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Geraghty Type Contractions in Relational Metric Space with Applications to Fractional Differential Equations

Ahmed Alamer , Nidal H. E. Eljaneid, Musaad S. Aldhabani *, Nifeen H. Altaweel and Faizan Ahmad Khan *

Department of Mathematics, University of Tabuk, Tabuk 71491, Saudi Arabia; aalamer@ut.edu.sa (A.A.); neljaneid@ut.edu.sa (N.H.E.E.); naltawil@ut.edu.sa (N.H.A.)
* Correspondence: maldhabani@ut.edu.sa (M.S.A.); fkhan@ut.edu.sa (F.A.K.)

Abstract: The present manuscript is devoted to investigating some existence and uniqueness results on fixed points by employing generalized contractions in the context of metric space endued with a weak class of transitive relation. Our results improve, modify, enrich and unify several existing fixed point theorems, The results proved in this study are utilized to find a unique solution of certain fractional boundary value problems.

Keywords: Λ-continuity; σ-self-closed relations; binary relations

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1. Introduction

In near past, FDE (abbreviation of 'fractional differential equations') are discussed owing to the impassable development and applicability of the area of fractional calculus. For deep discussions regarding fractional calculus and FDE, we refer to the classical books of Podlubny [1], Kilbas et al. [2] and Daftardar-Gejji [3]. Numerical solutions of FDE were discussed by Atanackovic and Stankovic [4] and more recently by Talib and Bohner [5]. Cernea [6] investigated existence theorems for certain Hadamard-type fractional integro-differential inclusion. Various authors have studied the solvability of certain classes of nonlinear FDE, e.g., Xiao [7], Zhang et al. [8], Cevikel and Aksoy [9], Laoubi et al. [10], Telli et al. [11], Dincel et al. [12], Area and Nieto [13], Jassim and Hussein [14], among others. In order to study the BVP (abbreviation of 'boundary value problem') for FDE, we refer the works of Jia et al. [15], Su et al. [16], Bouteraa and Benaicha [17], Zhang et al. [18], Luca [19], Shah et al. [20] and references therein. Very recently, Aljethi and Kiliçman [21] discussed the analytic properties of FDE and their applications to realistic data.

Metric fixed point theory occupies an important role in nonlinear functional analysis. The strength of metric fixed point theory lies in its wide range of applications to various domains such as optimization theory [22,23], variational inequalities [24,25] and FDE [26,27]. Recently, Ishtiaq et al. [28] obtained some fixed point theorems in intuitionistic fuzzy N_b metric space and utilized the same to solve a class of nonlinear fractional differential equations. The Banach contraction principle (BCP), being a stalwart and aesthetic tool of the domain of metric fixed point theory, was established by Banach in 1922. Various generalizations of BCP were established by enlarging the class of contraction mapping. One such generalization is due to Geraghty [29]. Let $\mathcal{B}$ denotes the class of functions $\beta : [0, \infty) \to [0, \infty)$ verifying

$$\beta(t_n) \to 1 \Rightarrow t_n \to 0.$$

Following Geraghty [29], a self-map $\mathcal{S}$ on a complete metric space $(\mathcal{M}, \sigma)$ verifying, for some $\beta \in \mathcal{B}$ and for all $r, s \in \mathcal{M}$,

$$\sigma(\mathcal{S}r, \mathcal{S}s) \leq \beta(\sigma(r,s))\sigma(r,s),$$

possesses a unique fixed point.

In 2015, Alam and Imdad [30] obtained an intense and flexible version of the BCP in a metric space endued with a binary relation. Following this novel work, various researchers have established a multiple of fixed point results in the framework of relational metric space under different types of contractivity conditions, viz., Boyd–Wong type contractions [31,32], Matkowski contractions [33,34], Meir–Keeler contractions [35], weak contractions [36,37], F-contractions [38], θ-contractions [39], almost contractions [40], rational type contractions [41], (ψ, ϕ)-contractions [42], (ψ, ϕ, θ)-contractions [43], etc. Mapping which involves such results verifies a weaker contraction condition relative to the usual contraction, as it must be held for comparative elements only. This restrictive nature enables such types of results to solve many complicated real world problems occurring in fractal spaces and fractional differential equations which employ specific auxiliary conditions, e.g., [44,45]. Very recently, Almarri et al. [46] established the relation–theoretic analogue of Geraghty's fixed point theorem [29], which also remains an improvement of the results of Harandi and Emami [47].

In short, the intention of this manuscript is two-fold:

1. To improve the results of Almarri et al. [46] by satisfying more generalized contraction conditions and to prove the existence and uniqueness results on fixed points in the framework of metric space endued with a locally $\mathcal{S}$-transitive binary relation.
2. By means of our fixed point results, to discuss the existence of a unique solution of the following BVP for an FDE in dependent variable u and independent variable x of the form:

$$-\mathcal{D}^{\eta}u(x) = h\Big(x, u(x), \mathcal{D}^{\gamma_1}u(x), \mathcal{D}^{\gamma_2}u(x), \ldots, \mathcal{D}^{\gamma_{n-1}}u(x)\Big)$$

$$\begin{cases} \mathcal{D}^{\gamma_i}u(0) = 0, \quad 1 \leq i \leq n-1, \\ \mathcal{D}^{\gamma_{n-1}+1}u(0) = 0, \\ \mathcal{D}^{\gamma_{n-1}}u(1) = \sum\limits_{j=1}^{m-2} c_j \mathcal{D}^{\gamma_{n-1}}u(\xi_j) \end{cases} \tag{1}$$

where

- $n \in \mathbb{N},\ n \geq 3$ and $n-1 < \eta \leq n$,
- $0 < \gamma_1 < \gamma_2 < \cdots < \gamma_{n-2} < \gamma_{n-1}$ and $n-3 < \gamma_{n-1} < \eta - 2$,
- $\mathcal{D}^{\eta}$ is standard Riemann–Liouville derivative,
- $h : [0,1] \times \mathbb{R}^n \to [0, \infty)$ is a continuous function,
- $c_j \in \mathbb{R}$ and $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-1} < 1$ verifying $0 < \sum\limits_{j=1}^{m-2} c_j \xi_j^{\eta-\gamma_{n-1}-1} < 1$.

2. Preliminaries

Throughout the text, $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{R}$ will denote, respectively, the set of natural numbers, that of whole numbers and that of real numbers. A binary relation or simply a relation Λ on a set $\mathcal{M}$ is a subset of $\mathcal{M}^2$. In the following definitions, $\mathcal{M}$ remains a set, σ is a metric on $\mathcal{M}$, Λ remains a relation on $\mathcal{M}$ and $\mathcal{S} : \mathcal{M} \to \mathcal{M}$ is a map.

Definition 1 ([30])**.** *The points $r, s \in \mathcal{M}$ are called Λ-comparative, denoted by $[r,s] \in \Lambda$, if $(r,s) \in \Lambda$ or $(s,r) \in \Lambda$.*

Definition 2 ([48])**.** *$\Lambda^{-1} := \{(r,s) \in \mathcal{M}^2 : (s,r) \in \Lambda\}$ is referred as transpose of Λ.*

Definition 3 ([48])**.** *$\Lambda^s := \Lambda \cup \Lambda^{-1}$, often called the symmetric closure of Λ, forms a symmetric relation.*

Clearly, $(r,s) \in \Lambda^s \Longleftrightarrow [r,s] \in \Lambda$. (cf. [30]).

Definition 4 ([48])**.** *Given $\mathcal{C} \subseteq \mathcal{M}$, the relation $\Lambda|_{\mathcal{C}} := \Lambda \cap \mathcal{C}^2$ is termed as the restriction of Λ on $\mathcal{C}$.*

Definition 5 ([30])**.** *Λ is termed as $\mathcal{S}$-closed if*

$$(\mathcal{S}r, \mathcal{S}s) \in \Lambda,$$

holds for each pair $r, s \in \mathcal{M}$ verifying $(r, s) \in \Lambda$.

Proposition 1 ([31])**.** *Λ remains $\mathcal{S}^n$-closed, whenever Λ is $\mathcal{S}$-closed.*

Definition 6 ([30])**.** *A sequence $\{r_n\} \subset \mathcal{M}$ verifying $(r_n, r_{n+1}) \in \Lambda \ \forall\ n \in \mathbb{N}_0$ is said to be Λ-preserving.*

Definition 7 ([49])**.** *A metric space $(\mathcal{M}, \sigma)$ is termed as Λ-complete if each Cauchy sequence in $\mathcal{M}$, which remains also Λ-preserving, is convergent.*

Definition 8 ([49])**.** *$\mathcal{S}$ is called Λ-continuous at $r \in \mathcal{M}$ if*

$$\mathcal{S}(r_n) \xrightarrow{\sigma} \mathcal{S}(r)$$

for any Λ-preserving sequence $\{r_n\} \subset \mathcal{M}$ verifying $r_n \xrightarrow{\sigma} r$.

Definition 9 ([49])**.** *A function is termed as Λ-continuous if it remains Λ-continuous at all points of $\mathcal{M}$.*

Definition 10 ([30])**.** *Λ is termed as σ-self-closed if each Λ-preserving convergent sequence in $(\mathcal{M}, \sigma)$ has a subsequence whose terms are Λ-comparative with the convergence limit.*

Definition 11 ([50])**.** *Given $r, s \in \mathcal{M}$, the set $\{\omega_0, \omega_1, \ldots, \omega_l\} \subset \mathcal{M}$ is termed as a path from r to s if:*

(i) $\omega_0 = r$ *and* $\omega_L = s$,
(ii) $(\omega_j, \omega_{j+1}) \in \Lambda, \quad 0 \leq j \leq L - 1.$

Definition 12 ([31])**.** *A subset $\mathcal{C} \subseteq \mathcal{M}$, in which any two elements join a path, is called an Λ-connected set.*

Definition 13 ([31])**.** *Λ is termed as locally $\mathcal{S}$-transitive if for each Λ-preserving sequence $\{r_n\} \subset \mathcal{M}$ (with range $\mathcal{P} := \{r_n : n \in \mathbb{N}_0\}$), the restriction $\Lambda|_{\mathcal{P}}$ remains transitive.*

3. Main Results

Let $\mathcal{A}$ denotes a class of bounded functions $\beta : [0, \infty) \to [0, \infty)$. Thus, $\exists\ \beta \in \mathcal{A}$ if and only if $\exists$ a constant $K > 0$ verifying

$$\sup_{t \in [0,\infty)} \beta(t) \leq K. \tag{2}$$

Remark 1. *$\mathcal{B} \subset \mathcal{A}$ but not conversely, e.g., for any $K > 0$, $\beta(t) = K \sin t \in \mathcal{A}$, while $\beta \notin \mathcal{B}$.*

Theorem 1. *Let $(\mathcal{M}, \sigma)$ be a metric space, Λ a relation on $\mathcal{M}$ while $\mathcal{S} : \mathcal{M} \to \mathcal{M}$ a map. Additionally,*

(i) *$(\mathcal{M}, \sigma)$ is Λ-complete,*
(ii) *$\exists\ r_0 \in \mathcal{M}$ verifying $(r_0, \mathcal{S}r_0) \in \Lambda$,*
(iii) *Λ remains $\mathcal{S}$-closed and locally $\mathcal{S}$-transitive,*
(iv) *$\mathcal{S}$ is Λ-continuous or Λ remains σ-self-closed,*
(v) *$\exists\ \beta \in \mathcal{A}$ with upper bound $K > 0$ and $\exists\ \lambda \in (0, 1/K)$ verifying*

$$\sigma(\mathcal{S}r, \mathcal{S}s) \leq \lambda\beta(\lambda\sigma(r, s))\sigma(r, s),\ \forall\ r, s \in \mathcal{M} \text{ with } (r, s) \in \Lambda.$$

Then $\mathcal{S}$ possesses a fixed point.

Proof. Define a sequence $\{r_n\} \subset \mathcal{M}$ verifying

$$r_n = \mathcal{S}^n(r_0) = \mathcal{S}(r_{n-1}), \quad \forall\, n \in \mathbb{N}. \tag{3}$$

Employing hypotheses (ii), (iii) and Proposition 1, one finds

$$(\mathcal{S}^n r_0, \mathcal{S}^{n+1} r_0) \in \Lambda.$$

Making use of (3), the above becomes

$$(r_n, r_{n+1}) \in \Lambda, \quad \forall\, n \in \mathbb{N}_0 \tag{4}$$

so that $\{r_n\}$ is Λ-preserving.

If $\exists\, n_0 \in \mathbb{N}_0$ satisfying $\sigma(r_{n_0}, r_{n_0+1}) = 0$, then by (3), one concludes that r_{n_0} is a fixed point of $\mathcal{S}$. Otherwise, in the case of $\sigma_n := \sigma(r_n, r_{n+1}) > 0, \forall\, n \in \mathbb{N}_0$, we use hypothesis (v) to obtain

$$\begin{aligned} \sigma_n = \sigma(r_n, r_{n+1}) = \sigma(\mathcal{S}r_{n-1}, \mathcal{S}r_n) &\leq \lambda\beta(\lambda\sigma(r_{n-1}, r_n))\sigma(r_{n-1}, r_n) \\ &\leq K\lambda\sigma(r_{n-1}, r_n) = K\lambda\sigma_{n-1} \end{aligned}$$

which by induction gives rise to

$$\sigma_n \leq K\lambda\sigma_{n-1} \leq (K\lambda)^2\sigma_{n-2} \leq \cdots \leq (K\lambda)^n\sigma_0 \longrightarrow 0, \text{ as } n \longrightarrow \infty$$

so that

$$\lim_{n\to\infty} \sigma_n = 0. \tag{5}$$

Now, one will have to show that $\{r_n\}$ is Cauchy. As $\{r_n\}$ is Λ-preserving (owing to (4)) and $\{r_n\} \subset \mathcal{S}(\mathcal{M})$ (owing to (3)), therefore, due to the locally $\mathcal{S}$-transitivity of Λ, one has $(r_n, r_m) \in \Lambda$, for $n < m$. Hence, applying condition (v), one obtains

$$\sigma(r_{n+1}, r_{m+1}) \leq \lambda\beta(\lambda\sigma(r_n, r_m))\sigma(r_n, r_m) \leq \lambda K\sigma(r_n, r_m). \tag{6}$$

Using triangle inequality and (6), one has

$$\begin{aligned} \sigma(r_n, r_m) &\leq \sigma(r_n, r_{n+1}) + \sigma(r_{n+1}, r_{m+1}) + \sigma(r_{m+1}, r_m) \\ &\leq \sigma_n + \lambda K\sigma(r_n, r_m) + \sigma_m \end{aligned}$$

thereby yielding

$$\sigma(r_n, r_m) \leq (1 - K\lambda)^{-1}[\sigma_n + \sigma_m]. \tag{7}$$

Letting $m, n \longrightarrow \infty$ in inequality (7) and using (5), one obtains

$$\sigma(r_n, r_m) \longrightarrow 0.$$

It follows that $\{r_n\}$ is a Cauchy sequence, which also remains Λ-preserving. Therefore, by Λ-completeness of the metric space $(\mathcal{M}, \sigma)$, $\exists$ an element $p \in \mathcal{M}$ verifying $\lim_{n\to\infty} r_n = p$.

Finally, one will have to use hypothesis (iii). Firstly, assume that $\mathcal{S}$ is Λ-continuous. As $\{r_n\}$ is Λ-preserving and $r_n \xrightarrow{\sigma} p$, due to Λ-continuity of $\mathcal{S}$ and (3), one obtains

$$p = \lim_{n\to\infty} r_{n+1} = \lim_{n\to\infty} \mathcal{S}(r_n) = \mathcal{S}(\lim_{n\to\infty} r_n) = \mathcal{S}(p).$$

Hence, we have finished. Otherwise, if Λ remains ϱ-self closed, then $\exists$ is a subsequence $\{r_{n_k}\}$ of $\{r_n\}$ verifying $[r_{n_k}, r] \in \Lambda, \forall\, k \in \mathbb{N}_0$. Consequently, for all $k \in \mathbb{N}_0$, we have either

$(r_{n_k}, r) \in \Lambda$ or $(r, r_{n_k}) \in \Lambda$. In the case of $(r_{n_k}, r) \in \Lambda$, making use of triangular inequality, assumption (v) and $r_{n_k} \xrightarrow{\varrho} r$, one obtains

$$\sigma(r_{n_k+1}, \mathcal{S}p) = \sigma(\mathcal{S}r_{n_k}, \mathcal{S}p)) \leq \lambda K \sigma(r_{n_k}, p) \to 0, \text{ as } k \to \infty$$

implying thereby

$$\lim_{k\to\infty} r_{n_k} = \mathcal{S}(p).$$

In case $(r, r_{n_k}) \in \Lambda$, one obtains the same conclusion by using symmetry of σ. Finally, owing to uniqueness of limit, one has $\mathcal{S}(p) = p$. □

Theorem 2. *Under the hypothesis of Theorem 1, if $\mathcal{S}(\mathcal{M})$ remains Λ^s-connected, then $\mathcal{S}$ possesses a unique fixed point.*

Proof. By Theorem 1, let r and s be two fixed points of $\mathcal{S}$; then

$$\mathcal{S}^n(r) = r \text{ and } \mathcal{S}^n(s) = s, \ \forall\, n \in \mathbb{N}_0.$$

Naturally, one has $r, s \in \mathcal{S}(\mathcal{M})$. The Λ^s-connectedness property of $\mathcal{S}(\mathcal{M})$ guarantees the existence of a path $\{\omega_0, \omega_1, \ldots, \omega_L\}$ between r to s, such that

$$\omega_0 = r, \omega_L = s \text{ and } [\omega_j, \omega_{j+1}] \in \Lambda, \ \forall\, j = 0, 1, \ldots, L-1. \tag{8}$$

Since Λ is $\mathcal{S}$-closed; therefore, one obtains

$$[\mathcal{S}^n \omega_j, \mathcal{S}^n \omega_{j+1}] \in \Lambda, \ \forall\, n \in \mathbb{N}_0 \text{ and } \forall\, j = 0, 1, \ldots, L-1. \tag{9}$$

Denote

$$\delta_n^j := \sigma(\mathcal{S}^n \omega_j, \mathcal{S}^n \omega_{j+1}), \ \forall\, n \in \mathbb{N}_0 \text{ and } \forall\, j = 0, 1, \ldots, L-1.$$

Making use of (9) and hypothesis (v), one obtains

$$\begin{aligned} \delta_n^j &= \sigma(\mathcal{S}^n \omega_j, \mathcal{S}^n \omega_{j+1}) \\ &= \sigma(\mathcal{S}(\mathcal{S}^{n-1}\omega_j), \mathcal{S}(\mathcal{S}^{n-1}\omega_{j+1})) \\ &\leq K\lambda \sigma(\mathcal{S}^{n-1}\omega_j, \mathcal{S}^{n-1}\omega_{j+1}) \\ &= K\lambda \delta_{n-1}^j \end{aligned}$$

yielding thereby

$$\delta_n^j \leq (K\lambda)^n \delta_0^j \to 0, \text{ as } n \to \infty$$

so that

$$\lim_{n\to\infty} \delta_n^j = 0. \tag{10}$$

Using triangle inequality, one obtains

$$\begin{aligned} \sigma(r, s) &= \sigma(\mathcal{S}^n \omega_0, \mathcal{S}^n \omega_L) \\ &\leq \delta_n^0 + \delta_n^1 + \cdots + \delta_n^{L-1} \\ &\to 0 \text{ as } n \to \infty \end{aligned}$$

so that $r = s$. Thus, $\mathcal{S}$ possesses a unique fixed point. □

In particular, for universal relation (i.e., $\Lambda = \mathcal{M}^2$), Theorem 2 produces the following result:

Corollary 1. *Let $(\mathcal{M}, \sigma)$ be a complete metric space while $\mathcal{S} : \mathcal{M} \to \mathcal{M}$ a function. If $\exists\, \beta \in \mathcal{A}$ with upper bound $K > 0$ and $\exists$ a constant $\lambda \in (0, 1/K)$, satisfying*

$$\sigma(\mathcal{S}r, \mathcal{S}s) \le \lambda\beta(\lambda\sigma(r,s))\sigma(r,s),\ \forall\, r,s \in \mathcal{M},$$

then $\mathcal{S}$ admits a unique fixed point.

Obviously, Corollary 1 improves and sharpens Geraghty's fixed point theorem [29].

4. An Application to Fractional Differential Equations

This section is devoted to finding a unique positive solution for the BVP (1) using our newly proved results.

Definition 14 ([1]). *Let $u : (0, \infty) \to \mathbb{R}$ be a function. Then*

$$I^{\eta} u(x) = \frac{1}{\Gamma(\eta)} \int_0^x (x - \xi)^{\eta - 1} u(\xi) d\xi$$

provided R.H.S. exists pointwise on $(0, \infty)$, this is called the Riemann–Liouville fractional integral of order $\eta > 0$ of u. Additionally,

$$\mathcal{D}_x^{\eta} u(x) = \frac{1}{\Gamma(n - \eta)} \left(\frac{d}{dx}\right)^n \int_0^x (x - \xi)^{n - \eta - 1} u(\xi) d\xi$$

where $n = [\eta] + 1$; provided R.H.S. exists pointwise on $(0, \infty)$, this is called the Riemann–Liouville fractional derivative of order $\eta > 0$ of u.

Proposition 2 ([1]). *We have the following:*

(i) *For $u \in L^1(0,1), \rho > \gamma > 0$,*

$$I^{\rho} I^{\gamma} u(x) = I^{\rho+\gamma} u(x), \quad \mathcal{D}_x^{\gamma} I^{\rho} u(x) = I^{\rho - \gamma} u(x), \quad \mathcal{D}_x^{\gamma} I^{\gamma} u(x) = u(x).$$

(ii) *For $\rho > 0, \gamma > 0$,*

$$\mathcal{D}_x^{\rho} x^{\gamma - 1} = \frac{\Gamma(\gamma)}{\Gamma(\gamma - \rho)} x^{\gamma - \rho - 1}$$

Proposition 3 ([1]). *If $\eta > 0$ and $f(u)$ remains integrable then*

$$I^{\eta} \mathcal{D}_x^{\eta} u(x) = f(u) + a_1 u^{\eta - 1} + a_2 u^{\eta - 2} + \cdots + a_n x^{\eta - n}$$

where $a_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$; $n = [\eta]$.

Lemma 1 ([8]). *If $u(x) = I^{\gamma_{n-1}} z(x)$, then BVP (1) is the equivalent to the following BVP:*

$$-\mathcal{D}^{\eta - \gamma_{n-1}} z(x) = h\Big(x, I^{\gamma_{n-1}} z(x), I^{\gamma_{n-1} - \gamma_1} z(x), \ldots, I^{\gamma_{n-1} - \gamma_{n-2}} z(x), z(x)\Big),$$

$$z(0) = z'(0) = 0, \quad z(1) = \sum_{j=1}^{m-2} a_j z(\xi_j). \tag{11}$$

Moreover, if $z \in C\big([0,1]; [0, \infty)\big)$ forms a solution of Problem (11), then $u(x) = I^{\gamma_{n-1}} z(x)$ forms a positive solution of Problem (1).

Given a constant $K > 0$, let us introduce the class Φ of monotonic increasing functions $\varphi : [0, \infty) \to [0, \infty)$ verifying

$$\varphi(t) \le Kt, \quad \forall\, t > 0.$$

Typical examples of functions $\varphi \in \Phi$ are $\varphi(t) = Kt$, $\varphi(t) = Kt^2/(1+t)$ and $\varphi(t) = (2K/\pi)t \arctan t$.

Theorem 3. *Let $h(x, u, u_2, \ldots, u_n)$ be increasing in arguments of u_i on $[0, \infty]$. Assume that $\exists$ n constants $\varepsilon_i > 0,\ i = 1, 2 \ldots, n$ verifying*

$$\max\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\} \leq (n\alpha)^{-1} \tag{12}$$

and $\exists$ a function $\varphi \in \Phi$ and constants $0 < \lambda_1 < \Gamma(\gamma_{n-1})/K$, $0 < \lambda_n < 1/K$, $0 < \lambda_i < \Gamma(\gamma_{n-1} - \gamma_i)/K, i = 1, 2, \leq, n-2$ verifying

$$|h(x, u_1, u_2, \ldots, u_n) - h(x, v_1, v_2, \ldots, v_n)| \leq \sum_{i=1}^{n} \varepsilon_i \varphi(\lambda_i(u_i - v_i)) \tag{13}$$

$\forall\ u_i, v_i \in [0, \infty),\ i = 1, 2, \ldots, n$ with $u_i \geq v_i$ and $x \in [0, 1]$. Then, Problem (1) admits a unique nonnegative solution.

Proof. Define

$$k(x,\xi) = \begin{cases} \frac{(x(1-\xi))^{\eta-\gamma_{n-1}-1} - (x-\xi)^{\eta-\gamma_{n-1}-1}}{\Gamma(\eta-\gamma_{n-1})}, & 0 \leq \xi \leq x \leq 1, \\ \frac{(x(1-\xi))^{\eta-\gamma_{n-1}-1}}{\Gamma(\eta-\gamma_{n-1})}, & 0 \leq x \leq \xi \leq 1. \end{cases}$$

As verified in [8], the Green function of (11) is

$$G(x,\xi) = k(x,\xi) + \frac{x^{\eta-\gamma_{n-1}-1}}{1 - \sum\limits_{j=1}^{m-2} c_j \xi_j^{\eta-\gamma_{n-1}-1}} \sum_{j=1}^{m-2} c_j k(\xi_j, \xi)$$

Additionally, it verifies the following property:

$$0 \leq G(x,\xi) \leq \frac{1}{\Gamma(\eta - \gamma_{n-1})} \left(1 + \frac{\sum\limits_{j=1}^{m-2} c_j}{1 - \sum\limits_{j=1}^{m-2} c_j \xi_j^{\eta-\gamma_{n-1}-1}} \right) = \alpha. \tag{14}$$

Clearly, the BVP (11) is equivalent to

$$z(x) = \int_0^x G(x,\xi) h(\xi, I^{\gamma_{n-1}} z(\xi), I^{\gamma_{n-1}-\gamma_1} z(\xi), \ldots, I^{\gamma_{n-1}-\gamma_{n-2}} z(\xi), z(\xi)) d\xi.$$

Denote

$$\mathcal{M} := \{z \in C([0,1] : z(x) \geq 0,\ x \in [0,1]\}.$$

On $\mathcal{M}$, define the metric σ and the relation Λ by

$$\sigma(u, v) = \sup_{0 \leq x \leq 1} |u(x) - v(x)|$$

and

$$\Lambda = \{(u, v) \in \mathcal{M}^2 : u(x) \geq v(x),\ \forall\ x \in [0,1]\}.$$

Define the operator $\mathcal{S} : \mathcal{M} \to C[0,1]$ by

$$(\mathcal{S}z)(x) = \int_0^1 G(x,\xi) h(\xi, I^{\gamma_{n-1}} z(\xi), I^{\gamma_{n-1}\gamma_1} z(\xi), \ldots, I^{\gamma_{n-1}\gamma_{n-2}} z(\xi), z(\xi)) d\xi, \quad z \in \mathcal{M}. \tag{15}$$

Then from the assumption on h and (14), one has

$$\mathcal{S}(\mathcal{M}) \subset \mathcal{M}.$$

It follows that $\mathcal{S}$ forms a self-mapping on $\mathcal{M}$.

Now, one will verify all the hypotheses of Theorems 1 and 2.

(i) $\mathcal{M}$, being cone in $C[0,1]$, is a closed set of $C[0,1]$. Again, as $C[0,1]$ is complete, $(\mathcal{M}, \sigma)$ forms a complete metric space. Consequently, the metric space $(\mathcal{M}, \sigma)$ is also Γ-complete.

(ii) The zero function 0 verifies $(0, \mathcal{S}0) \in \Lambda$.

(iii) Take $z, w \in \mathcal{M}$ verifying $(z, w) \in \Lambda$, which thereby implies $z(x) \geq w(x), \forall\, x \in C[0,1]$. One has

$$\begin{aligned}
(\mathcal{S}z)(x) &= \int_0^1 G(x,\xi) h\Big(\xi, I^{\gamma_{n-1}} z(\xi), I^{\gamma_{n-1}-\gamma_1} z(\xi), \ldots, I^{\gamma_{n-1}-\gamma_{n-2}} z(\xi), z(\xi)\Big) d\xi \\
&\geq \int_0^1 G(x,\xi) h\Big(\xi, I^{\gamma_{n-1}} w(\xi), I^{\gamma_{n-1}-\gamma_1} w(\xi), \ldots, I^{\gamma_{n-1}-\gamma_{n-2}} w(\xi), w(\xi)\Big) d\xi \\
&= (\mathcal{S}w)(x)
\end{aligned}$$

so that $(\mathcal{S}z, \mathcal{S}w) \in \Lambda$, which yields that Λ is $\mathcal{S}$-closed. Further, the relation Λ being transitive is also locally $\mathcal{S}$-transitive.

(iv) As proved in [47], Λ is σ-self-closed.

(v) Obviously,

$$z(\xi) - w(\xi) \leq \sigma(z, w).$$

Therefore, one has

$$I^{\gamma_{n-1}} z(\xi) - I^{\gamma_{n-1}} w(\xi) \leq \int_0^x \frac{(x-\xi)^{\gamma_{n-1}-1} |z(\xi) - w(\xi)|}{\Gamma(\gamma_{n-1})} d\xi \leq \frac{\sigma(z,w)}{\Gamma(\gamma_{n-1})}. \tag{16}$$

$$\begin{aligned}
I^{\gamma_{n-1}-\gamma_i} z(\xi) - I^{\gamma_{n-1}-\gamma_i} w(\xi) &\leq \int_0^x \frac{(x-\xi)^{\gamma_{n-1}-\gamma_i-1} |z(\xi) - w(\xi)|}{\Gamma(\gamma_{n-1}-\gamma_i)} d\xi \\
&\leq \frac{\sigma(z,w)}{\Gamma(\gamma_{n-1}-\gamma_i)}, i = 1, 2, \ldots, n-2.
\end{aligned} \tag{17}$$

Take $z, w \in \mathcal{M}$ verifying $(z, w) \in \Lambda$ implying thereby $z(x) \geq w(x), \forall\, x \in C[0,1]$. Using (13) and (15)–(17), one obtains

$$\begin{aligned}
\sigma(\mathcal{S}z, \mathcal{S}w) &= \max_{x \in [0,1]} |\mathcal{S}z(x) - \mathcal{S}w(x)| \\
&\leq \eta \int_0^1 |h(\xi, I^{\gamma_{n-1}} z(\xi), I^{\gamma_{n-1}-\gamma_1} z(\xi), \ldots, I^{\gamma^{n-1}}\gamma_{n-2} z(\xi), z(\xi)) \\
&\quad - h(\xi, I^{\gamma_{n-1}} w(\xi), I^{\gamma_{n-1}-\gamma_1} w(\xi), \ldots, I^{\gamma_{n-1}-\gamma_{n-2}} w(\xi), w(\xi))| d\xi \\
&\leq \eta \int_0^1 \Big[\varepsilon_1 \varphi\big(\lambda_1 (I^{\gamma_{n-1}} z(\xi) - I^{\gamma_{n-1}} w(\xi))\big) + \varepsilon_2 \varphi\big(\lambda_2 (I^{\gamma_{n-1}-\gamma_1} z(\xi) - I^{\gamma_{n-1}-\gamma_1} w(\xi))\big) \\
&\quad + \cdots + \varepsilon_{n-1} \varphi\big(\lambda_{n-1} (I^{\gamma_{n-1}\gamma_{n-2}} z(\xi) - I^{\gamma_{n-1}-\gamma_{n-2}} w(\xi))\big) \\
&\quad + \varepsilon_n \varphi\big(\lambda_n (z(\xi) - w(\xi))\big)\Big] d\xi.
\end{aligned} \tag{18}$$

Set

$$\lambda := \max\left\{\frac{\lambda_1}{\Gamma(\gamma_{n-1})}, \frac{\lambda_2}{\Gamma(\gamma_{n-1}-\gamma_1)}, \cdots, \frac{\lambda_{n-1}}{\Gamma(\gamma_{n-1}-\gamma_{n-2})}, \lambda_n\right\}.$$

Then the inequality (18) reduces to

$$\sigma(\mathcal{S}z, \mathcal{S}w) \leq n\eta \max\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\} \varphi(\lambda \sigma(z, w))$$

which in view of (12) becomes

$$\sigma(\mathcal{S}z,\mathcal{S}w) \le \varphi(\lambda\sigma(z,w)). \tag{19}$$

Define an auxiliary function $\beta : [0,\infty) \to [0,\infty)$ by

$$\beta(t) = \begin{cases} 0, & \text{if } t = 0 \\ \varphi(t)/t, & \text{if } t > 0. \end{cases}$$

As $\varphi \in \Phi$, then $\beta(t) \in \mathcal{A}$. Now, we claim that

$$\sigma(\mathcal{S}z,\mathcal{S}w) \le \beta(\lambda\sigma(z,w))\lambda\sigma(z,w). \tag{20}$$

In case $z = w$, the inequality (20) is obviously satisfied. For $z \ne w$, from (19), one obtains

$$\sigma(\mathcal{S}z,\mathcal{S}w) \le \frac{\varphi(\lambda\sigma(z,w))}{\lambda\sigma(z,w)}\lambda\sigma(z,w) = \beta(\lambda\sigma(z,w))\lambda\sigma(z,w). \tag{21}$$

Thus, in both the cases, the contractivity condition (20) holds for $(z,w) \in \Lambda$.

Hence, the hypotheses (i)–(v) of Theorem 1 are satisfied. Let $u, v \in \mathcal{M}$ be arbitrary. Denote $\vartheta := \max\{\mathcal{S}u, \mathcal{S}v\} \in \mathcal{M}$. Then, $\mathcal{S}(\mathcal{M})$ is Λ^s-connected as $(\mathcal{S}u,\vartheta) \in \Lambda$ and $(\mathcal{S}v,\vartheta) \in \Lambda$, $\{\mathcal{S}u,\vartheta,\mathcal{S}v\}$ admit a path in Λ^s between $\mathcal{S}(u)$ and $\mathcal{S}(v)$. Consequently, by Theorem 2, $\mathcal{S}$ possesses a unique fixed point, say $\overline{z} \in \mathcal{M}$, which remains a unique solution of Problem (11). Therefore, by Lemma 1, $\overline{u} \in C[0,1]$ (whereas $\overline{u}(x) = I^{\gamma_{n-1}}\overline{z}(x)$) forms the unique nonnegative solution of the BVP (1). □

Theorem 4. *Under the hypotheses of Theorem 3, if $\exists\, x_0 \in [0,1]$ such that $h(x_0,0,\ldots,0) \ne 0$, then the unique solution of BVP (1) remains positive.*

Proof. In lieu of Theorem 3, the BVP (1) admits a unique nonnegative solution $\overline{u} \in C[0,1]$. One has to prove the nonnegative solution is also positive, i.e., $\overline{u}(x) > 0$ for each $x \in (0,1)$. By contrast, assume that $0 < x^* < 1$ verifying $u(x^*) = 0$ and

$$\overline{u}(x^*) = \int_0^1 G(x^*,\xi)h\Big(\xi, I^{\gamma_{n-1}}z(\xi), I^{\gamma_{n-1}-\gamma_1},\ldots, I^{\gamma_{n-1}-\gamma_{n-2}}z(\xi), z(\xi)\Big)d\xi = 0.$$

Then, one has

$$\begin{aligned} 0 = \overline{u}(x^*) &= \int_0^x G(x^*,\xi)h\Big(\xi, I^{\gamma_{n-1}}z(\xi), I^{\gamma_{n-1}-\gamma_1}z(\xi),\ldots, I^{\gamma_{n-1}-\gamma_{n-2}}z(\xi), z(\xi)\Big)d\xi \\ &\ge \int_0^1 G(x^*,\xi)h(\xi,0,\ldots,0)d\xi \ge 0 \end{aligned}$$

yielding thereby

$$\int_0^1 G(x^*,\xi)h(\xi,0,\ldots,0)d\xi = 0$$

so that

$$G(x^*,\xi)h(\xi,0,\ldots,0) = 0, \quad a.e. \quad \xi \in [0,1].$$

However, $G(x^*,\xi) > 0, \xi \in (0,1)$. Therefore, one has

$$h(\xi,0,\ldots,0) = 0, \quad a.e. \quad \xi \in [0,1]. \tag{22}$$

On the other hand, as $h(x_0,0,\ldots,0) \ne 0, x_0 \in [0,1]$, one has $h(x_0,0,\ldots,0) > 0$. Owing to the continuity of h, $\exists$ a set Ω verifying $x_0 \in \Omega$ and the Lebesgue measure $\mu(\Omega) > 0$

such that $h(x, 0, \ldots, 0) > 0$ for any $x \in \Omega$ contradicts (22). Consequently, one has $\overline{u}(x) > 0$, i.e., $\overline{u}(x)$ forms a positive solution of (1). $\square$

Example 1. *Consider the following fractional BVP:*

$$\begin{aligned} &\mathcal{D}^{5/2}u(x) = e^x + \frac{1}{10}u(x) + \sin^2\left(\mathcal{D}^{1/8}u(x)\right) + \frac{1}{3}\cos^2\left(\mathcal{D}^{1/4}u(x)\right), 0 < x < 1, \\ &\mathcal{D}^{1/4}u(0) = \mathcal{D}^{5/4}u(0), \mathcal{D}^{1/4}u(1) = \frac{1}{4}u(1) = \frac{1}{4}\mathcal{D}^{1/4}u\left(\frac{1}{4}\right) + \frac{1}{2}\mathcal{D}^{1/4}u\left(\frac{3}{4}\right). \end{aligned} \tag{23}$$

Here , one has

$$\sum_{j=1}^{m-2} c_j \xi_j^{\eta-\gamma_{n-1}-1} = \frac{1}{4}\left(\frac{1}{4}\right)^{5/4} + \frac{1}{2}\left(\frac{3}{4}\right)^{5/4} = 0.39316 < 1$$

and

$$\alpha = \frac{1}{\Gamma(\eta - \gamma_{n-1})}\left(1 + \frac{\sum\limits_{j=1}^{m-2} c_j}{1 - \sum\limits_{j=1}^{m-2} c_j \xi_j^{\eta-\gamma_{n-1}-1}}\right) = 0.10964$$

which give rise to

$$(n\alpha)^{-1} = 3.0405.$$

Consider $K = 15$ and $\varphi(t) = 15t$. Take

$$h(x, u_2, u_2, u_3) = e^x + \frac{1}{10}u_1 + \frac{1}{2}\sin^2 u_2 + \frac{1}{3}\cos^2 u_3, (x, u_1, u_2, u_3) \in [0,1] \times [0,\infty)^3.$$

Then, for all $u_1 \geq v_1, u_2 \geq v_2, u_3 \geq v_3$, one has

$$\begin{aligned} |h(x, u_1, u_2, u_3) - h(x, u_1, u_2, u_3))| &= \left|\frac{u_1 - v_1}{10} + \frac{\sin^2 u_2 - \sin^2 v_2}{2} + \frac{\cos^2 u_3 - \cos^2 v_3}{3}\right| \\ &\leq \frac{u_1 - v_1}{10} + \frac{u_2 - v_2}{2} + \frac{u_3 - v_3}{3} \\ &= \frac{2}{75} \times 15 \times \frac{1}{4} \times (u_1 - v_1) + \frac{1}{10} \times 15 \times \frac{1}{3}(u_2 - v_3) \\ &\quad + \frac{4}{9} \times 15 \times \frac{1}{20}(u_3 - v_3) \\ &= \frac{2}{75}\varphi\left(\frac{1}{4}(u_1 - v_1)\right) + \frac{1}{10}\varphi\left(\frac{1}{3}(u_3 - v_3)\right) \\ &\quad + \frac{4}{9}\varphi\left(\frac{1}{20}(u_3 - v_3)\right) \\ &= \sum_{i=1}^{3} \varepsilon_i \varphi(\lambda_i(u_i - v_i)), \end{aligned}$$

where

$$\varepsilon_1 = \frac{2}{75}, \quad \varepsilon_2 = \frac{1}{10}, \quad \varepsilon_3 = \frac{4}{9}, \quad \lambda_1 = \frac{1}{4}, \quad \lambda_2 = \frac{1}{3}, \quad \lambda_3 = \frac{1}{20}.$$

Therefore , one has $\varphi \in \Phi$. Hence, all the hypotheses of Theorem 3 hold. Moreover, $h(0, 0, \ldots, 0) = 4/3 \neq 0$. By Theorems 3 and 4, the BVP (23) has a unique positive solution.

5. Conclusions

In particular for universal relation, Theorem 2 deduces an enriched version of Geraghty's fixed point theorem [29]. Under the restriction $\Lambda = \preceq$, the partial order, Theorems 1 and 2 deduce the corresponding results of Zhou et al. [26]. Furthermore, Theorems 1 and 2 also improve the fixed point results contained in Almarri et al. [46] and Harandi and Emami [47].

In Example 1, we have $\beta(t) = \varphi(t)/t = K = 15$. This implies that $\beta \in \mathcal{B}$. Therefore, the unique positive solution of BVP (23) cannot be determined by using the fixed point theorem of Almarri et al. [46]. However, one can obtain the unique positive solution of BVP (23) via Theorem 2. This substantiates the utility of Theorem 2 over the result of Almarri et al. [46].

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