

For a fixed value $q \geq 1$, we recall the Lagrange polynomials with the set of Radau points $\tau_1, \tau_2, \cdots, \tau_q$ on $(0, 1]$; that is, we have

$$l_k(\tau) = \prod_{j=1, j \neq k}^{q} \frac{(\tau - \tau_j)}{(\tau_k - \tau_j)}, \quad k = 1, 2, \ldots, q.$$

It is obvious that $l_k(\tau)$ has the degree $q-1$, $l_k(\tau_m) = 0$ for $m \neq k$, and $l_k(\tau_m) = 1$ for $m = k$. By setting $t(\tau) = t_n + \tau\lambda_n$, $\tau \in [0, 1]$, the interval $[0, 1]$ is mapped to $\overline{\mathcal{I}}_n = [t_n, t_{n+1}]$, $n = 0, 1, \cdots, N-1$. Then, the quadrature rule in Equation (6) is adapted to the interval $\overline{\mathcal{I}}_n$ with its abscissae and weights as follows:

$$\begin{aligned} &t_{n,k} = t_n + \tau_k \lambda_n, \quad k = 1, 2, \cdots, q\ (t_{n,q} = t_{n+1}), \\ &l_{n,k}(t) = l_k(\tau), \ (t = t_n + \tau\lambda_n), \\ &\omega_{n,k} = \int_{t_n}^{t_{n+1}} l_{n,k}(t) \mathrm{d}t = \lambda_n \int_0^1 l_k(\tau) \mathrm{d}\tau = \lambda_n \omega_k, \quad k = 1, 2, \ldots, q. \end{aligned}$$

Therefore, $U|_{\mathcal{I}_n}$ is uniquely determined by the function $U^{n,k} = U^{n,k}(x) = U(x, t_{n,k}) \in \mathcal{S}^n_h$ $(k = 1, 2, \cdots, q)$ such that

$$U(x,t) = \sum_{k=1}^{q} l_{n,k}(t) U^{n,k}(x), \quad (x,t) \in \Omega \times \mathcal{I}_n. \tag{12}$$

Now, if $\Psi = \Psi(x) \in \mathcal{S}^n_h$, then the function $\Phi = l_{n,k}\Psi$ is an element of $\mathcal{V}^n_{h\lambda}$. By applying Lemma 5, we obtain that Equation (11) is equivalent to

$$\begin{aligned}&\delta_{q,k}\langle U^{n,q},\Psi\rangle-\sum_{j=1}^{q}w_{n,j}l'_{n,k}(t_{n,j})\langle U^{n,j},\Psi\rangle+(\nu+i\eta)w_{n,k}\langle BU^{n,k},\Psi\rangle\\&+(\kappa+i\zeta)\int_{\mathcal{I}_n}l_{n,k}(t)\langle f(U),\Psi\rangle\mathrm{d}t-\gamma w_{n,k}\langle U^{n,k},\Psi\rangle=l_{n,k}(t_n)\langle U^n,\Psi\rangle,\\&\qquad\qquad\Psi\in\mathcal{S}_h^n,\quad k=1,2,\ldots,q,\end{aligned}\tag{13}$$

where $\delta_{q,k}$ is the Kronecker delta function which satisfies $\delta_{q,k}=0$ if $q\neq k$ and $\delta_{q,k}=1$ if $q=k$. Here, $l'_{n,k}(t_{n,k})$ denotes the derivative of $l_{n,k}(t)$ with respect to t at point $t_{n,k}$.

We introduce $q\times q$ matrices N and M, defined by

$$N_{jk}=w_{n,k}l'_{n,j}(t_{n,k})=\omega_k l'_j(\tau_k),\quad M=e_qe_q^T-N,$$

where $e_q=(0,0,\cdots,0,1)^T\in\mathbb{R}^q$. It is clear that N and M are independent of λ_n, and if $Y=(y^{n,1},y^{n,2},\cdots,y^{n,q})^T\in\mathbb{R}^q$, then

$$Y^TMY=\sum_{j=1}^{q}\delta_{q,j}y^{n,q}y^{n,j}-\sum_{j,k=1}^{q}w_{n,k}l'_{n,j}(t_{n,k})y^{n,j}y^{n,k}.$$

Lemma 6 ([25])**.** *Let $\hat{M}=D^{-1/2}MD^{1/2}$ with $D=\mathrm{diag}\{\tau_1,\tau_2,\cdots,\tau_q\}$. Then, the following holds:*

$$X^T\hat{M}X\geq\mu(\sum_{k=1}^{q}{x_k}^2),\quad\forall X=(x_1,x_2,\cdots,x_q)^T\in\mathbb{R}^q,$$

where $\mu=\frac{1}{2}\min\{\frac{\omega_1}{\tau_1},\frac{\omega_2}{\tau_2},\cdots,\frac{\omega_{q-1}}{\tau_{q-1}},1+\omega_q\}>0$.

In order to take advantage of the positivity of $\hat{M}$, we choose $\Phi=\tau_k^{-\frac{1}{2}}l_{n,k}(t)\Psi(x)\in\mathcal{V}_{h\lambda}^n$ and take $\tilde{U}^{n,j}=\tau_j^{-\frac{1}{2}}U^{n,j}(x)\in\mathcal{S}_h^n$ into the scheme in Equation (13). Then, we obtain

$$\begin{aligned}&\delta_{q,k}\langle\tilde{U}^{n,q},\Psi\rangle-\sum_{j=1}^{q}w_{n,j}\tau_j^{\frac{1}{2}}\tau_k^{-\frac{1}{2}}l'_{n,k}(t_{n,j})\langle\tilde{U}^{n,j},\Psi\rangle+(\nu+i\eta)w_{n,k}\langle B\tilde{U}^{n,k},\Psi\rangle\\&+(\kappa+i\zeta)\int_{\mathcal{I}_n}\tau_k^{-\frac{1}{2}}l_{n,k}(t)\langle f(U),\Psi\rangle\mathrm{d}t-\gamma w_{n,k}\langle\tilde{U}^{n,k},\Psi\rangle-\tau_k^{-\frac{1}{2}}l_{n,k}(t_n)\langle U^n,\Psi\rangle=0,\\&\qquad\qquad\Psi\in\mathcal{S}_h^n,\quad k=1,2,\ldots,q.\end{aligned}\tag{14}$$

Now, we are ready to prove the following result:

Theorem 1. *Let U^n be given in $\mathcal{S}_h^{n-1}$. Then, for sufficiently small values of λ_n, there exists $\{\tilde{U}^{n,j}\}_{j=1}^q$ in $(\mathcal{S}_h^n)^q$, satisfying Equation (14). Therefore, Equation (11) has a solution $U\in\mathcal{V}_{h\lambda}^n$. Furthermore, U is unique.*

Proof. Note that $(\mathcal{S}_h^n)^q$ is a Hilbert space with the inner product

$$\langle\langle V,Z\rangle\rangle=\sum_{k=1}^{q}\langle v_k,z_k\rangle$$

for every $V=(v_1,v_2,\cdots,v_q)^T\in(\mathcal{S}_h^n)^q$, $Z=(z_1,z_2,\cdots,z_q)^T\in(\mathcal{S}_h^n)^q$. And we denote the corresponding norm $(\sum_{k=1}^{q}\|v_k\|^2)^{1/2}$ with $|||V|||$.

Next, we shall use Lemma 4 to show that the map $G : (\mathcal{S}_h^n)^q \to (\mathcal{S}_h^n)^q$, which is defined by

$$\begin{aligned}
&\langle G(V)_k, \Psi\rangle = \delta_{q,k}\langle v_q, \Psi\rangle - \sum_{j=1}^{q} w_{n,j}\tau_j^{\frac{1}{2}}\tau_k^{-\frac{1}{2}} l'_{n,k}(t_{n,j})\langle v_j, \Psi\rangle \\
&+ (\nu + i\eta) w_{n,k}\langle Bv_k, \psi\rangle + (\kappa + i\zeta)\int_{\mathcal{I}_n} \tau_k^{-\frac{1}{2}} l_{n,k}(t)\langle f(\sum_{j=1}^{q} \tau_j^{\frac{1}{2}} l_{n,j} v_j), \Psi\rangle \mathrm{d}t \\
&- \gamma w_{n,k}\langle v_k, \Psi\rangle - \tau_k^{-\frac{1}{2}} l_{n,k}(t_n)\langle U^n, \Psi\rangle, \quad \forall \Psi \in \mathcal{S}_h^n, \quad k = 1, 2, \ldots, q,
\end{aligned} \tag{15}$$

has a zero point in $(\mathcal{S}_h^n)^q$ (i.e., Equation (14) has a solution).

Note that if $f : \mathbb{C} \to \mathbb{C}$ is continuous, then G is continuous on $(\mathcal{S}_h^n)^q$. We take $\Psi = v_k$ in Equation (15) and sum it from $k = 1$ to q to obtain

$$\begin{aligned}
\mathrm{Re}\langle\langle G(V), V\rangle\rangle \geq\ & \mathrm{Re}\left\{\sum_{k=1}^{q} \delta_{q,k}\langle v_q, v_k\rangle - \sum_{k,j=1}^{q} w_{n,j}\tau_j^{\frac{1}{2}}\tau_k^{-\frac{1}{2}} l'_{n,k}(t_{n,j})\langle v_j, v_k\rangle\right\} \\
& - \sqrt{\kappa^2 + \zeta^2}\int_{\mathcal{I}_n} \sum_{k=1}^{q} |\tau_k^{-\frac{1}{2}} l_{n,k}(t)\langle f(\sum_{j=1}^{q} \tau_j^{\frac{1}{2}} l_{n,j} v_j), v_k\rangle| \mathrm{d}t \\
& - |\gamma| \sum_{k=1}^{q} w_{n,k}\langle v_k, v_k\rangle - \sum_{k=1}^{q} |\tau_k^{-\frac{1}{2}} l_{n,k}(t_n)\langle U^n, v_k\rangle|.
\end{aligned} \tag{16}$$

From Lemma 6, we have

$$\begin{aligned}
& \mathrm{Re}\left\{\sum_{k=1}^{q} \delta_{q,k}\langle v_q, v_k\rangle - \sum_{k,j=1}^{q} w_{n,j}\tau_j^{\frac{1}{2}}\tau_k^{-\frac{1}{2}} l'_{n,k}(t_{n,j})\langle v_j, v_k\rangle\right\} \\
=\ & V^T \hat{M} V \geq \mu(\sum_{j=1}^{q} v_j{}^2) = \mu|||V|||^2.
\end{aligned} \tag{17}$$

By using the Cauchy–Schwarz inequality and the properties of Equation (9), we obtain

$$\begin{aligned}
& \sqrt{\kappa^2 + \zeta^2}\int_{\mathcal{I}_n} \sum_{k=1}^{q} |\tau_k^{-\frac{1}{2}} l_{n,k}(t)\langle f(\sum_{j=1}^{q} \tau_j^{\frac{1}{2}} l_{n,j} v_j), v_k\rangle| \mathrm{d}t \\
\leq\ & \sqrt{\kappa^2 + \zeta^2} \sum_{k=1}^{q} \int_{\mathcal{I}_n} \tau_k^{-\frac{1}{2}} l_{n,k}(t) \parallel f(\sum_{j=1}^{q} \tau_j^{\frac{1}{2}} l_{n,j} v_j) \parallel \parallel v_k \parallel \mathrm{d}t \\
\leq\ & C_1\sqrt{\kappa^2 + \zeta^2} \sum_{k,j=1}^{q} \int_{\mathcal{I}_n} \tau_k^{-\frac{1}{2}}\tau_j^{\frac{1}{2}} l_{n,k}(t) l_{n,j}(t) \parallel v_j \parallel \parallel v_s \parallel \mathrm{d}t \\
\leq\ & C_1\sqrt{\kappa^2 + \zeta^2} \sum_{i,j,k=1}^{q} w_{n,k}\tau_i^{-\frac{1}{2}}\tau_j^{\frac{1}{2}} l_{n,i}(t_{n,k}) l_{n,j}(t_{n,k}) \parallel v_j \parallel \parallel v_i \parallel \\
\leq\ & C_1\sqrt{\kappa^2 + \zeta^2}\lambda_n \sum_{k=1}^{q} w_k \parallel v_k \parallel^2 \leq \hat{C}_1\lambda_n|||V|||^2,
\end{aligned} \tag{18}$$

where $\hat{C}_1 = C_1\sqrt{\kappa^2 + \zeta^2}, C_1 > 0$. There also exists a constant $\hat{C}_2 > 0$ such that

$$|\gamma| \sum_{k=1}^{q} w_{n,k}\langle v_k, v_k\rangle \leq \hat{C}_2\lambda_n|||V|||^2. \tag{19}$$

Additionally, we can find $C_3 > 0$ such that

$$\sum_{k=1}^{q} |\tau_k^{-\frac{1}{2}} l_{n,k}(t_n)\langle U^n, v_k\rangle| \leq \sum_{k=1}^{q} |\tau_k^{-\frac{1}{2}} l_{n,k}(t_n) \parallel U^n \parallel \parallel v_k \parallel \leq C_3 \parallel U^n \parallel \; |||V|||. \tag{20}$$

Let λ_n be sufficiently small and satisfy $\mu - (\hat{C}_1 + \hat{C}_2)\lambda_n > 0$. By combining Equations (16–20) and setting $\delta = \frac{2C_3}{\mu - (\hat{C}_1 + \hat{C}_2)\lambda_n} \parallel U^n \parallel$, and in the case where $|||V||| = \delta$, we obtain

$$\mathrm{Re}\langle\langle G(V), V\rangle\rangle \geq \delta C_3 \parallel U^n \parallel > 0.$$

Under Lemma 4, there exists at least one fixed point $V \in (S_h^n)^q$ such that $G(V) = \mathbf{0}$, and existence is now proven.

Let $V = (v_1, v_2, \cdots, v_q)^T$ and $V^* = (v_1^*, v_2^*, \cdots, v_q^*)^T$ be the solutions of Equation (15). Setting $\Psi = v_k - v_k^*$ and summing from $k = 1$ to q yields that

$$\begin{aligned} 0 &\leq \langle\langle G(V - V^*), V - V^*\rangle\rangle \\ &= \sum_{k=1}^{q} \delta_{q,k}\langle v_q - v_q^*, v_k - v_k^*\rangle - \sum_{k,j=1}^{q} w_{n,j}\tau_j^{\frac{1}{2}}\tau_k^{-\frac{1}{2}} l'_{n,k}(t_{n,j})\langle v_j - v_j^*, v_k - v_k^*\rangle \\ &\quad + (\nu + i\eta)\sum_{k=1}^{q} w_{n,k}\langle B(v_k - v_k^*), v_k - v_k^*\rangle - \gamma \sum_{k=1}^{q} w_{n,k}\langle v_k - v_k^*, v_k - v_k^*\rangle. \\ &\quad + (\kappa + i\zeta)\sum_{k=1}^{q}\int_{\mathcal{I}_n} \tau_k^{-\frac{1}{2}} l_{n,k}(t)\langle f(\sum_{j=1}^{q}\tau_j^{\frac{1}{2}} l_{n,j}(v_j - v_j^*)), v_j - v_j^*\rangle \mathrm{d}t. \end{aligned} \tag{21}$$

Through similar analysis of the right of Equation (21), we can obtain that

$$(\mu - C\lambda_n)|||V - V^*|||^2 \leq 0.$$

If λ_n is small enough, then $V = V^*$ is obtained. The uniqueness is proven. □

In the following, C will denote any constant independent of h_n and λ_n. It may be different in different places. Next, we analyze the stability of the scheme in Equation (11).

Theorem 2. *For sufficiently small λ_n values, the discontinuous Galerkin finite element scheme (Equation (11)) is stable. Then, there exists $C > 0$ such that*

$$\max_{\mathcal{I}_n} \parallel U \parallel \leq C \parallel U^0 \parallel . \tag{22}$$

Proof. By using $\Psi = \tilde{U}^{n,k}$ in Equation (14) and summing it from 1 to q, the following result is derived:

$$\begin{aligned} &\sum_{k=1}^{q} \delta_{q,k}\langle \tilde{U}^{n,q}, \tilde{U}^{n,k}\rangle - \sum_{k,j=1}^{q} w_{n,j}\tau_j^{\frac{1}{2}}\tau_k^{-\frac{1}{2}} l'_{n,k}(t_{n,j})\langle \tilde{U}^{n,j}, \tilde{U}^{n,k}\rangle \\ &+ (\nu + i\eta)\sum_{k=1}^{q} w_{n,k}\langle B\tilde{U}^{n,k}, \tilde{U}^{n,k}\rangle + (\kappa + i\zeta)\int_{\mathcal{I}_n}\sum_{k=1}^{q}\tau_k^{-\frac{1}{2}} l_{n,k}(t)\langle f(U), \tilde{U}^{n,k}\rangle \mathrm{d}t \\ &= \sum_{k=1}^{q}\tau_k^{-\frac{1}{2}} l_{n,k}(t_n)\langle U^n, \tilde{U}^{n,k}\rangle + \gamma\sum_{k=1}^{q} w_{n,k}\langle \tilde{U}^{n,k}, \tilde{U}^{n,k}\rangle. \end{aligned} \tag{23}$$

Let $\tilde{U} = (\tilde{U}^{n,1}, \tilde{U}^{n,2}, \cdots, \tilde{U}^{n,q})^T$. It follows from Lemma 6 that

$$\mathrm{Re}\left\{\sum_{k=1}^{q} \delta_{q,k}\langle \tilde{U}^{n,q}, \tilde{U}^{n,k}\rangle - \sum_{k,j=1}^{q} w_{n,j}\tau_j^{\frac{1}{2}}\tau_k^{-\frac{1}{2}} l'_{n,k}(t_{n,j})\langle \tilde{U}^{n,j}, \tilde{U}^{n,k}\rangle\right\} \geq \mu|||\tilde{U}|||^2. \tag{24}$$

From Equation (9), we can obtain

$$\left|(\kappa + i\zeta)\int_{\mathcal{I}_n}\sum_{k=1}^{q}\tau_k^{-\frac{1}{2}}l_{n,k}(t)\langle f(U),\tilde{U}^{n,k}\rangle \mathrm{d}t\right| \leq C\lambda_n|||\tilde{U}|||^2. \tag{25}$$

This is similar to Equation (20) in that

$$\left|\sum_{k=1}^{q}\tau_k^{-\frac{1}{2}}l_{n,k}(t_n)(U^n,\tilde{U}^{n,k})\right| \leq C \parallel U^n \parallel \ |||\tilde{U}||| \leq C \parallel U^n \parallel^2 + \frac{\mu}{4}|||\tilde{U}|||^2. \tag{26}$$

Also, we have

$$\left|\gamma\sum_{k=1}^{q}w_{n,k}(\tilde{U}^{n,k},\tilde{U}^{n,k})\right| \leq C\lambda_n|||\tilde{U}|||^2. \tag{27}$$

By taking the real part of Equation (23) and using Equations (24–27), and for sufficiently small λ_n values, we can find

$$|||\tilde{U}|||^2 \leq C \parallel U^n \parallel^2 . \tag{28}$$

We define the space-time norm $|||\cdot|||_n$ as

$$|||U|||_n = \left(\int_{\mathcal{I}_n}\langle U,U\rangle \mathrm{d}t\right)^{\frac{1}{2}}.$$

From Equation (12), we have

$$\begin{aligned} |||U|||_n^2 &= \sum_{k,j=1}^{q}\int_{\mathcal{I}_n}\tau_k^{\frac{1}{2}}\tau_j^{\frac{1}{2}}l_{n,k}(t)l_{n,j}(t)\langle\tilde{U}^{n,k},\tilde{U}^{n,j}\rangle \mathrm{d}t \\ &= \lambda_n\sum_{k=1}^{q}w_k\tau_k \parallel \tilde{U}^{n,k} \parallel^2 \leq C\lambda_n|||\tilde{U}|||^2. \end{aligned} \tag{29}$$

Moreover, through Equation (28), we have

$$|||U|||_n^2 \leq C\lambda_n \parallel U^n \parallel^2 . \tag{30}$$

In the same way, by using $\Phi = U$ in Equation (11), we can find that

$$\parallel U^{n+1} \parallel^2 \ \leq \ \parallel U^n \parallel^2 + C|||U|||_n^2.$$

After substituting Equation (30) into the above formula, we obtain

$$\parallel U^{n+1} \parallel^2 \ \leq \ C(1+\lambda_n) \parallel U^n \parallel^2 . \tag{31}$$

By iterating Equation (31) from n to 1, there exists $C_0 > 0$ such that

$$\parallel U^{n+1} \parallel^2 \ \leq \ C_0 \parallel U^0 \parallel^2 .$$

Therefore, we have

$$|||U|||_n \leq C\lambda_n^{\frac{1}{2}} \parallel U^0 \parallel . \tag{32}$$

By applying the L^∞–L^2 space inverse inequality ([44], page 29)

$$\max_{\mathcal{I}_n}|g(t)| \leq C\lambda_n^{-\frac{1}{2}}\left(\int_{\mathcal{I}_n}|g(t)|^2\mathrm{d}t\right)^{\frac{1}{2}}, \quad \forall\, g(t) \in P_{q-1}(\mathcal{I}_n) \tag{33}$$

to Equation (32), the desired result (Equation (22)) is proven completely. □

5. Error Estimate

This section will analyze the error of the fully discrete space-time finite element scheme in Equation (11). To accomplish this, we define the elliptic projection operator $P_h^E : H_0^{\frac{\alpha}{2}}(\Omega) \to \mathcal{S}_h^n$ with the property that

$$\langle (-\triangle)^{\frac{\alpha}{2}}(u - P_h^E u), v\rangle = 0, \quad \forall v \in \mathcal{S}_h^n.$$

According to Nitsche's method ([44], page 29), we can obtain the following L^2 norm error estimate for the elliptic projection of u:

Lemma 7 ([41])**.** *If $u \in H_0^{\frac{\alpha}{2}}(\Omega) \cap H^\beta(\Omega)$, then there exists $C > 0$ such that*

$$\begin{aligned} &\|u - P_h^E u\| \le Ch_n^\beta \|u\|_\beta, \quad \frac{\alpha}{2} \le \beta \le r,\ \alpha \ne \frac{3}{2}; \\ &\|u - P_h^E u\| \le Ch_n^{\beta-\epsilon} \|u\|_\beta, \quad \alpha = \frac{3}{2},\ 0 < \epsilon < \frac{1}{2}, \end{aligned} \tag{34}$$

where $r - 1$ is the degree of the space $\mathcal{S}_h^n$ which is introduced in Section 3.

For the points $t_{n,k}$ of $\mathcal{I}_n = (t_n, t_{n+1}]$, we introduce the usual Lagrange interpolation operator $\hat{I}_n^{q-1} : C(\mathcal{I}_n) \to P_{q-1}(\mathcal{I}_n)$ such that

$$\hat{I}_n^{q-1} g(t_{n,k}) = g(t_{n,k}),\ \forall g(t) \in C(\mathcal{I}_n),\ k = 1, 2, \ldots, q,$$

where $q \ge 1$, $q - 1$ is the degree of the polynomial with respect to t in the finite element space $\mathcal{V}_{h\lambda}$. Let $W(x,t) = \hat{I}_n^{q-1} P_h^E u(x,t), (x,t) \in \Omega \times \mathcal{I}_n$, and $|||u|||_{n,\beta} = (\int_{\mathcal{I}_n} \| u \|_\beta^2 \, \mathrm{d}t)^{\frac{1}{2}}$ denote the norm for a space $L^2(\mathcal{I}_n; H^\beta(\Omega))$. According to Lemma 7 and [41], the following estimations hold:

$$\begin{aligned} &\max_{\mathcal{I}_n} \| u - W \| \le C\lambda_n^q \max_{\mathcal{I}_n} \| u^{(q)} \| + Ch_n^\beta \max_{\mathcal{I}_n} \| u \|_\beta, \quad \frac{\alpha}{2} \le \beta \le r,\ \alpha \ne \frac{3}{2}; \\ &|||u - W|||_n \le C\lambda_n^q |||u^{(q)}|||_n + Ch_n^\beta |||u|||_{n,\beta}, \quad \frac{\alpha}{2} \le \beta \le r,\ \alpha \ne \frac{3}{2}. \end{aligned} \tag{35}$$

Suppose that the exact solution of Equation (2) satisfies the following regularity conditions:

$$u, u_t \in L^\infty((0,T]; H^{r+1}(\Omega)), \quad u^{(q+1)} \in L^\infty((0,T]; L^2(\Omega)).$$

Now, we present our error analysis result for the scheme in Equation (11).

Theorem 3. *Let u and U be the solutions of Equations (2) and (11), respectively. Then, we have*

$$\begin{aligned} &\max_{t\in[0,T]} \|u(x,t) - U(x,t)\| \\ \le\ & C\left\{ \max_n \lambda_n^q \max_{\mathcal{I}_n} (\|u^{(q+1)}\| + \|u^{(q)}\| + \|(Bu)^{(q)}\|) + N_C(n) \max_n \|[\Gamma^n]\| \right\} \\ &+ C \max_n h_n^\beta \max_{\mathcal{I}_n} (||u||_\beta + ||u_t||_\beta), \quad \frac{\alpha}{2} \le \beta \le r,\ \alpha \ne \frac{3}{2}, \end{aligned} \tag{36}$$

$$
\begin{aligned}
&\max_{t\in[0,T]} \|u(x,t)-U(x,t)\| \\
\le\ & C\Big\{\max_n \lambda_n^q \max_{\mathcal{I}_n}(\|u^{(q+1)}\|+\|u^{(q)}\|+\|(Bu)^{(q)}\|)+N_C(n)\max_n\|[\Gamma^n]\|\Big\} \\
&+C\max_n h_n^{\beta-\epsilon}\max_{\mathcal{I}_n}(\|u\|_\beta+\|u_t\|_\beta), \quad \frac{\alpha}{2}\le\beta\le r,\ \alpha=\frac{3}{2},\ 0<\epsilon<\frac{1}{2},
\end{aligned}
\tag{37}
$$

where $N_C(n)$ *denotes the number of time slabs in which* $\mathcal{S}_h^{n-1}\neq\mathcal{S}_h^n$, $[\Gamma^n]=\Gamma_+^n-\Gamma_-^n$, $\Gamma_\pm^n=\lim_{t\to t_n^\pm}\Gamma(x,t)$, *and* $\Gamma(x,t)=(u-P_h^E u)(x,t)$, $n=1,2,\ldots,N-1$.

Proof. The error e_u between the finite element solution U and exact solution u is rewritten as

$$e_u=U-u=(U-W)+(W-u)=\Theta+\Lambda.$$

The estimation of Λ is described in Equation (35), and we only need to estimate Θ. We first have the basic error equation from Equation (11):

$$
\begin{aligned}
&\langle\Theta^{n+1},\Phi^{n+1}\rangle-\int_{\mathcal{I}_n}\langle\Theta,\Phi_t\rangle\mathrm{d}t+(\nu+i\eta)\int_{\mathcal{I}_n}\langle B\Theta,\Phi\rangle\mathrm{d}t-\gamma\int_{\mathcal{I}_n}\langle\Theta,\Phi\rangle\mathrm{d}t \\
&=(\kappa+i\zeta)\int_{\mathcal{I}_n}\langle f(W)-f(U),\Phi\rangle\mathrm{d}t+\langle\Theta^n,\Phi_+^n\rangle-(\kappa+i\zeta)\int_{\mathcal{I}_n}\langle f(W),\Phi\rangle\mathrm{d}t \\
&\quad+\int_{\mathcal{I}_n}\langle W,\Phi_t\rangle\mathrm{d}t+\int_{\mathcal{I}_n}[\gamma\langle\Theta,\Phi\rangle-(\nu+i\eta)\langle BW,\Phi\rangle]\mathrm{d}t \\
&\quad-\langle W^{n+1},\Phi^{n+1}\rangle+\langle W^n,\Phi_+^n\rangle, \quad \forall\Phi\in\mathcal{V}_h^n.
\end{aligned}
\tag{38}
$$

We put $\Phi=l_{n,k}(t)\Psi(x)$, where $\Psi(x)\in\mathcal{S}_h^n$ and $t\in(t_n,t_{n+1}]$. By taking the decompositions $W=\sum_{j=1}^q l_{n,j}(t)P_h^E u(x,t_{n,j})$ and $\Theta=\sum_{j=1}^q l_{n,j}(t)\Theta^{n,j}$, where $\Theta^{n,j}=U^{n,j}(x,t_{n,j})-P_h^E u(x,t_{n,j})$, Equation (38) is transferred to

$$
\begin{aligned}
&\delta_{q,k}\langle\Theta^{n,q},\Psi\rangle-\sum_{j=1}^q w_{n,j}l'_{n,k}(t_{n,j})\langle\Theta^{n,j},\Psi\rangle+(\nu+i\eta)w_{n,k}\langle B\Theta^{n,k},\Psi\rangle \\
&=\gamma w_{n,k}\langle\Theta^{n,k},\Psi\rangle+(\kappa+i\zeta)\int_{\mathcal{I}_n}\langle f(W)-f(U),l_{n,k}\Psi\rangle\mathrm{d}t+l_{n,k}(t_n)\langle\Theta^n,\Psi\rangle \\
&\quad-(\kappa+i\zeta)\int_{\mathcal{I}_n}\langle f(W),l_{n,k}\Psi\rangle\mathrm{d}t+\sum_{j=1}^q w_{n,j}l'_{n,k}(t_{n,j})\langle P_h^E u(x,t_{n,j}),\Psi\rangle \\
&\quad+\gamma w_{n,k}\langle P_h^E u(x,t_{n,j}),\Psi\rangle-(\nu+i\eta)w_{n,k}\langle BP_h^E u(x,t_{n,j}),\Psi\rangle \\
&\quad-\delta_{q,k}\langle P_h^E u(x,t_{n,q}),\Psi\rangle+l_{n,k}(t_n)\langle W^n,\Psi\rangle, \quad \forall\Psi\in\mathcal{S}_h^n,\ k=1,2,\ldots,q.
\end{aligned}
\tag{39}
$$

Since the exact solution u satisfies Equation (11), and by the definition of P_h^E, we have the following error equation:

$$
\begin{aligned}
&\delta_{q,k}\langle\Theta^{n,q},\Psi\rangle-\sum_{j=1}^q w_{n,j}l'_{n,k}(t_{n,j})\langle\Theta^{n,j},\Psi\rangle+(\nu+i\eta)w_{n,k}\langle B\Theta^{n,k},\Psi\rangle \\
&=\gamma w_{n,k}\langle\Theta^{n,k},\Psi\rangle+(\kappa+i\zeta)\int_{\mathcal{I}_n}\langle f(W)-f(U),l_{n,k}\Psi\rangle\mathrm{d}t+l_{n,k}(t_n)\langle\Theta^n,\Psi\rangle \\
&\quad+(\kappa+i\zeta)\int_{\mathcal{I}_n}\langle f(u)-f(W),l_{n,k}\Psi\rangle\mathrm{d}t+\langle\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4,\Psi\rangle \\
&\quad+\langle[\Gamma^n],l_{n,k}(t_n)\Psi\rangle, \quad k=1,2,\ldots,q,\ \forall\ \Psi\in\mathcal{S}_h^n,
\end{aligned}
\tag{40}
$$

where

$$\begin{aligned}
\Sigma_1 &= \delta_{q,k}\Gamma^{n,q} - \sum_{j=1}^{q} w_{n,j} l'_{n,k}\Gamma^{n,j} - l_{n,k}(t_n)\Gamma^n_+,\\
\Sigma_2 &= \sum_{j=1}^{q} w_{n,j} l'_{n,k}(t_{n,j}) u^{n,j} - \int_{\mathcal{I}_n} l'_{n,k}(t) u \mathrm{d}t,\\
\Sigma_3 &= -(\nu + i\eta) w_{n,k} B u^{n,k} + (\nu + i\eta)\int_{\mathcal{I}_n} l_{n,k} B u \mathrm{d}t,\\
\Sigma_4 &= -\gamma w_{n,k} u^{n,k} + \gamma \int_{\mathcal{I}_n} l_{n,k} u \mathrm{d}t,\\
[\Gamma^n] &= \Gamma^n_+ - \Gamma^n_- = u^n_+ - P^E_h u^n_+ - (u^n_- - P^E_h u^n_-),
\end{aligned}$$

where $u^n_\pm = \lim\limits_{t\to t_n^\pm} u(x,t)$ and $u^n_+ = u^n_-$.

If $S_h^{n-1} = S_h^n$, then $\|[\Gamma^n]\| = 0$; otherwise, $\|[\Gamma^n]\| \neq 0$. However, according to Lemma 7 and [41], we have

$$\|[\Gamma^n]\| \le \|u^n_+ - P^E_h u^n_+\| + \|u^n_- - P^E_h u^n_-\| \le C h_n^\beta \|u\|_\beta + C h_{n-1}^\beta \|u\|_\beta, \quad \frac{\alpha}{2} \le \beta \le r, \ \alpha \neq \frac{3}{2}.$$

Here, we set $h_0 = h_1$ for simplicity's sake.

Next, we consider the bounds of the norms of $\Sigma_j (j = 1,2,3,4)$. Since

$$\delta_{q,k} - \sum_{j=1}^{q} w_{n,j} l'_{n,k}(t_{n,j}) - l_{n,k}(t_n) = \delta_{q,k} - \int_{\mathcal{I}_n} l'_{n,k}(t)\mathrm{d}t - l_{n,k}(t_n) = 0,$$

then there exist constants $C_{j,k}$ and C such that

$$\begin{aligned}
\|\Sigma_1\| &= \|\sum_{j=1}^{q} C_{j,k} \int_{t_{n,j-1}}^{t_{n,j}} \Gamma_t(\xi)\mathrm{d}\xi\|\\
&\le C\int_{\mathcal{I}_n} \|u_t - P^E_h u_t\|\mathrm{d}\xi \le C\lambda_n^{\frac{1}{2}} h_n^\beta |||u_t|||_{n,\beta}, \quad \frac{\alpha}{2} \le \beta \le r, \ \alpha \neq \frac{3}{2}.
\end{aligned} \tag{41}$$

Let $\hat{\mathcal{I}}_n^q : C(\overline{\mathcal{I}}_n) \to P_q(\overline{\mathcal{I}}_n)\,(\overline{\mathcal{I}}_n = [t_n, t_{n+1}])$ be the Lagrange interpolation operator on the points $t_{n,k}$ such that $\hat{I}_n^q g(t_{n,k}) = g(t_{n,k})$, $k = 1,2,\ldots,q$, and $\hat{I}_n^q g(t_n) = g(t_n^+) := \lim\limits_{\epsilon\to 0^+} g(t_n + \epsilon)$, where $g(t) \in C(\mathcal{I}_n)$. For every $x \in \Omega$, $l'_{n,k}(t)\hat{I}_n^q u$ is a polynomial with the degree $2q-2$. Moreover, we have

$$\begin{aligned}
\|\Sigma_2\| &= \|\sum_{j=1}^{q} w_{n,j} l'_{n,k}(t_{n,j}) u^{n,j} - \int_{\mathcal{I}_n} l'_{n,k}(t_{n,j}) u \mathrm{d}t\|\\
&= \|\int_{\mathcal{I}_n} l'_{n,k}(t)(\hat{I}_n^q - I)u\mathrm{d}t\| \le C|||(\hat{I}_n^q - I)u|||_n \left(\int_{\mathcal{I}_n} |l'_{n,k}(t)|^2 \mathrm{d}t\right)^{\frac{1}{2}}\\
&\le C\left(\lambda_n^{-1}\int_0^1 |l'(\tau)|^2\mathrm{d}\tau\right)^{\frac{1}{2}} \cdot \lambda_n^{q+1}|||u^{(q+1)}|||_n \le C\lambda_n^{q+\frac{1}{2}}|||u^{(q+1)}|||_n.
\end{aligned} \tag{42}$$

In the same way, we have

$$\|\Sigma_3\| \le C\lambda_n^{q+\frac{1}{2}}|||(Bu)^{(q)}|||_n, \quad \|\Sigma_4\| \le C\lambda_n^{q+\frac{1}{2}}|||u^{(q)}|||_n. \tag{43}$$

By letting $\Psi = \Theta^{n,k}$ in the error equation (Equation (40)) and summing it from 1 to q, we obtain

$$\begin{aligned} &\frac{1}{2}(\|\Theta^{n+1}\|^2 + \|\Theta_+^n\|^2) + (\nu + i\eta)\int_{\mathcal{I}_n}\langle B\Theta, \Theta\rangle \mathrm{d}t - \gamma\int_{\mathcal{I}_n}\langle \Theta, \Theta\rangle \mathrm{d}t \\ = \;& (\kappa + i\zeta)\int_{\mathcal{I}_n}\langle f(W) - f(U) + f(u) - f(W), \Theta\rangle \mathrm{d}t + \langle \Theta^n, \Theta_+^n\rangle \\ &+ \sum_{k=1}^{q}\langle \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4, \Theta^{n,k}\rangle + \langle [\Gamma^n], \Theta_+^n\rangle. \end{aligned} \tag{44}$$

According to Equation (9) and the Cauchy–Schwarz and Young inequalities, we have

$$\begin{aligned} &\left|(\kappa + i\zeta)\int_{\mathcal{I}_n}\langle f(W) - f(U) + f(u) - f(W), \Theta\rangle \mathrm{d}t\right| \\ \leq\; & |(\kappa + i\zeta)|\int_{\mathcal{I}_n}|\langle f(W) - f(U), \Theta\rangle| \mathrm{d}t + |(\kappa + i\zeta)|\int_{\mathcal{I}_n}|\langle f(u) - f(W), \Theta\rangle| \mathrm{d}t \\ \leq\; & C(|||\Theta|||_n^2 + |||W - u|||_n^2). \end{aligned} \tag{45}$$

It follows from Equations (41–43) that

$$\begin{aligned} &\left|\sum_{k=1}^{q}\langle \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4, \Theta^{n,k}\rangle\right| \\ \leq\; & \sum_{k=1}^{q}(\|\Sigma_1\| + \|\Sigma_2\| + \|\Sigma_3\| + \|\Sigma_4\|)\|\Theta^{n,k}\| \\ \leq\; & C\left[h_n^{\beta}|||u_t|||_{n,\beta} + \lambda_n^q|||u^{(q+1)}|||_n + |||(Bu)^{(q)}|||_n + |||u^{(q)}|||_n\right] + |||\Theta|||_n^2, \\ & \qquad\qquad \frac{\alpha}{2} \leq \beta \leq r, \quad \alpha \neq \frac{3}{2}. \end{aligned} \tag{46}$$

In addition, we have

$$\begin{aligned} |\langle \Theta^n, \Theta_+^n\rangle| &\leq \|\Theta^n\|^2 + \frac{\|\Theta_+^n\|^2}{4}, \\ |\langle [\Gamma^n], \Theta_+^n\rangle| &\leq \|[\Gamma^n]\|^2 + \frac{\|\Theta_+^n\|^2}{4}. \end{aligned} \tag{47}$$

By taking the real part of Equation (44) and combining Equations (45–47), we obtain

$$\|\Theta^{n+1}\|^2 \leq C\Big\{|||\Theta|||_n^2 + \|\Theta^n\|^2 + \|[\Gamma^n]\|^2 + \mathfrak{L}^2(h_n, \lambda_n)\Big\}, \tag{48}$$

where

$$\begin{aligned} \mathfrak{L}(h_n, \lambda_n) = \;& \lambda_n^q\Big(|||u^{(q+1)}|||_n + |||u^{(q)}|||_n + |||(Bu)^{(q)}|||_n\Big) \\ &+ h_n^{\beta}(|||u_t|||_{n,\beta} + |||u|||_{n,\beta}), \quad \frac{\alpha}{2} \leq \beta \leq r, \;\; \alpha \neq \frac{3}{2}. \end{aligned}$$

In order to obtain the estimation of $|||\Theta|||_n$, we take $\Theta = \sum_{j=1}^{q} \tau_j^{\frac{1}{2}} l_{n,j} \hat{\Theta}^{n,j}$, where $\hat{\Theta}^{n,j} = \tau_j^{-\frac{1}{2}} \Theta^{n,j}$ and $\Psi = \hat{\Theta}^{n,k}$ in Equation (40). After multiplying both sides of the equation by $\tau_k^{-\frac{1}{2}}$ and then summing k from 1 to q, we have

$$\begin{aligned} &\sum_{k=1}^{q} \delta_{q,k} \langle \hat{\Theta}^{n,q}, \hat{\Theta}^{n,k} \rangle - \sum_{k,j=1}^{q} w_{n,j} l'_{n,k}(t_{n,j}) \tau_j^{\frac{1}{2}} \tau_k^{-\frac{1}{2}} \langle \hat{\Theta}^{n,j}, \hat{\Theta}^{n,k} \rangle \\ &+ (\nu + i\eta) \sum_{k=1}^{q} w_{n,k} \langle B\hat{\Theta}^{n,k}, \hat{\Theta}^{n,k} \rangle - \gamma \sum_{k=1}^{q} w_{n,k} \langle \hat{\Theta}^{n,k}, \hat{\Theta}^{n,k} \rangle \\ = \; &(\kappa + i\zeta) \sum_{k=1}^{q} \int_{\mathcal{I}_n} \tau_k^{-\frac{1}{2}} \langle f(W) - f(U) + f(u) - f(W), l_{n,k} \hat{\Theta}^{n,k} \rangle \mathrm{d}t \\ &+ \sum_{k=1}^{q} \left\{ \langle \Theta^n, l_{n,k}(t^n) \hat{\Theta}^{n,k} \rangle + \langle \sum_{j=1}^{4} \Sigma_j, \hat{\Theta}^{n,k} \rangle + \langle [\Gamma^n], l_{n,k}(t_n) \hat{\Theta}^{n,k} \rangle \right\}. \end{aligned} \tag{49}$$

Let $\hat{\Theta}^n = (\hat{\Theta}^{n,1}, \hat{\Theta}^{n,2}, \cdots, \hat{\Theta}^{n,q})^T$. Similarly, it holds that

$$\mu |||\hat{\Theta}^n||| \leq C(\|\Theta^n\| + \|[\Gamma^n]\|) + C\lambda_n^{\frac{1}{2}}(|||\Theta|||_n + \mathfrak{L}(h_n, \lambda_n)) \tag{50}$$

From Equation (29), and for sufficiently small λ_n values, we obtain

$$|||\Theta|||_n \leq C\lambda_n^{\frac{1}{2}}(\|\Theta^n\| + \|[\Gamma^n]\|) + C\lambda_n \mathfrak{L}(h_n, \lambda_n). \tag{51}$$

By observing Equations (48) and (51), the following inequality is obtained:

$$\|\Theta^{n+1}\|^2 \leq C\{(1 + \lambda_n)(\|\Theta^n\|^2 + \|[\Gamma^n]\|^2) + (1 + \lambda_n^2)\mathfrak{L}^2(h_n, \lambda_n)\}. \tag{52}$$

Iterating Equation (52) n times implies

$$\|\Theta^{n+1}\|^2 \leq C\|\Theta^0\|^2 + C\sum_{j=0}^{n}(\|[\Gamma^j]\|^2 + \mathfrak{L}^2(h_j, \lambda_j)). \tag{53}$$

By substituting Equation (53) into Equation (51), we find

$$|||\Theta|||_n \leq C\lambda_n^{\frac{1}{2}} \left(\|\Theta^0\|^2 + \sum_{j=0}^{n} \|[\Gamma^j]\|^2 + \sum_{j=0}^{n} \mathfrak{L}^2(h_j, \lambda_j) \right)^{\frac{1}{2}}. \tag{54}$$

By using the inverse inequality in Equation (33), we obtain

$$\begin{aligned} \max_{\mathcal{I}_n} \|\Theta\| \leq \; &C h_0^{\beta} \|u^0\|_{\beta} + C \left(\sum_{j=0}^{n} \|[\Gamma^j]\|^2 \right)^{\frac{1}{2}} \\ &+ C \left(\sum_{j=0}^{n} \left(\lambda_j^{2q} (|||u^{(q+1)}|||_j^2 + |||u^{(q)}|||_j^2 + |||(Bu)^{(q)}|||_j^2) + h_j^{2\beta} (|||u_t|||_{j,\beta}^2 + |||u|||_{j,\beta}^2) \right) \right)^{\frac{1}{2}}. \end{aligned} \tag{55}$$

From Equation (35) for the estimation of $\Lambda = W - u$, the desired result (Equation (36)) follows. Equation (37) can be found through a similar process. □

6. Numerical Tests

In this section, some numerical results are provided to evaluate the effectiveness of the fully discrete space-time finite element scheme in Equation (11) for the fractional Ginzburg–Landau equation.

First, we consider the numerical accuracy of the proposed scheme for the integer-order Ginzburg–Landau equation ($\alpha = 2$). In this case, the exact solution of the equation exists, and it is precisely given by [45]

$$u(x,t) = a(x)\exp(i(d\ln a(x)) - i\omega t),$$

where

$$d = \frac{\sqrt{1+4\nu^2}-1}{2\nu}, \quad \omega = -\frac{d(1+4\nu^2)}{2\nu}, \quad F = \sqrt{\frac{d\sqrt{1+4\nu^2}}{-2\kappa}}, \quad a(x) = F\mathrm{sech}(x).$$

We take

$$\nu = 0.1, \quad \eta = 1, \quad \kappa = -\frac{\nu(3\sqrt{1+4\nu^2}-1)}{2(2+9\nu^2)}, \quad \zeta = 1, \quad \gamma = 0, \tag{56}$$

and $\Omega = [-16, 16]$, $T = 1$ here. We can see that the modulus of the initial value

$$u(x,0) = 1.01233\,\mathrm{sech}(x)\ \exp(0.0990195i\ln a(x)) \tag{57}$$

asymptotically equals zero as $|x| \geq 10$. Thus, $u(x,t)$ can be negligible outside Ω, and we can set $u(x,t) = 0$ for $x \in \mathbb{R} \setminus \Omega$ and $t \in (0,1]$. The finite element space is composed of linear piecewise polynomials in both the temporal and spatial directions. We make the space step size $h_n = 0.01$ sufficiently small, and then the error is defined by

$$E(\lambda_n) = \max_{1 \leq n \leq \frac{T}{\lambda_n}} \|u(x,t_n) - U(x,t_n)\|.$$

For a fixed h_n value, let λ_n^1 and λ_n^2 stand for two different time steps. Then, we have the convergence rate in time

$$R_t \approx \log_{\frac{\lambda_n^1}{\lambda_n^2}} \frac{E(\lambda_n^1)}{E(\lambda_n^2)}.$$

To find the accuracy for the space, we fix the time step size to $\lambda_n = 0.001$. Then, the error is defined by

$$E(h_n) = \max_{1 \leq n \leq \frac{T}{\lambda_n}} \|u(x,t_n) - U(x,t_n)\|.$$

If h_n^1 and h_n^2 are the different space steps, then the convergence rate in space is

$$R_s \approx \log_{\frac{h_n^1}{h_n^2}} \frac{E(h_n^1)}{E(h_n^2)}.$$

The results are all listed in Table 1. This table shows that the scheme has almost second-order accuracy in the temporal and spatial directions. The presented numerical results support the validity of the fully discrete space-time finite element scheme (Equation (11)).

Table 1. The errors and convergent orders in $L^\infty(L^2(\Omega))$ norm when $\alpha = 2$.

$h_n = 0.01$			$\lambda_n = 0.001$		
λ_n	$E(\lambda_n)$	R_t	h_n	$E(h_n)$	R_s
0.5000	3.4264×10^{-1}	—	0.5000	1.9121×10^{-1}	—
0.2500	1.0346×10^{-1}	1.7276	0.2500	5.9959×10^{-2}	1.6731
0.1250	2.8578×10^{-2}	1.8561	0.1250	1.5311×10^{-2}	1.9694
0.0625	8.1853×10^{-3}	1.8038	0.0625	3.6402×10^{-3}	2.0724

When $1 < \alpha < 2$, it is difficult to obtain the exact solution explicitly, so we use the numerical solution $\widetilde{\widetilde{U}}$ obtained with a smaller step size $h_n = 0.01$ and $\lambda_n = 0.001$ instead of the exact solution for different values of α to check the accuracy of the scheme. We consider the fractional Ginzburg–Landau equation with the following initial condition:

$$u(x,0) = \frac{\exp(-2x^2)}{\exp(-x) + \exp(x)}. \tag{58}$$

It is evident that $u(x,0)$ decays to zero with the spatial variable x away from the origin. By setting $\nu = 1/8$, $\eta = 1$, $\kappa = 1$, $\zeta = 2$, and $\gamma = 1$, the error in both the temporal and spatial directions is as follows:

$$E(\lambda_n, h_n) = \max_{1 \le n \le \frac{T}{\lambda_n}} \|\widetilde{\widetilde{U}}(x,t_n) - U(x,t_n)\|.$$

Here, we always use $\lambda_n = 0.1h_n$. By denoting $p = \frac{h_n^1}{h_n^2} = \frac{\lambda_n^1}{\lambda_n^2}$, then the convergence rate is

$$R_{ts} \approx \log_p \frac{E(\lambda_n^1, h_n^1)}{E(\lambda_n^2, h_n^2)}.$$

The numerical results are listed in Table 2. It shows that the scheme has second-order accuracy in space and time, further indicating our proposed scheme's effectiveness and reliability.

Table 2. The errors and convergent orders in the $L^\infty(L^2(\Omega))$ norm with $\lambda_n = 0.1h_n$.

$\alpha = 1.3$			$\alpha = 1.6$		$\alpha = 1.9$	
λ_n	$E(\lambda_n, h_n)$	R_{ts}	$E(\lambda_n, h_n)$	R_{ts}	$E(\lambda_n, h_n)$	R_{ts}
0.10000	3.6931×10^{-1}	—	6.4859×10^{-1}	—	5.4731×10^{-1}	—
0.05000	1.5184×10^{-1}	1.2823	2.6499×10^{-1}	1.2914	1.8383×10^{-1}	1.5740
0.02500	5.3618×10^{-2}	1.5018	8.9322×10^{-2}	1.5689	5.9724×10^{-2}	1.6221
0.01250	1.5466×10^{-2}	1.7936	2.7021×10^{-2}	1.7249	1.5153×10^{-2}	1.9787
0.00625	3.8906×10^{-3}	1.9910	7.1296×10^{-3}	1.9221	4.0874×10^{-3}	1.8903

Secondly, there is the dissipative mechanism of the fractional Laplacian. We take the initial condition to be $u(x,0)$ as in Equation (58) and set $\Omega = [-15, 15]$, $\nu = 0.1$, $\eta = \kappa = \zeta = \gamma = 1$. The profiles of the numerical solutions at $t = 1$ with different values for α are given in Figure 1. We can see that the wave shape changed with the values of α. The changing trend is consistent with that in [7].

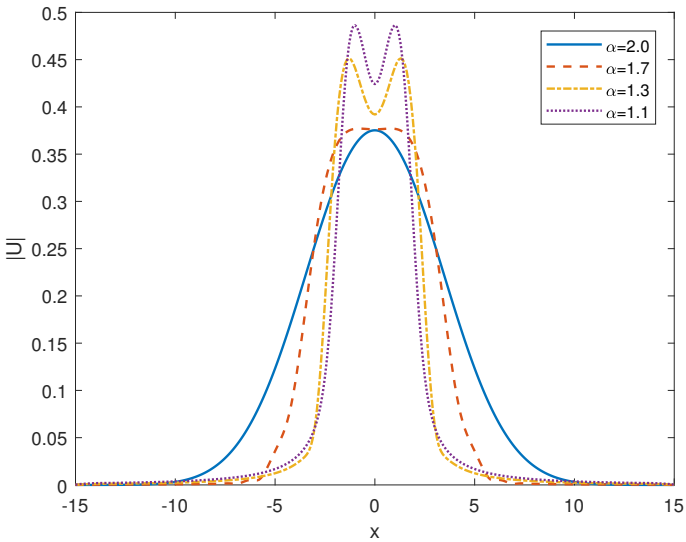

Figure 1. The profiles of $|U|$ at $t = 1$ for different fractional orders α.

Thirdly, we focus on the influence of the parameter γ for wave shape evolution in fractional cases. We choose $\alpha = 1.5, \Omega = [-10, 10], h_n = 0.2$, and $\lambda_n = 0.025$ and then set $\gamma = -3, -1, 0, 1, 3$ to compute the numerical solution to $T = 1$. In this case, we choose $\nu = \eta = \kappa = \zeta = 1$. Figures 2–4 present the numerical solutions. The figures show that the parameter γ affects the wave shape of the solutions dramatically. If $\gamma \leq 0$, then the solution $|U|$ decays and it increases when $\gamma > 0$. These results are promising and in agreement with the results in [7,17,46].

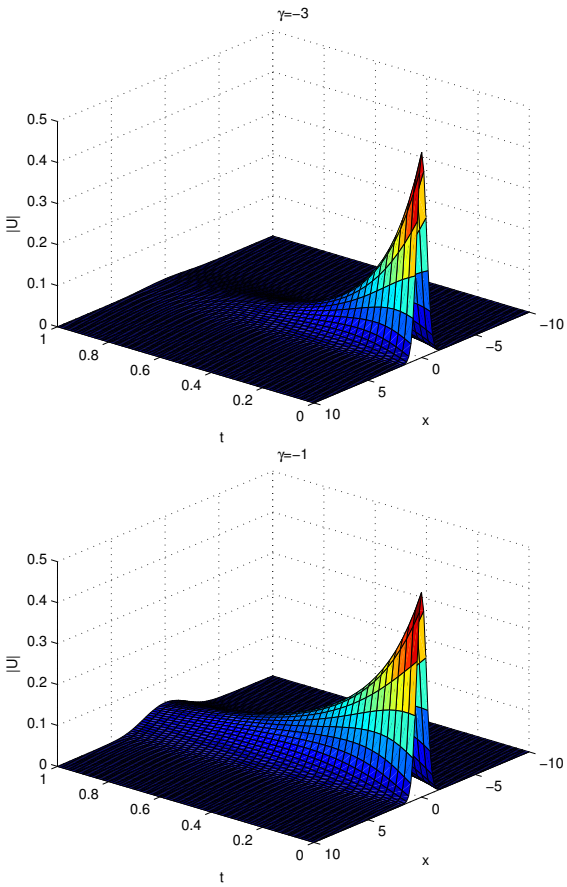

Figure 2. The numerical solutions of $|U|$ with $\gamma = -3$ and $\gamma = -1$.

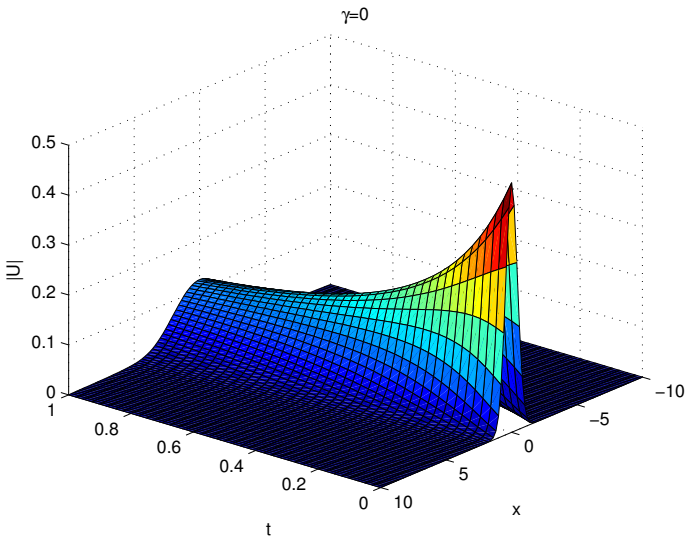

Figure 3. The numerical solutions of $|U|$ with $\gamma = 0$.

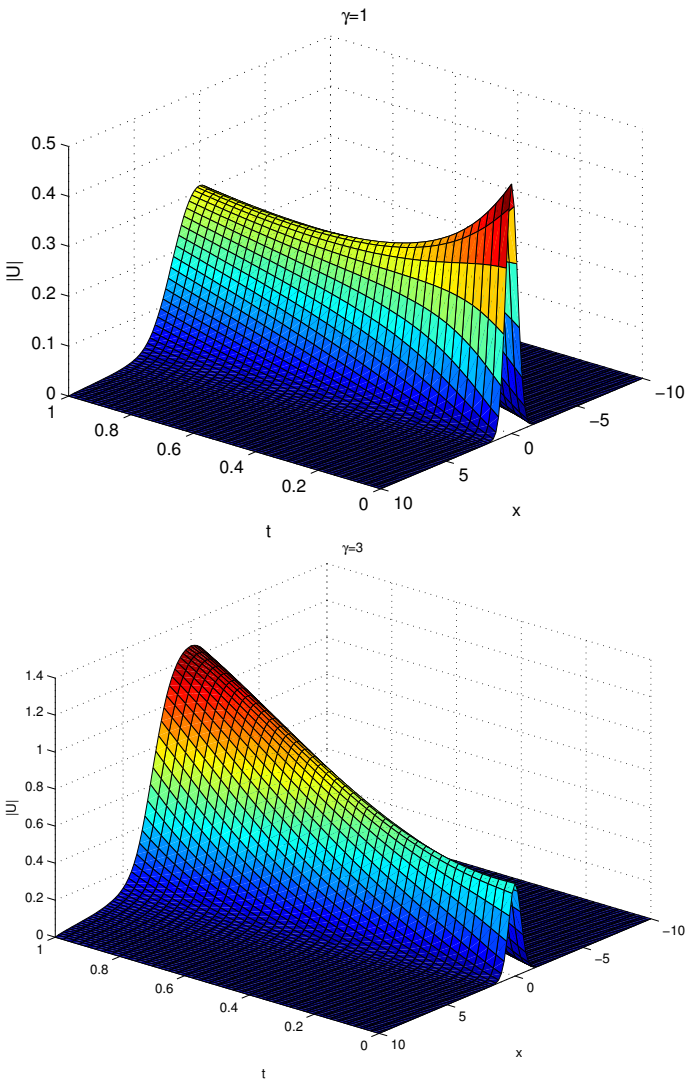

Figure 4. The numerical solution of $|U|$ with $\gamma = 1$ and $\gamma = 3$.

Fourth, we pay attention to the inviscid limit behavior of the discrete solution. According to [47], we know that the solution of the equation converges to the solution of the

fractional Schrödinger equation (i.e., $\nu = \kappa = 0$). Here we choose $\eta = 1$, $\zeta = -2$, $\gamma = 0$ and then make ν and κ smaller and smaller to observe the asymptotics for different values of α. From Figure 5, we see that the results are in accordance with the results in [7,47].

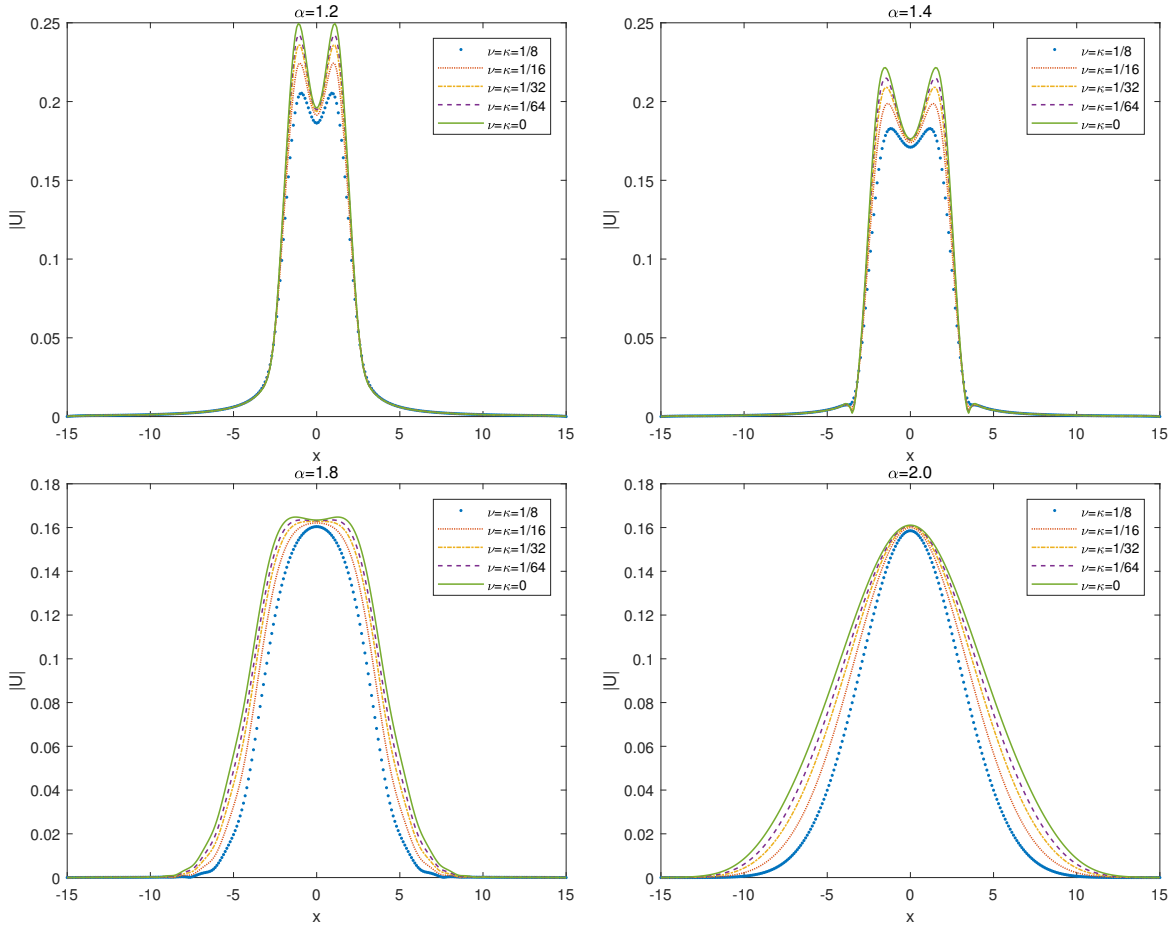

Figure 5. The profiles of $|U|$ at $t = 1$ with diminishing ν and κ for different α.

Finally, we construct two experiments to show that the fully discrete space-time finite element solutions may be discontinuous at different time nodes. We use the same parameters as those in Equation (56) ($\alpha = 2$ and $\nu = \eta = \kappa = \zeta = 1$ for $\alpha = 1.6$). By setting $\lambda_n = 0.05$ and $h_n = 0.1$ for the two cases, the modules of the jump $[U]^n$ of the discrete solution at time nodes t_n are shown in Figure 6. From this, we can see that the discrete solutions are discontinuous at some time nodes.

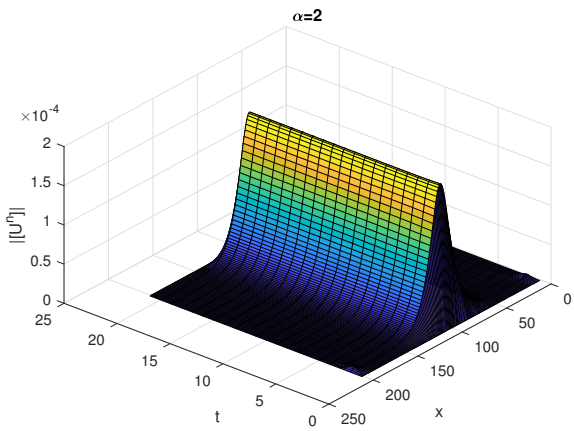

Figure 6. *Cont.*

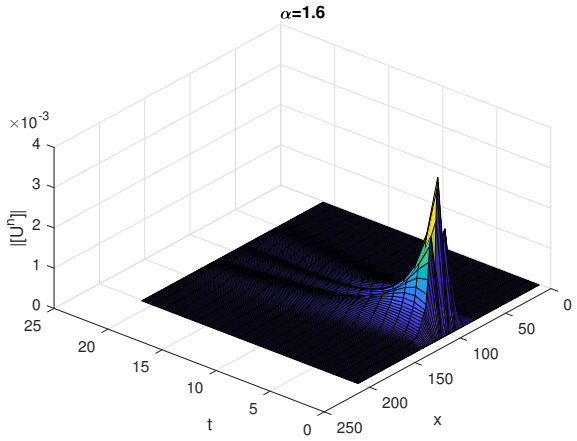

Figure 6. The figures of $|[U^n]|$ when $\alpha = 2$ and $\alpha = 1.6$.

7. Conclusions

This paper extends the space-time finite element method to the fractional Ginzurg–Landau equation. The presented method is a fully discrete Galerkin finite element method that solves the equation with the unified finite-element framework in both the temporal and spatial directions. This is more flexible for dealing with discontinuous problems because our finite element scheme permits discontinuity in time. The well-posedness and error estimate of the discrete solution are proven under weak restriction on the space-time mesh, which only demands that the time step λ_n be small enough. The numerical examples illustrate the effectiveness of the space-time finite element method for the equation.

When the equation admits solutions that form singularities in finite time, appropriate adaptive methods seem to be a good choice. One reason for considering the discontinuous space-time finite element method is the need for flexible schemes suitable for computation on unstructured meshes. In this work, we have illustrated the availability of the discontinuous Galerkin method to the fractional Ginzurg–Landau equation. This provides a guarantee for our forthcoming work to design adaptive algorithms.

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