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Certain New Reverse Hölder- and Minkowski-Type Inequalities for Modified Unified Generalized Fractional Integral Operators with Extended Unified Mittag–Leffler Functions

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Abstract: In this article, we obtain certain novel reverse Hölder- and Minkowski-type inequalities for modified unified generalized fractional integral operators (FIOs) with extended unified Mittag–Leffler functions (MLFs). The predominant results of this article generalize and extend the existing fractional Hölder- and Minkowski-type integral inequalities in the literature. As applications, the reverse versions of weighted Radon-, Jensen- and power mean-type inequalities for modified unified generalized FIOs with extended unified MLFs are also investigated.

Keywords: Hölder's inequalities; Minkowski's inequalities; weighted Radon-type inequalities; Jensen-type inequalities; power mean-type inequalities; fractional integral operators

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1. Introduction

Let us begin with the following well-known Young inequality [1,2]:

$$\mathcal{A}^{1-\mathbb{v}}\mathcal{B}^{\mathbb{v}} \leqslant (1-\mathbb{v})\mathcal{A} + \mathbb{v}\mathcal{B} \ \text{ for } \forall \mathcal{A}, \mathcal{B} > 0 \text{ and } \mathbb{v} \in [0,1]. \tag{1}$$

The foregoing formula (1) is also sometimes known as $\mathbb{v}$-weighted arithmetic-geometric mean inequality. For example, by employing the Kantorovich constant, Zuo et al. [3] showed the refined version of the above classical Young inequality. In the paper [4], Sababheh and Choi obtained some multiple refined versions of Young-type inequalities containing real numbers and matrix operators. By making use of Marichev-Saigo-Maeda fractional integral operators (FIOs), the author [5] obtained some new weighted Young-type Marichev-Saigo-Maeda FIO inequalities. In 2002, Tominaga [6] established the reverse inequality of Young with Specht's ratio as

$$(1-\mathbb{v})\mathcal{A} + \mathbb{v}\mathcal{B} \leqslant S(\mathcal{A}/\mathcal{B})\mathcal{A}^{1-\mathbb{v}}\mathcal{B}^{\mathbb{v}} \ \text{ for } \forall \mathcal{A}, \mathcal{B} > 0 \text{ and } \mathbb{v} \in [0,1], \tag{2}$$

where $S(h)$ stands for the Specht's ratio given by $S(h) = h^{1/(h-1)}/\left(e\log h^{1/(h-1)}\right)$ for $h > 0$ and $h \neq 1$. For some characteristics of Specht's ratio, the reader can see the reference [6]. In the same paper [6], Tominaga showed the following converse difference inequality for the Young inequality

$$(1-\mathbb{v})\mathcal{A} + \mathbb{v}\mathcal{B} - \mathcal{A}^{1-\mathbb{v}}\mathcal{B}^{\mathbb{v}} \leqslant L(\mathcal{A},\mathcal{B})\log S(\mathcal{A}/\mathcal{B}) \ \text{ for } \forall \mathcal{A}, \mathcal{B} > 0 \text{ and } \mathbb{v} \in [0,1], \tag{3}$$

where $L(*,\star)$ represents the logarithmic mean defined by $L(x,y) = (y-x)/(\log y - \log x)$, $x \neq y$ and $L(x,x) \equiv x$ for two positive real constants x and y.

In 2012, Furuichi [7] presented the following refined Young inequalities associating $\mathbb{v}$-weighted geometric mean with $\mathbb{v}$-weighted arithmetic mean:

$$\mathcal{A}^{1-\mathbb{v}}\mathcal{B}^{\mathbb{v}} \leqslant S((\mathcal{A}/\mathcal{B})^{\mathbb{r}})\mathcal{A}^{1-\mathbb{v}}\mathcal{B}^{\mathbb{v}} \leqslant (1-\mathbb{v})\mathcal{A} + \mathbb{v}\mathcal{B} \text{ for } \forall \mathcal{A},\mathcal{B} > 0, \tag{4}$$

where $\mathbb{v} \in [0,1]$, $\mathbb{r} = \min\{\mathbb{v}, 1-\mathbb{v}\}$.

For $1/\mathbb{p} + 1/\mathbb{q} = 1$ with $\mathbb{p} > 1$, $\mathbb{r} = \min\{1/\mathbb{p}, 1/\mathbb{q}\}$, the above inequalities (2), (3) and (4) can be represented as

$$\mathcal{A}^{\frac{1}{\mathbb{p}}}\mathcal{B}^{\frac{1}{\mathbb{q}}} \leqslant S\left(\left(\frac{\mathcal{A}}{\mathcal{B}}\right)^{\mathbb{r}}\right)\mathcal{A}^{\frac{1}{\mathbb{p}}}\mathcal{B}^{\frac{1}{\mathbb{q}}} \leqslant \frac{\mathcal{A}}{\mathbb{p}} + \frac{\mathcal{B}}{\mathbb{q}} \leqslant S\left(\frac{\mathcal{A}}{\mathcal{B}}\right)\mathcal{A}^{\frac{1}{\mathbb{p}}}\mathcal{B}^{\frac{1}{\mathbb{q}}} \text{ for } \forall \mathcal{A},\mathcal{B} > 0,\ \mathbb{v} \in [0,1], \tag{5}$$

$$\frac{\mathcal{A}}{\mathbb{p}} + \frac{\mathcal{B}}{\mathbb{q}} - \mathcal{A}^{\frac{1}{\mathbb{p}}}\mathcal{B}^{\frac{1}{\mathbb{q}}} \leqslant L(\mathcal{A},\mathcal{B}) \log S\left(\frac{\mathcal{A}}{\mathcal{B}}\right) \text{ for } \forall \mathcal{A},\mathcal{B} > 0 \text{ and } \mathbb{v} \in [0,1]. \tag{6}$$

The famous classical Hölder's and Minkowski's inequalities declares that (see [1,2])

$$\sum_{i=1}^{n} \mathfrak{A}_i\mathfrak{B}_i \leqslant \left(\sum_{i=1}^{n} \mathfrak{A}_i^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}} \left(\sum_{i=1}^{n} \mathfrak{B}_i^{\mathbb{q}}\right)^{\frac{1}{\mathbb{q}}}, \tag{7}$$

$$\left(\sum_{i=1}^{n} (\mathfrak{A}_i + \mathfrak{B}_i)^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}} \leqslant \left(\sum_{i=1}^{n} \mathfrak{A}_i^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}} + \left(\sum_{i=1}^{n} \mathfrak{B}_i^{\mathbb{q}}\right)^{\frac{1}{\mathbb{q}}}, \tag{8}$$

where $1/\mathbb{p} + 1/\mathbb{q} = 1$ with $\mathbb{p} > 1$, $\{\mathfrak{A}_i\}_{i=1}^{n}$ and $\{\mathfrak{B}_i\}_{i=1}^{n}$ are nonnegative real sequences.

The integral analogues of the preceding Hölder's inequality (7) and Minkowski's inequality (8) are given as

$$\int_a^b \mathfrak{F}(x)\mathfrak{G}(x)dx \leqslant \left(\int_a^b \mathfrak{F}^{\mathbb{p}}(x)dx\right)^{\frac{1}{\mathbb{p}}} \left(\int_a^b \mathfrak{G}^{\mathbb{q}}(x)dx\right)^{\frac{1}{\mathbb{q}}}, \tag{9}$$

$$\left(\int_a^b (\mathfrak{F}(x) + \mathfrak{G}(x))^{\mathbb{p}}dx\right)^{\frac{1}{\mathbb{p}}} \leqslant \left(\int_a^b \mathfrak{F}^{\mathbb{p}}(x)dx\right)^{\frac{1}{\mathbb{p}}} + \left(\int_a^b \mathfrak{G}^{\mathbb{q}}(x)dx\right)^{\frac{1}{\mathbb{q}}}, \tag{10}$$

where $1/\mathbb{p} + 1/\mathbb{q} = 1$ with $\mathbb{p} > 1$, $\mathfrak{F}$ and $\mathfrak{G}$ express two nonnegative continuous functions on $[a,b]$. The mentioned sum forms (7) and (8) and integral analogs (9) and (10) of Hölder's and Minkowski's inequalities have attracted the attention of a large number of scholars. For instance, Manjegani [8] presented some extensions of Hölder-type trace inequalities for operators. By employing the fractional quantum integrals, Yang [9] gave some fractional quantum Hölder- and Minkowski-type integral inequalities. Based on the local FIOs, Chen et al. [10] investigated the Hölder-type functional inequalities and the reverse version. Furthermore, Minkowski- and Dresher-type inequalities for local FIOs were also presented. By means of the generalized proportional FIOs, Rahman et al. [11] introduced reverse nonlocal fractional Minkowski-type inequalities and some related inequalities.

In 2016, using Specht's ratio, Zhao and Cheung [12] investigated a new reverse version of the foregoing Hölder's inequality. They proved that the following inequality held for $1/\mathbb{p} + 1/\mathbb{q} = 1$ with $\mathbb{p} > 1$,

$$\left(\int_a^b \mathfrak{F}^{\mathbb{p}}(x)dx\right)^{\frac{1}{\mathbb{p}}} \left(\int_a^b \mathfrak{G}^{\mathbb{q}}(x)dx\right)^{\frac{1}{\mathbb{q}}} \leqslant \int_a^b S\left(\frac{Y\mathfrak{F}^{\mathbb{p}}(x)}{X\mathfrak{G}^{\mathbb{q}}(x)}\right)\mathfrak{F}(x)\mathfrak{G}(x)dx, \tag{11}$$

where $\mathfrak{F}$ and $\mathfrak{G}$ demonstrate two continuous positive functions on $[a,b]$,

$$X = \int_a^b \mathfrak{F}^{\mathbb{p}}(x)dx \quad \text{and} \quad Y = \int_a^b \mathfrak{G}^{\mathbb{q}}(x)dx. \tag{12}$$

By substituting $\mathfrak{F}, g$ for $\mathfrak{F}^{\mathbb{p}}, g^{\mathbb{q}}$, respectively, the inequality (12) can be given as

$$\left(\int_a^b \mathfrak{F}(x)dx\right)^{\frac{1}{\mathbb{p}}}\left(\int_a^b \mathfrak{G}(x)dx\right)^{\frac{1}{\mathbb{q}}} \leqslant \int_a^b S\left(\frac{Y\mathfrak{F}(x)}{X\mathfrak{G}(x)}\right)\mathfrak{F}^{\frac{1}{\mathbb{p}}}(x)\mathfrak{G}^{\frac{1}{\mathbb{q}}}(x)dx, \tag{13}$$

where $1/\mathbb{p} + 1/\mathbb{q} = 1$ with $\mathbb{p} > 1$, $\mathfrak{F}$ and $\mathfrak{G}$ stand for two continuous positive functions on $[a, b]$,

$$X = \int_a^b \mathfrak{F}(x)dx \quad \text{and} \quad Y = \int_a^b \mathfrak{G}(x)dx. \tag{14}$$

Based on Specht's ratio and the diamond-α integral on an arbitrary time scale, El-Deeb et al. [13] obtained some new reverse versions of Hölder-type inequalities on an arbitrary time scale, which can be seen as the extensions of the inequalities (11) and (13).

In 2020, Benaissa and Budak [14] improved the reverse version of the Hölder's inequality mentioned above. They showed that, for $\hbar, \ell > 0$, $1/\mathbb{p} + 1/\mathbb{q} = 1$ with $\mathbb{p} > 1$,

$$\left(\int_a^b w(x)\mathfrak{F}^{\hbar}(x)dx\right)^{\frac{1}{\mathbb{p}}}\left(\int_a^b w(x)\mathfrak{G}^{\ell}(x)dx\right)^{\frac{1}{\mathbb{q}}} \leqslant \left(\frac{\mathbb{M}}{\mathbb{m}}\right)^{\frac{1}{\mathbb{p}\mathbb{q}}} \int_a^b w(x)\mathfrak{F}^{\frac{\hbar}{\mathbb{p}}}(x)\mathfrak{G}^{\frac{\ell}{\mathbb{q}}}(x)dx, \tag{15}$$

where $\mathfrak{F}$ and g stand for two continuous positive functions on $[a, b]$ satisfying $0 < \mathbb{m} \leqslant \mathfrak{F}^{\hbar}(x)/\mathfrak{G}^{\ell}(x) \leqslant \mathbb{M}$ for all $x \in [a, b]$, and w is continuous positive weight function. When $w(x) = 1$, Zhao and Cheung [12] presented the special case of the above reverse Hölder-type inequality (15). Agarwal et al. [15], Yin and Qi [16], and Zakarya et al. [17] considered the reverse Hölder-type inequalities for Δ-integral and diamond-α integral on an arbitrary time scales similar to the inequality (15), respectively. In 2022, using the diamond-α integral on an arbitrary time scale and introducing two parameters, Benaissa [18] obtained a generalized form of the anterior reverse Hölder's inequality.

In 2010, Set et al. [19] investigated the new reverse analog of a Minkowski-type inequality and the related result. They showed that, for $1 \leqslant \mathbb{p}$, f and g are continuous positive functions on $[a, b]$ satisfying $0 < \mathbb{m} \leqslant \mathfrak{F}(t)/\mathfrak{G}(t) \leqslant \mathbb{M}$ for all $t \in [a, b]$, then

$$\left(\int_a^b \mathfrak{F}^{\mathbb{p}}(x)dx\right)^{\frac{1}{\mathbb{p}}} + \left(\int_a^b \mathfrak{G}^{\mathbb{p}}(x)dx\right)^{\frac{1}{\mathbb{p}}} \leqslant \mathbb{c}_1\left(\int_a^b (\mathfrak{F}(x)+\mathfrak{G}(x))^{\mathbb{p}}dx\right)^{\frac{1}{\mathbb{p}}}, \tag{16}$$

$$\left(\int_a^b \mathfrak{F}^{\mathbb{p}}(x)dx\right)^{\frac{2}{\mathbb{p}}} + \left(\int_a^b \mathfrak{G}^{\mathbb{p}}(x)dx\right)^{\frac{2}{\mathbb{p}}} \geqslant \mathbb{c}_2\left(\int_a^b \mathfrak{F}^{\mathbb{p}}(x)dx\right)^{\frac{1}{\mathbb{p}}}\left(\int_a^b \mathfrak{G}^{\mathbb{p}}(x)dx\right)^{\frac{1}{\mathbb{p}}}, \tag{17}$$

where

$$\mathbb{c}_1 = \frac{\mathbb{M}(\mathbb{m}+1)+(\mathbb{M}+1)}{(\mathbb{m}+1)(\mathbb{M}+1)} \quad \text{and} \quad \mathbb{c}_2 = \frac{(\mathbb{m}+1)(\mathbb{M}+1)}{\mathbb{M}} - 2. \tag{18}$$

Based on the Riemann–Liouville FIOs, Zoubir [20] considered the reverse Minkowski-type fractional integral inequalities similar to the inequalities (16) and (18). Later, Chinchane and Pachpatte [21] and Taf and Brahim [22], Chinchane and Pachpatte [23], Chinchane [24], Sousa and Oliveira [25], Rashid [26], and Aljaaidi et al. [27] investigated the reverse fractional Minkowski's inequalities for the Hadamard FIOs, Saigo FIOs, generalized k-FIOs, Katugampola FIOs, generalized $\mathcal{K}$-FIOs and proportional FIOs, respectively.

In this paper, inspired by the mentioned papers, we will consider certain new reverse Hölder- and Minkowski-type inequalities for modified unified generalized FIOs with extended unified Mittag–Leffler functions (MLFs). The principal results of this article generalize and extend the existing fractional Hölder- and Minkowski-type integral inequalities in the literature. As applications, the reverse analogues of weighted Radon-, Jensen- and power mean-type inequalities for modified unified generalized FIOs with extended unified MLFs are also given.

2. Preliminaries

In this section, we will first present the generalized Q function, which can be seen as the generalization of the canonical MLF.

Definition 1 ([28]). *Let $Q^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\cdot;\cdot,\cdot)$ be the generalized Q function defined by*

$$Q^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(t;\underline{\mathscr{A}},\underline{\mathscr{B}})=\sum_{l=0}^{\infty}\frac{\prod_{i=1}^{n}B(\mathscr{B}_i,l)(\lambda)_{\rho l}(\theta)_{kl}t^l}{\prod_{i=1}^{n}B(\mathscr{A}_i,l)(\gamma)_{\delta l}(\mu)_{\nu l}\Gamma(\alpha l+\beta)},\tag{19}$$

where $k\in(0,1)\cup\mathbb{N}$, the generalized Pochhammer symbol $(\lambda)_{\rho l}=(\Gamma(\lambda+\rho l)/\Gamma(\lambda))$, $\Gamma(\cdot)$ and $B(\cdot,\cdot)$ denotes the well-known gamma function and beta function, respectively, $\underline{\mathscr{A}}=(\mathscr{A}_1,\mathscr{A}_2,\cdots,\mathscr{A}_n)$, $\underline{\mathscr{B}}=(\mathscr{B}_1,\mathscr{B}_2,\cdots,\mathscr{B}_n)$, $\alpha,\beta,\gamma,\delta,\mu,\nu,\lambda,\rho,\theta,\mathscr{A}_i,\mathscr{B}_i\in\mathbb{C}$, $\min\{\mathscr{R}(\alpha),\mathscr{R}(\beta),\mathscr{R}(\gamma),\mathscr{R}(\delta),\mathscr{R}(\lambda),\mathscr{R}(\theta),\mathscr{R}(\rho)\}>0$, and $\mathscr{R}(\alpha)$ demonstrates the real part of complex number α.

Based on the generalized Q function above, Zhou et al. [29] presented the following generalized FIOs.

Definition 2 ([29]). *The generalized FIOs ${}_{Q}I^{\omega,\lambda,\rho,\theta,k,n}_{u^+,\alpha,\beta,\gamma,\delta,\mu,\nu}\psi(x;\underline{\mathscr{A}},\underline{\mathscr{B}})$ and ${}_{Q}I^{\omega,\lambda,\rho,\theta,k,n}_{v^-,\alpha,\beta,\gamma,\delta,\mu,\nu}\psi(x;\underline{\mathscr{A}},\underline{\mathscr{B}})$ with the generalized Q function (19) are introduced as*

$${}_{Q}I^{\omega,\lambda,\rho,\theta,k,n}_{u^+,\alpha,\beta,\gamma,\delta,\mu,\nu}\psi(x;\underline{\mathscr{A}},\underline{\mathscr{B}})=\int_u^x(x-s)^{\beta-1}Q^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(x-s)^{\alpha};\underline{\mathscr{A}},\underline{\mathscr{B}})\psi(s)ds,\tag{20}$$

$${}_{Q}I^{\omega,\lambda,\rho,\theta,k,n}_{v^-,\alpha,\beta,\gamma,\delta,\mu,\nu}\psi(x;\underline{\mathscr{A}},\underline{\mathscr{B}})=\int_x^v(s-x)^{\beta-1}Q^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(s-x)^{\alpha};\underline{\mathscr{A}},\underline{\mathscr{B}})\psi(s)ds.\tag{21}$$

In 2018, Andrić et al. [30] first introduced an extended generalized MLF as follows.

Definition 3 ([30]). *Assume that $\alpha,\beta,\gamma,\lambda,\theta\in\mathbb{C}$, $\min\{\mathscr{R}(\alpha),\mathscr{R}(\beta),\mathscr{R}(\gamma),\mathscr{R}(\lambda),\mathscr{R}(\theta)\}>0$, $\mathscr{R}(\theta)>\mathscr{R}(\lambda)$ and $0<k\le r+\mathscr{R}(\alpha)$ for $p\ge0$, $r>0$. Then, define the extended generalized MLF $E^{\lambda,\theta,k,r}_{\alpha,\beta,\gamma}(t;p)$ by the following series*

$$E^{\lambda,\theta,k,r}_{\alpha,\beta,\gamma}(t;p)=\sum_{l=0}^{\infty}\frac{B_p(\gamma+lk,\theta-\lambda)}{B(\lambda,\theta-\lambda)}\frac{(\theta)_{lk}}{(\gamma)_{lr}}\frac{t^n}{\Gamma(\alpha l+\beta)},\tag{22}$$

where, $\min\{\mathscr{R}(x),\mathscr{R}(y),\mathscr{R}(p)\}>0$, $B_p(*,\star)$ denotes a generalization of the beta function by

$$B_p(x,y)=\int_0^1 s^{x-1}(1-s)^{y-1}e^{-\frac{p}{s(1-s)}}ds.\tag{23}$$

Second, Andrić et al. [30] gave the following extended generalized FIOs with the extended generalized MLF.

Definition 4 ([30]). *The extended generalized FIOs $\varepsilon^{\omega,\lambda,\theta,k,r}_{u^+,\alpha,\beta,\gamma}\psi(x;p)$ and $\varepsilon^{\omega,\lambda,\theta,k,r}_{v^-,\alpha,\beta,\gamma,\delta,\mu,\nu}\psi(x;p)$ with the above extended generalized MLF (22) are presented by the following forms*

$$\varepsilon^{\omega,\lambda,\theta,k,r}_{u^+,\alpha,\beta,\gamma}\psi(x;p)=\int_u^x(x-s)^{\beta-1}E^{\lambda,\theta,k,r}_{\alpha,\beta,\gamma}(\omega(x-s)^{\alpha};p)\psi(s)ds,\tag{24}$$

$$\varepsilon^{\omega,\lambda,\theta,k,r}_{v^-,\alpha,\beta,\gamma}\psi(x;p)=\int_x^v(s-x)^{\beta-1}E^{\lambda,\theta,k,r}_{\alpha,\beta,\gamma}(\omega(s-x)^{\alpha};p)\psi(s)ds.\tag{25}$$

By employing the extended generalized MLF above, Farid [31,32] introduced the following unified FIOs with regard to an increasing function.

Definition 5 ([31,32]). *Suppose that $\psi,\xi:[u,v]\to\mathbb{R}$, $0<u<v$, are two continuous functions so that ψ is positive satisfying $\psi\in L_1[u,v]$, and ξ is strictly increasing and differentiable. Also let ϕ be a positive*

function so that ϕ/x be an increasing on $[u,+\infty)$ and $\omega,\alpha,\beta,\gamma,\lambda,\theta \in \mathbb{C}$, $\min\{\mathscr{R}(\alpha),\mathscr{R}(\beta),\mathscr{R}(\gamma),\mathscr{R}(\lambda),\mathscr{R}(\theta)\} > 0$, $\mathscr{R}(\theta) > \mathscr{R}(\lambda)$ with $p \geq 0$, $r > 0$ and $0 < k \leq r + \mathscr{R}(\alpha)$. Then, for $x \in [u,v]$, the left and right-side unified FIOs $\left({}^{\phi}_{\xi}F^{\omega,\lambda,\theta,k,r}_{u^+,\alpha,\beta,\gamma}\psi\right)(x;p)$ and $\left({}^{\phi}_{\xi}F^{\omega,\lambda,\theta,k,r}_{v^-,\alpha,\beta,\gamma}\psi\right)(x;p)$ are introduced by

$$\left({}^{\phi}_{\xi}F^{\omega,\lambda,\theta,k,r}_{u^+,\alpha,\beta,\gamma}\psi\right)(x;p) = \int_u^x \frac{\phi(\xi(x)-\xi(s))}{\xi(x)-\xi(s)} E^{\lambda,\theta,k,r}_{\alpha,\beta,\gamma}(\omega(\xi(x)-\xi(s))^{\alpha};p)\psi(s)d(\xi(s)), \tag{26}$$

$$\left({}^{\phi}_{\xi}F^{\omega,\lambda,\theta,k,r}_{v^-,\alpha,\beta,\gamma}\psi\right)(x;p) = \int_x^v \frac{\phi(\xi(s)-\xi(x))}{\xi(s)-\xi(x)} E^{\lambda,\theta,k,r}_{\alpha,\beta,\gamma}(\omega(\xi(s)-\xi(x))^{\alpha};p)\psi(s)d(\xi(s)). \tag{27}$$

In 2005, Raina gave the following definition of Mittag–Leffler-like function (MLLF).

Definition 6 ([33]). *Let $\mathcal{E}^{\sigma,k}_{\rho,\lambda}(x)$ and $\Gamma_k(\varrho)$ be the MLLF and k-gamma function introduced by*

$$\mathcal{E}^{\sigma,k}_{\rho,\lambda}(x) = \sum_{n=0}^{\infty} \frac{\sigma(n)x^n}{k\Gamma_k(\rho k n+\lambda)}, \quad \rho,\lambda > 0, \quad \text{and} \quad \Gamma_k(\varrho) = \int_0^{\infty} \exp\left(-\frac{t^k}{k}\right) t^{\varrho-1}dt, \tag{28}$$

where $|x| < \mathcal{R}$, the coefficient $\sigma(n)$ is a bounded positive sequence for $n \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$ and a positive constant $\mathcal{R}$. The k-gamma function satisfies the relations $\Gamma_k(\varrho+k) = \varrho\Gamma_k(\varrho)$, $\Gamma_k(\varrho) = k^{\varrho/k-1}\Gamma_k(\varrho/k)$, and $\Gamma(\varrho) = \lim_{k\to 1}\Gamma_k(\varrho)$.

In 2022, taking advantage of the MLLF, the author [34,35] introduced the following generalized FIOs.

Definition 7 ([34,35]). *Assume that ψ,ξ are two continuous functions from $[u,v]$ to $\mathbb{R}$ for $0 < u < v$ so that $\psi \in L_1[u,v]$ is positive, and ξ is a strictly increasing and differentiable function. Also suppose that ϕ/x is an increasing function on $[u,+\infty)$ for a positive function ϕ and $\omega \in \mathbb{R}$, $\rho,\lambda,\mu > 0$. Then for $x \in [u,v]$, the left and right-side generalized FIOs ${}^{\phi}_{\xi}\mathcal{F}^{\Omega,\omega,\sigma,k}_{u^+,\alpha,\rho,\lambda}\psi(x)$ and ${}^{\phi}_{\xi}\mathcal{F}^{\Omega,\omega,\sigma,k}_{v^-,\alpha,\rho,\lambda}\psi(x)$ with the MLLF (28) are given by*

$${}^{\phi}_{\xi}\mathcal{F}^{\Omega,\omega,\sigma,k}_{u^+,\alpha,\rho,\lambda}\psi(x) = \aleph^{-1}(x)\int_u^x \frac{\phi(\xi(x)-\xi(s))}{\xi(x)-\xi(s)}\aleph(s)\mathcal{E}^{\sigma,k}_{\rho,\lambda}(\omega(\xi(x)-\xi(s))^{\alpha})\psi(s)d(\xi(s)), \tag{29}$$

$${}^{\phi}_{\xi}\mathcal{F}^{\Omega,\omega,\sigma,k}_{v^-,\alpha,\rho,\lambda}\psi(x) = \aleph^{-1}(x)\int_x^v \frac{\phi(\xi(s)-\xi(x))}{\xi(s)-\xi(x)}\aleph(s)\mathcal{E}^{\sigma,k}_{\rho,\lambda}(\omega(\xi(s)-\xi(x))^{\alpha})\psi(s)d(\xi(s)), \tag{30}$$

where $\aleph(s)$ stands for a weighted function satisfying $\aleph(s) > 0$ for all $s \in [u,v]$.

In 2021, Zhang et al. [36] introduced a new MLF unifying the generalized Q function (19) and extended MLF (22). Moreover, the following FIOs involving the unified MLF as its kernel were established as

Definition 8 ([36]). *Let $\underline{\mathscr{A}} = (\mathscr{A}_1,\mathscr{A}_2,\cdots,\mathscr{A}_n)$, $\underline{\mathscr{B}} = (\mathscr{B}_1,\mathscr{B}_2,\cdots,\mathscr{B}_n)$, $\underline{\mathscr{C}} = (\mathscr{C}_1,\mathscr{C}_2,\cdots,\mathscr{C}_n)$, where $\mathscr{A}_i,\mathscr{B}_i,\mathscr{C}_i$, $i = 1,2,\ldots,n$, such that $\forall i$, $\mathscr{R}(\mathscr{A}_i),\mathscr{R}(\mathscr{B}_i),\mathscr{R}(\mathscr{C}_i) > 0$. Furthermore, let $\alpha,\beta,\gamma,\delta,\mu,\nu,\rho,\theta,t \in \mathbb{C}$, $\min\{\mathscr{R}(\alpha),\mathscr{R}(\beta),\mathscr{R}(\gamma),\mathscr{R}(\delta),\mathscr{R}(\lambda),\mathscr{R}(\theta)\} > 0$, $k \in (0,1)\cup\mathbb{N}$ with $p \geqslant 0$, and $k + \mathscr{R}(\rho) < \mathscr{R}(\delta+\nu+\alpha)$ with $\mathscr{I}(\rho) = \mathscr{I}(\delta+\nu+\alpha)$, $\mathscr{I}(\alpha)$ denotes the imaginary part of complex number α. Then, we present the unified MLF by*

$$M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(t;\underline{\mathscr{A}},\underline{\mathscr{B}},\underline{\mathscr{C}},p) = \sum_{l=0}^{\infty} \frac{\prod_{i=1}^{n} B_p(\mathscr{B}_i,\mathscr{A}_i)(\lambda)_{\rho l}(\theta)_{kl}t^l}{\prod_{i=1}^{n} B(\mathscr{C}_i,\mathscr{A}_i)(\gamma)_{\delta l}(\mu)_{\nu l}\Gamma(\alpha l+\beta)}. \tag{31}$$

Definition 9 ([36]). *The generalized FIOs $I^{\omega,\lambda,\rho,\theta,k,n}_{u^+,\alpha,\beta,\gamma,\mu,\nu}\psi(x;\underline{\mathscr{A}},\underline{\mathscr{B}},\underline{\mathscr{C}},p)$ and $I^{\omega,\lambda,\rho,\theta,k,n}_{v^-,\alpha,\beta,\gamma,\mu,\nu}\psi(x;\underline{\mathscr{A}},\underline{\mathscr{B}},\underline{\mathscr{C}},p)$ with the unified MLF (31) are obtained by*

$$I^{\omega,\lambda,\rho,\theta,k,n}_{u^+,\alpha,\beta,\gamma,\mu,\nu}\psi(x;\underline{\mathscr{A}},\underline{\mathscr{B}},\underline{\mathscr{C}},p) = \int_u^x (x-s)^{\beta-1} \cdot M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(x-s)^{\alpha};\underline{\mathscr{A}},\underline{\mathscr{B}},\underline{\mathscr{C}},p)\psi(s)ds, \quad (32)$$

$$I^{\omega,\lambda,\rho,\theta,k,n}_{v^-,\alpha,\beta,\gamma,\mu,\nu}\psi(x;\underline{\mathscr{A}},\underline{\mathscr{B}},\underline{\mathscr{C}},p) = \int_x^v (s-x)^{\beta-1} \cdot M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(s-x)^{\alpha};\underline{\mathscr{A}},\underline{\mathscr{B}},\underline{\mathscr{C}},p)\psi(s)ds. \quad (33)$$

In 2022, making use of the unified MLF above, Gao et al. [37] presented the unified generalized FIOs as follows.

Definition 10 ([37]). *Assume that ψ,ξ are two continuous functions from $[u,v]$ to $\mathbb{R}$ for $0<u<v$ so that $\psi\in L_1[u,v]$ is positive, and ξ is a strictly increasing and differentiable function. Also suppose that ϕ/x is an increasing function on $[u,+\infty)$ for a positive function ϕ and $\omega\in\mathbb{R}$. Let $\underline{\mathscr{A}}=(\mathscr{A}_1,\mathscr{A}_2,\cdots,\mathscr{A}_n)$, $\underline{\mathscr{B}}=(\mathscr{B}_1,\mathscr{B}_2,\cdots,\mathscr{B}_n)$, $\underline{\mathscr{C}}=(\mathscr{C}_1,\mathscr{C}_2,\cdots,\mathscr{C}_n)$, where $\mathscr{A}_i,\mathscr{B}_i,\mathscr{C}_i$, $i=1,2,\ldots,n$, such that $\forall i$, $\mathscr{R}(\mathscr{A}_i),\mathscr{R}(\mathscr{B}_i),\mathscr{R}(\mathscr{C}_i)>0$. Furthermore, let $\omega,\alpha,\beta,\gamma,\delta,\mu,\nu,\rho,\theta,t\in\mathbb{C}$, $\min\{\mathscr{R}(\alpha),\mathscr{R}(\beta),\mathscr{R}(\gamma),\mathscr{R}(\delta),\mathscr{R}(\lambda),\mathscr{R}(\theta)\}>0$, $k\in(0,1)\cup\mathbb{N}$ with $p\geqslant 0$, and $k+\mathscr{R}(\rho)<\mathscr{R}(\delta+\nu+\alpha)$ with $\mathscr{I}(\rho)=\mathscr{I}(\delta+\nu+\alpha)$. Then, for $x\in[u,v]$, the left and right-side unified generalized FIOs $({}^{\phi}_{\xi}\Omega^{\omega,\lambda,\rho,\theta,k,n}_{u^+,\alpha,\beta,\gamma,\mu,\nu}\psi)(x;p)$ and $({}^{\phi}_{\xi}\Omega^{\omega,\lambda,\rho,\theta,k,n}_{v^-,\alpha,\beta,\gamma,\mu,\nu}\psi)(x;p)$ with the unified MLF (31) are given by*

$$({}^{\phi}_{\xi}\Omega^{\omega,\lambda,\rho,\theta,k,n}_{u^+,\alpha,\beta,\gamma,\mu,\nu}\psi)(x;p) = \int_u^x \frac{\phi(\xi(x)-\xi(s))}{\xi(x)-\xi(s)} \cdot M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\xi(x)-\xi(s))^{\alpha};p)\psi(s)d(\xi(s)), \quad (34)$$

$$({}^{\phi}_{\xi}\Omega^{\omega,\lambda,\rho,\theta,k,n}_{v^-,\alpha,\beta,\gamma,\mu,\nu}\psi)(x;p) = \int_x^v \frac{\phi(\xi(s)-\xi(x))}{\xi(s)-\xi(x)} \cdot M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\xi(s)-\xi(x))^{\alpha};p)\psi(s)d(\xi(s)). \quad (35)$$

In 2022, by means of the modified extended beta function, Abubakar et al. [38] gave the following extended unified MLF, which can be seen as the extensions of gamma, beta, and hypergeometric MLFs.

Definition 11 ([38]). *Let $\underline{\mathscr{A}}=(\mathscr{A}_1,\mathscr{A}_2,\cdots,\mathscr{A}_n)$, $\underline{\mathscr{B}}=(\mathscr{B}_1,\mathscr{B}_2,\cdots,\mathscr{B}_n)$, $\underline{\mathscr{C}}=(\mathscr{C}_1,\mathscr{C}_2,\cdots,\mathscr{C}_n)$, where $\mathscr{A}_i,\mathscr{B}_i,\mathscr{C}_i$, $i=1,2,\ldots,n$, such that $\forall i$, $\mathscr{R}(\mathscr{A}_i),\mathscr{R}(\mathscr{B}_i),\mathscr{R}(\mathscr{C}_i)>0$. Furthermore, let $\alpha,\beta,\gamma,\delta,\mu,\nu,\rho,\theta,t\in\mathbb{C}$, $\min\{\mathscr{R}(\alpha),\mathscr{R}(\beta),\mathscr{R}(\gamma),\mathscr{R}(\delta),\mathscr{R}(\lambda),\mathscr{R}(\theta)\}>0$, $k\in(0,1)\cup\mathbb{N}$ with $p\geqslant 0$, and $k+\mathscr{R}(\rho)<\mathscr{R}(\delta+\nu+\alpha)$ with $\mathscr{I}(\rho)=\mathscr{I}(\delta+\nu+\alpha)$. Then the extended unified MLF is presented by*

$${}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(t;\underline{\mathscr{A}},\underline{\mathscr{B}},\underline{\mathscr{C}},\sigma) = \sum_{l=0}^{\infty}\frac{\prod_{i=1}^{n}B^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}(\mathscr{B}_i,\mathscr{A}_i,\varsigma)(\lambda)_{\rho l}(\theta)_{kl}t^l}{\prod_{i=1}^{n}B(\mathscr{C}_i,\mathscr{A}_i)(\gamma)_{\delta l}(\mu)_{\nu l}\Gamma(\alpha l+\beta)}, \quad (36)$$

where $B^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}(*,\cdot,\star)$ denotes the modified extended beta function defined by

$$B^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}(\tau_1,\tau_2,\varsigma) = \int_0^1 s^{\tau_1-1}(1-s)^{\tau_2-1}\varsigma^{\left(-\frac{\sigma_1}{s^{\varrho_1}}-\frac{\sigma_2}{(1-s)^{\varrho_2}}\right)}ds, \quad (37)$$

for $\min\{\mathscr{R}(\tau_1),\mathscr{R}(\tau_2),\mathscr{R}(\sigma_1),\mathscr{R}(\sigma_2),\mathscr{R}(\varrho_1),\mathscr{R}(\varrho_2)\}>0$, $\varsigma\in(0,\infty)\backslash\{1\}$.

Finally, the following definition of modified unified generalized FIOs will be introduced based on the extended unified MLF.

Definition 12. *Assume that ψ, ξ are two continuous functions from $[u, v]$ to $\mathbb{R}$ for $0 < u < v$ so that $\psi \in L_1[u, v]$ is positive, and ξ is a strictly increasing and differentiable function. Also suppose that ϕ/x is an increasing function on $[u, +\infty)$ for a positive function ϕ and $\omega \in \mathbb{R}$. Let $\underline{\mathscr{A}} = (\mathscr{A}_1, \mathscr{A}_2, \cdots, \mathscr{A}_n)$, $\underline{\mathscr{B}} = (\mathscr{B}_1, \mathscr{B}_2, \cdots, \mathscr{B}_n)$, $\underline{\mathscr{C}} = (\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_n)$, where $\mathscr{A}_i, \mathscr{B}_i, \mathscr{C}_i$, $i = 1, 2, n, \ldots$, such that $\forall i$, $\mathscr{R}(\mathscr{A}_i), \mathscr{R}(\mathscr{B}_i), \mathscr{R}(\mathscr{C}_i) > 0$. Furthermore, let $\omega, \alpha, \beta, \gamma, \delta, \mu, \nu, \rho, \theta$, $t \in \mathbb{C}$, $\min\{\mathscr{R}(\alpha), \mathscr{R}(\beta), \mathscr{R}(\gamma), \mathscr{R}(\delta), \mathscr{R}(\lambda), \mathscr{R}(\theta)\} > 0$, $k \in (0, 1) \cup \mathbb{N}$ with $p \geqslant 0$, and $k + \mathscr{R}(\rho) < \mathscr{R}(\delta + \nu + \alpha)$ with $\mathscr{I}(\rho) = \mathscr{I}(\delta + \nu + \alpha)$. Then, for $x \in [u, v]$, the left and right-side modified unified generalized FIOs $({}^{\phi}_{\xi}\Theta^M_{u^+}\psi)(x;\sigma)$ and $({}^{\phi}_{\xi}\Theta^M_{v^-}\psi)(x;\sigma)$ with the extended unified MLF (36) are defined by*

$$
\begin{aligned}
({}^{\phi}_{\xi}\Theta^M_{u^+}\psi)(x;\sigma) =& ({}^{\phi,\varrho_1,\varrho_2}_{\xi,\sigma_1,\sigma_2}\Theta^{\omega,\lambda,\rho,\theta,k,n}_{u^+,\alpha,\beta,\gamma,\mu,\nu}\psi)(x;\sigma) \\
=& \aleph^{-1}(x)\int_u^x \aleph(t)\mathscr{M}^t_x({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}, \xi, \phi)\psi(t)d(\xi(t)), \quad (38)\\
({}^{\phi}_{\xi}\Theta^M_{v^-}\psi)(x;\sigma) =& ({}^{\phi,\varrho_1,\varrho_2}_{\xi,\sigma_1,\sigma_2}\Theta^{\omega,\lambda,\rho,\theta,k,n}_{v^-,\alpha,\beta,\gamma,\mu,\nu}\psi)(x;\sigma) \\
=& \aleph^{-1}(x)\int_x^v \aleph(t)\mathscr{M}^x_t({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}, \xi, \phi)\psi(t)d(\xi(t)), \quad (39)
\end{aligned}
$$

where $\aleph(s)$ stands for a weighted function satisfying $\aleph(t) > 0$ for all $t \in [u, v]$ and the kernel function $\mathscr{M}^t_x({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}, \xi, \phi)$ is given as

$$
\begin{aligned}
\mathscr{M}^t_x({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}, \xi, \phi) =& \frac{\phi(\xi(x) - \xi(t))}{\xi(x) - \xi(t)} \\
&\cdot {}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\xi(x) - \xi(t))^{\alpha}; \underline{\mathscr{A}}, \underline{\mathscr{B}}, \underline{\mathscr{C}}, \sigma). \quad (40)
\end{aligned}
$$

Remark 1. *The unified MLF (31) can be seen as the generalization of the generalized Q function (19) and extended MLF (22); however, the unified MLF (31) can be seen as the special case of the extended unified MLF (36). Therefore, the modified unified generalized FIOs (38) and (39) includes the FIOs (20) and (21), (24) and (25), the unified FIOs (26) and (27), the unified generalized FIOs (34) and (35). From ([34], Remarks 9 and 10) and ([35], Remarks 2.2 and 2.3), we point out that the raised previously unified FIOs (26) and (27) can produce a great number of existent FIOs according to distinct setting values of the relevant parameters and functions.*

For the sake of convenience, we always assume that all of the modified unified generalized FIOs exist throughout the article.

3. Reverse Hölder Type Inequalities

In this section, we will establish some new reverse Hölder-type inequalities for modified unified generalized FIOs.

Theorem 1. *Assume that f, g, w are three continuous positive functions on $[u, v]$. Let $1/\mathbb{p} + 1/\mathbb{q} = 1$ satisfying $\mathbb{p} > 1$. Then, for $x \in [u, v]$, we have the following FIO inequalities*

$$
\begin{aligned}
({}^{\phi}_{\xi}\Theta^M_{u^+}wfg)(x;\sigma) \leqslant& \left({}^{\phi}_{\xi}\Theta^M_{u^+}S\left(\left(\frac{\mathscr{G}_1 f^{\mathbb{p}}}{\mathscr{F}_1 g^{\mathbb{q}}}\right)^{\mathbb{r}}\right)wfg\right)(x;\sigma) \\
\leqslant& ({}^{\phi}_{\xi}\Theta^M_{u^+}wf^{\mathbb{p}})^{\frac{1}{\mathbb{p}}}(x;\sigma)({}^{\phi}_{\xi}\Theta^M_{u^+}wg^{\mathbb{q}})^{\frac{1}{\mathbb{q}}}(x;\sigma) \leqslant \left({}^{\phi}_{\xi}\Theta^M_{u^+}S\left(\frac{\mathscr{G}_1 f^{\mathbb{p}}}{\mathscr{F}_1 g^{\mathbb{q}}}\right)wfg\right)(x;\sigma), \quad (41)
\end{aligned}
$$

$$
\begin{aligned}
({}^{\phi}_{\xi}\Theta^M_{u^+}wf^{\mathbb{p}})^{\frac{1}{\mathbb{p}}}(x;\sigma)({}^{\phi}_{\xi}\Theta^M_{u^+}wg^{\mathbb{q}})^{\frac{1}{\mathbb{q}}}(x;\sigma) - ({}^{\phi}_{\xi}\Theta^M_{u^+}wfg)(x;\sigma) \leqslant& ({}^{\phi}_{\xi}\Theta^M_{u^+}wf^{\mathbb{p}})^{\frac{1}{\mathbb{p}}}(x;\sigma) \\
\cdot ({}^{\phi}_{\xi}\Theta^M_{u^+}wg^{\mathbb{q}})^{\frac{1}{\mathbb{q}}}(x;\sigma)\left({}^{\phi}_{\xi}\Theta^M_{u^+}L\left(\frac{wf^{\mathbb{p}}}{\mathscr{F}_1}, \frac{wg^{\mathbb{q}}}{\mathscr{G}_1}\right)\log S\left(\frac{\mathscr{G}_1 f^{\mathbb{p}}}{\mathscr{F}_1 g^{\mathbb{q}}}\right)\right)&(x;\sigma), \quad (42)
\end{aligned}
$$

where $\mathbb{r} = \min\{1/\mathbb{p}, 1/\mathbb{q}\}$, $\mathscr{F}_1 = \left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w f^{\mathbb{p}}\right)(x;\sigma)$, $\mathscr{G}_1 = \left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w g^{\mathbb{q}}\right)(x;\sigma)$, $L(*,\star)$ *and* $S(\cdot)$ *denote the logarithmic mean and Specht's ratio, respectively.*

Proof. Let $\mathcal{A} = w(t)f^{\mathbb{p}}(t)/\mathscr{F}_1$ and $\mathcal{B} = w(t)g^{\mathbb{q}}(t)/\mathscr{G}_1$ in (5), then

$$\frac{w(t)f(t)g(t)}{\mathscr{F}_1^{\frac{1}{\mathbb{p}}}\mathscr{G}_1^{\frac{1}{\mathbb{q}}}} \leqslant S\left(\left(\frac{\mathscr{G}_1 f^{\mathbb{p}}(t)}{\mathscr{F}_1 g^{\mathbb{q}}(t)}\right)^{\mathbb{r}}\right)\frac{w(t)f(t)g(t)}{\mathscr{F}_1^{\frac{1}{\mathbb{p}}}\mathscr{G}_1^{\frac{1}{\mathbb{q}}}} \leqslant \frac{w(t)f^{\mathbb{p}}(t)}{\mathbb{p}\mathscr{F}_1} + \frac{w(t)g^{\mathbb{q}}(t)}{\mathbb{q}\mathscr{G}_1} \leqslant S\left(\frac{\mathscr{G}_1 f^{\mathbb{p}}(t)}{\mathscr{F}_1 g^{\mathbb{q}}(t)}\right)\frac{w(t)f(t)g(t)}{\mathscr{F}_1^{\frac{1}{\mathbb{p}}}\mathscr{G}_1^{\frac{1}{\mathbb{q}}}}. \quad (43)$$

Multiplying simultaneously both sides of (43) by $\aleph^{-1}(x)\aleph(t)\mathscr{M}_x^t\left({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}, \xi, \phi\right)\xi'(t)$ $w(t)$ and integrating the acquired inequality with regard to t from u to x, we claim based on the operator (38)

$$\frac{\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} wfg\right)(x;\sigma)}{\mathscr{F}_1^{\frac{1}{\mathbb{p}}}\mathscr{G}_1^{\frac{1}{\mathbb{q}}}} \leqslant \frac{\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} S\left(\left(\frac{\mathscr{G}_1 f^{\mathbb{p}}}{\mathscr{F}_1 g^{\mathbb{q}}}\right)^{\mathbb{r}}\right) wfg\right)(x;\sigma)}{\mathscr{F}_1^{\frac{1}{\mathbb{p}}}\mathscr{G}_1^{\frac{1}{\mathbb{q}}}} \leqslant \frac{\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} wf^{\mathbb{p}}\right)(x;\sigma)}{\mathbb{p}\mathscr{F}_1} + \frac{\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} wg^{\mathbb{q}}\right)(x;\sigma)}{\mathbb{q}\mathscr{G}_1} \leqslant \frac{\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} S\left(\frac{\mathscr{G}_1 f^{\mathbb{p}}}{\mathscr{F}_1 g^{\mathbb{q}}}\right) wfg\right)(x;\sigma)}{\mathscr{F}_1^{\frac{1}{\mathbb{p}}}\mathscr{G}_1^{\frac{1}{\mathbb{q}}}}. \quad (44)$$

According to the definitions of $\mathscr{F}_1$ and $\mathscr{G}_1$, the inequalities (44) can be rewritten as

$$\frac{\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} wfg\right)(x;\sigma)}{\mathscr{F}_1^{\frac{1}{\mathbb{p}}}\mathscr{G}_1^{\frac{1}{\mathbb{q}}}} \leqslant \frac{\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} S\left(\left(\frac{\mathscr{G}_1 f^{\mathbb{p}}}{\mathscr{F}_1 g^{\mathbb{q}}}\right)^{\mathbb{r}}\right) wfg\right)(x;\sigma)}{\mathscr{F}_1^{\frac{1}{\mathbb{p}}}\mathscr{G}_1^{\frac{1}{\mathbb{q}}}} \leqslant 1 \leqslant \frac{\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} S\left(\frac{\mathscr{G}_1 f^{\mathbb{p}}}{\mathscr{F}_1 g^{\mathbb{q}}}\right) wfg\right)(x;\sigma)}{\mathscr{F}_1^{\frac{1}{\mathbb{p}}}\mathscr{G}_1^{\frac{1}{\mathbb{q}}}}, \quad (45)$$

which are the desired inequalities (41). Let $\mathcal{A} = w(t)f^{\mathbb{p}}(t)/\mathscr{F}_1$ and $\mathcal{B} = w(t)g^{\mathbb{q}}(t)/\mathscr{G}_1$ in (6), then

$$\frac{w(t)f^{\mathbb{p}}(t)}{\mathbb{p}\mathscr{F}_1} + \frac{w(t)g^{\mathbb{q}}(t)}{\mathbb{q}\mathscr{G}_1} - \frac{w(t)f(t)g(t)}{\mathscr{F}_1^{\frac{1}{\mathbb{p}}}\mathscr{G}_1^{\frac{1}{\mathbb{q}}}} \leqslant L\left(\frac{w(t)f^{\mathbb{p}}(t)}{\mathscr{F}_1}, \frac{w(t)g^{\mathbb{q}}(t)}{\mathscr{G}_1}\right)\log S\left(\frac{\mathscr{G}_1 f^{\mathbb{p}}(t)}{\mathscr{F}_1 g^{\mathbb{q}}(t)}\right). \quad (46)$$

Multiplying simultaneously both sides of (46) by $\aleph^{-1}(x)\aleph(t)\mathscr{M}_x^t\left({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}, \xi, \phi\right)\xi'(t)$ $w(t)$ and integrating the achieved inequality in regard to t from u to x, we gain based on the operator (38)

$$\frac{\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} wf^{\mathbb{p}}\right)(x;\sigma)}{\mathbb{p}\mathscr{F}_1} + \frac{\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} wg^{\mathbb{q}}\right)(x;\sigma)}{\mathbb{q}\mathscr{G}_1} - \frac{\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} wfg\right)(x;\sigma)}{\mathscr{F}_1^{\frac{1}{\mathbb{p}}}\mathscr{G}_1^{\frac{1}{\mathbb{q}}}} \leqslant \left({}^{\phi}_{\xi}\Theta^{M}_{u^+} L\left(\frac{wf^{\mathbb{p}}}{\mathscr{F}_1}, \frac{wg^{\mathbb{q}}}{\mathscr{G}_1}\right)\log S\left(\frac{\mathscr{G}_1 f^{\mathbb{p}}}{\mathscr{F}_1 g^{\mathbb{q}}}\right)\right)(x;\sigma), \quad (47)$$

which are the anticipated inequalities (42). This completes the proof. □

Remark 2. *It follows from* (41) *that* $\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wfg\right)(x;\sigma) \leqslant \left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wf^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma)\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wg^{\mathbb{q}}\right)^{\frac{1}{\mathbb{q}}}(x;\sigma)$ *for* $1/\mathbb{p}+1/\mathbb{q}=1$ *with* $\mathbb{p}>1$*, which is Hölder-type inequalities for modified unified generalized FIOs. The reverse of the above inequality holds also when* $0<\mathbb{p}<1$ *and when* $\mathbb{p}<0$ *or* $\mathbb{q}<0$*.*

Theorem 2. *Suppose that* f,g,w *are three continuous positive functions on* $[u,v]$*. Let* $1/\mathbb{p}+1/\mathbb{q}=1$ *satisfying* $\mathbb{p}<0$ *(or* $\mathbb{q}<0$*). Then, for* $x\in[u,v]$*, we have the following FIO inequalities*

$$\begin{aligned}
&\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wf^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma)\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wg^{\mathbb{q}}\right)^{\frac{1}{\mathbb{q}}}(x;\sigma)\\
&\quad\leqslant \left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wf^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma)\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}S\left(\left(\frac{\mathscr{G}_2f^{\mathbb{p}}}{\mathscr{F}_2fg}\right)^{\mathbb{r}}\right)wg^{\mathbb{q}}\right)^{\frac{1}{\mathbb{q}}}(x;\sigma)\leqslant\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wfg\right)^{\mathbb{q}}(x;\sigma)\\
&\qquad\leqslant \left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wf^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma)\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}S\left(\frac{\mathscr{G}_2f^{\mathbb{p}}}{\mathscr{F}_2fg}\right)wg^{\mathbb{q}}\right)^{\frac{1}{\mathbb{q}}}(x;\sigma),
\end{aligned}\tag{48}$$

$$\begin{aligned}
&\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wfg\right)^{\mathbb{q}}(x;\sigma)-\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wf^{\mathbb{p}}\right)^{\frac{\mathbb{q}}{\mathbb{p}}}(x;\sigma)\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wg^{\mathbb{q}}\right)(x;\sigma)\\
&\quad\leqslant\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wfg\right)^{\mathbb{q}}(x;\sigma)\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}L\left(\frac{wf^{\mathbb{p}}}{\mathscr{F}_2},\frac{wfg}{\mathscr{G}_2}\right)\log S\left(\frac{\mathscr{G}_2f^{\mathbb{p}}}{\mathscr{F}_2fg}\right)\right)(x;\sigma),
\end{aligned}\tag{49}$$

where $\mathbb{r}=\min\{\mathbb{q},1-\mathbb{q}\}$ *(or* $\mathbb{r}=\min\{\mathbb{p},1-\mathbb{p}\}$*),* $\mathscr{F}_2=\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wf^{\mathbb{p}}\right)(x;\sigma)$*,* $\mathscr{G}_2=\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wfg\right)(x;\sigma)$*,* $L(*,\star)$ *and* $S(\cdot)$ *denote the logarithmic mean and Specht's ratio, respectively.*

Proof. Suppose that $\mathbb{p}<0$ (if $\mathbb{q}<0$, the idea is really the same). Let $\mathbb{P}=-\mathbb{p}/\mathbb{q}$ and $\mathbb{Q}=1/\mathbb{q}$, then we obtain $1/\mathbb{P}+1/\mathbb{Q}=1$ satisfying $\mathbb{P}>1$ and $\mathbb{Q}>1$. Assume that F,G,w are three continuous positive functions on $[u,v]$ with $\mathbb{R}=\min\{1/\mathbb{P},1/\mathbb{Q}\}$. It follows from (41) and (42) that

$$\begin{aligned}
&\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wFG\right)(x;\sigma)\leqslant\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}S\left(\left(\frac{\mathscr{G}_2F^{\mathbb{P}}}{\mathscr{F}_2G^{\mathbb{Q}}}\right)^{\mathbb{R}}\right)wFG\right)(x;\sigma)\\
&\quad\leqslant\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wF^{\mathbb{P}}\right)^{\frac{1}{\mathbb{P}}}(x;\sigma)\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wG^{\mathbb{Q}}\right)^{\frac{1}{\mathbb{Q}}}(x;\sigma)\leqslant\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}S\left(\frac{\mathscr{G}_2F^{\mathbb{P}}}{\mathscr{F}_2G^{\mathbb{Q}}}\right)wFG\right)(x;\sigma),
\end{aligned}\tag{50}$$

$$\begin{aligned}
&\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wF^{\mathbb{P}}\right)^{\frac{1}{\mathbb{P}}}(x;\sigma)\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wG^{\mathbb{Q}}\right)^{\frac{1}{\mathbb{Q}}}(x;\sigma)-\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wFG\right)(x;\sigma)\leqslant\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wF^{\mathbb{P}}\right)^{\frac{1}{\mathbb{P}}}(x;\sigma)\\
&\quad\cdot\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wG^{\mathbb{Q}}\right)^{\frac{1}{\mathbb{Q}}}(x;\sigma)\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}L\left(\frac{wF^{\mathbb{P}}}{\mathscr{F}_2},\frac{wG^{\mathbb{Q}}}{\mathscr{G}_2}\right)\log S\left(\frac{\mathscr{G}_2F^{\mathbb{P}}}{\mathscr{F}_2G^{\mathbb{Q}}}\right)\right)(x;\sigma),
\end{aligned}\tag{51}$$

where $\mathscr{F}_2=\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wF^{\mathbb{P}}\right)(x;\sigma)$ and $\mathscr{G}_2=\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wG^{\mathbb{q}}\right)(x;\sigma)$.

Letting $F=f^{-\mathbb{q}}$ and $G=f^{\mathbb{q}}g^{\mathbb{q}}$ in the inequalities (50) and (51), we obtain

$$\begin{aligned}
&\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wg^{\mathbb{q}}\right)(x;\sigma)\leqslant\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}S\left(\left(\frac{\mathscr{G}_2f^{\mathbb{p}}}{\mathscr{F}_2fg}\right)^{\mathbb{r}}\right)wg^{\mathbb{q}}\right)(x;\sigma)\\
&\quad\leqslant\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wf^{\mathbb{p}}\right)^{\frac{-\mathbb{q}}{\mathbb{p}}}(x;\sigma)\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wfg\right)^{\mathbb{q}}(x;\sigma)\leqslant\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}S\left(\frac{\mathscr{G}_2f^{\mathbb{p}}}{\mathscr{F}_2fg}\right)wg^{\mathbb{q}}\right)(x;\sigma),
\end{aligned}\tag{52}$$

$$\begin{aligned}
&\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wf^{\mathbb{p}}\right)^{\frac{-\mathbb{q}}{\mathbb{p}}}(x;\sigma)\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wfg\right)^{\mathbb{q}}(x;\sigma)-\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wg^{\mathbb{q}}\right)(x;\sigma)\leqslant\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wf^{\mathbb{p}}\right)^{\frac{-\mathbb{q}}{\mathbb{p}}}(x;\sigma)\\
&\quad\cdot\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wfg\right)^{\mathbb{q}}(x;\sigma)\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}L\left(\frac{wf^{\mathbb{p}}}{\mathscr{F}_2},\frac{wfg}{\mathscr{G}_2}\right)\log S\left(\frac{\mathscr{G}_2f^{\mathbb{p}}}{\mathscr{F}_2fg}\right)\right)(x;\sigma).
\end{aligned}\tag{53}$$

Multiplying simultaneously both sides of (52) and (53) by $\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wf^{\mathbb{p}}\right)^{\frac{\mathbb{q}}{\mathbb{p}}}(x;\sigma)$, we obtain

$$\begin{aligned}
&\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w f^{\mathbb{p}}\right)^{\frac{\mathbb{q}}{\mathbb{p}}}(x;\sigma)\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w g^{\mathbb{q}}\right)(x;\sigma)\\
&\quad\leqslant \left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w f^{\mathbb{p}}\right)^{\frac{\mathbb{q}}{\mathbb{p}}}(x;\sigma)\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} S\left(\left(\frac{\mathscr{G}_2 f^{\mathbb{p}}}{\mathscr{F}_2 f g}\right)^{\mathbb{r}}\right) w g^{\mathbb{q}}\right)(x;\sigma) \leqslant \left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w f g\right)^{\mathbb{q}}(x;\sigma)\\
&\qquad\leqslant \left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w f^{\mathbb{p}}\right)^{\frac{\mathbb{q}}{\mathbb{p}}}(x;\sigma)\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} S\left(\frac{\mathscr{G}_2 f^{\mathbb{p}}}{\mathscr{F}_2 f g}\right) w g^{\mathbb{q}}\right)(x;\sigma), \qquad (54)
\end{aligned}$$

$$\begin{aligned}
&\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w f g\right)^{\mathbb{q}}(x;\sigma) - \left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w f^{\mathbb{p}}\right)^{\frac{\mathbb{q}}{\mathbb{p}}}(x;\sigma)\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w g^{\mathbb{q}}\right)(x;\sigma)\\
&\quad\leqslant \left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w f g\right)^{\mathbb{q}}(x;\sigma)\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} L\left(\frac{w f^{\mathbb{p}}}{\mathscr{F}_2}, \frac{w f g}{\mathscr{G}_2}\right)\log S\left(\frac{\mathscr{G}_2 f^{\mathbb{p}}}{\mathscr{F}_2 f g}\right)\right)(x;\sigma). \qquad (55)
\end{aligned}$$

which are the desired inequalities (48) and (49). This completes the proof. □

By substituting f, g for $f^{\mathbb{p}}, g^{\mathbb{q}}$ in (41) and (48), respectively, we derive the following corollary.

Corollary 1. (a) *Under the assumptions of Theorem 1, inequalities (41) can be rewritten as*

$$\begin{aligned}
&\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w f^{\frac{1}{\mathbb{p}}} g^{\frac{1}{\mathbb{q}}}\right)(x;\sigma) \leqslant \left({}^{\phi}_{\xi}\Theta^{M}_{u^+} S\left(\left(\frac{\mathscr{G}_1^* f}{\mathscr{F}_1^* g}\right)^{\mathbb{r}}\right) w f^{\frac{1}{\mathbb{p}}} g^{\frac{1}{\mathbb{q}}}\right)(x;\sigma)\\
&\quad\leqslant \left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w f\right)^{\frac{1}{\mathbb{p}}}(x;\sigma)\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w g\right)^{\frac{1}{\mathbb{q}}}(x;\sigma) \leqslant \left({}^{\phi}_{\xi}\Theta^{M}_{u^+} S\left(\frac{\mathscr{G}_1^* f}{\mathscr{F}_1^* g}\right) w f^{\frac{1}{\mathbb{p}}} g^{\frac{1}{\mathbb{q}}}\right)(x;\sigma). \qquad (56)
\end{aligned}$$

where $\mathscr{F}_1^* = \left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w f\right)(x;\sigma)$ *and* $\mathscr{G}_1^* = \left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w g\right)(x;\sigma)$.

(b) *Under the conditions of Theorem 2, inequalities (48) can be rewritten as*

$$\begin{aligned}
&\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w f\right)^{\frac{1}{\mathbb{p}}}(x;\sigma)\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w g\right)^{\frac{1}{\mathbb{q}}}(x;\sigma)\\
&\leqslant \left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w f\right)^{\frac{1}{\mathbb{p}}}(x;\sigma)\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} S\left(\left(\frac{\mathscr{G}_2^* f}{\mathscr{F}_2^* f^{\frac{1}{\mathbb{p}}} g^{\frac{1}{\mathbb{q}}}}\right)^{\mathbb{r}}\right) w g\right)^{\frac{1}{\mathbb{q}}}(x;\sigma) \leqslant \left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w f^{\frac{1}{\mathbb{p}}} g^{\frac{1}{\mathbb{q}}}\right)^{\mathbb{q}}(x;\sigma)\\
&\qquad\leqslant \left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w f\right)^{\frac{1}{\mathbb{p}}}(x;\sigma)\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} S\left(\frac{\mathscr{G}_2^* f}{\mathscr{F}_2^* f^{\frac{1}{\mathbb{p}}} g^{\frac{1}{\mathbb{q}}}}\right) h g^{\mathbb{q}}\right)^{\frac{1}{\mathbb{q}}}(x;\sigma), \qquad (57)
\end{aligned}$$

where $\mathscr{F}_2^* = \left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w f\right)(x;\sigma)$ *and* $\mathscr{G}_2^* = \left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w f^{\frac{1}{\mathbb{p}}} g^{\frac{1}{\mathbb{q}}}\right)(x;\sigma)$.

Theorem 3. *Let* $1/\mathbb{p} + 1/\mathbb{q} = 1$ *with* $\mathbb{p} > 1$. *Assume that* f, g, w *are three continuous positive functions on* $[u, v]$ *satisfying* $0 < \mathbb{m} \leqslant f^{\mathbb{p}}(t)/g^{\mathbb{q}}(t) \leqslant \mathbb{M}$ *for all* $t \in [u, v]$. *Then, for* $x \in [u, v]$, *we have the following fractional integral inequalities*

$$\left(\frac{\mathbb{m}}{\mathbb{M}}\right)^{\frac{1}{\mathbb{p}\mathbb{q}}}\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w f g\right)(x;\sigma) \leqslant \left({}^{\phi}_{\xi}\Theta^{M}_{u^+} S\left(\frac{\mathscr{G}_1 f^{\mathbb{p}}}{\mathscr{F}_1 g^{\mathbb{q}}}\right) w f g\right)(x;\sigma), \qquad (58)$$

where $S(\cdot)$ *denotes the Specht's ratio,* $\mathscr{F}_1 = \left({}^{\phi}_{\xi}\Theta^{M}_{u^+} h f^{\mathbb{p}}\right)(x;\sigma)$ *and* $\mathscr{G}_1 = \left({}^{\phi}_{\xi}\Theta^{M}_{u^+} h g^{\mathbb{q}}\right)(x;\sigma)$.

Proof. It follows from (41) that

$$\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w f^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma)\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w g^{\mathbb{q}}\right)^{\frac{1}{\mathbb{q}}}(x;\sigma) \leqslant \left({}^{\phi}_{\xi}\Theta^{M}_{u^+} S\left(\frac{\mathscr{G}_1 f^{\mathbb{p}}}{\mathscr{F}_1 g^{\mathbb{q}}}\right) w f g\right)(x;\sigma). \qquad (59)$$

Since $0 < \mathbb{m} \leqslant f^{\mathbb{p}}(t)/g^{\mathbb{q}}(t) \leqslant \mathbb{M}$, then we observe

$$\mathbb{M}^{-1/\mathbb{p}} f(t) \leqslant g^{\mathbb{q}/\mathbb{p}}(t) \text{ and } \mathbb{m}^{1/\mathbb{q}} g(t) \leqslant f^{\mathbb{p}/\mathbb{q}}(t). \tag{60}$$

Multiplying the above inequality (60) by $g(t)$ and $f(t)$, respectively, then we obtain

$$\mathbb{M}^{-1/\mathbb{p}} f(t)g(t) \leqslant g^{\mathbb{q}/\mathbb{p}+1}(t) = g^{\mathbb{q}}(t) \text{ and } \mathbb{m}^{1/\mathbb{q}} f(t)g(t) \leqslant f^{\mathbb{p}/\mathbb{q}+1}(t) = f^{\mathbb{p}}(t). \tag{61}$$

Multiplying simultaneously the inequalities (61) by $\aleph^{-1}(x)\aleph(t)\mathscr{M}_x^t({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}, \xi, \phi)$ $\xi'(t)w(t)$ and integrating the acquired inequalities with regard to t from u to x, we claim based on the operator (38)

$$\begin{aligned} \frac{1}{\mathbb{M}^{\frac{1}{\mathbb{p}}}} \big({}^{\phi}_{\xi}\Theta^M_{u^+} wfg\big)(x;\sigma) &\leqslant \big({}^{\phi}_{\xi}\Theta^M_{u^+} wg^{\mathbb{q}}\big)(x;\sigma), \\ \mathbb{m}^{\frac{1}{\mathbb{q}}} \big({}^{\phi}_{\xi}\Theta^M_{u^+} wfg\big)(x;\sigma) &\leqslant \big({}^{\phi}_{\xi}\Theta^M_{u^+} wf^{\mathbb{p}}\big)(x;\sigma). \end{aligned} \tag{62}$$

Combining (59) and (62) yields the following inequality

$$\left(\frac{\mathbb{m}}{\mathbb{M}}\right)^{\frac{1}{\mathbb{p}\mathbb{q}}} \big({}^{\phi}_{\xi}\Theta^M_{u^+} wfg\big)(x;\sigma) \leqslant \left({}^{\phi}_{\xi}\Theta^M_{u^+} S\left(\frac{\mathscr{G}_1 f^{\mathbb{p}}}{\mathscr{F}_1 g^{\mathbb{q}}}\right) wfg\right)(x;\sigma), \tag{63}$$

which is the desired inequality (58). This completes the proof. □

Theorem 4. *Let $1/\mathbb{p} + 1/\mathbb{q} = 1$ with $\mathbb{p} > 1$. Suppose that f, g, w are three continuous positive functions on $[u, v]$ satisfying $0 < \mathbb{m} \leqslant f(t)/g(t) \leqslant \mathbb{M}$ for all $t \in [u, v]$. Then, for $x \in [u, v]$, we have the following FIO inequalities*

$$\left(\frac{\mathbb{m}}{\mathbb{M}}\right)^{\frac{1}{\mathbb{p}\mathbb{q}}} \left({}^{\phi}_{\xi}\Theta^M_{u^+} wf^{\frac{1}{\mathbb{p}}} g^{\frac{1}{\mathbb{q}}}\right)(x;\sigma) \leqslant \left({}^{\phi}_{\xi}\Theta^M_{u^+} S\left(\frac{\mathscr{G}_1^* f}{\mathscr{F}_1^* g}\right) wf^{\frac{1}{\mathbb{p}}} g^{\frac{1}{\mathbb{q}}}\right)(x;\sigma), \tag{64}$$

$$\frac{\mathbb{m}^{\frac{1}{\mathbb{p}^2}}}{\mathbb{M}^{\frac{1}{\mathbb{q}^2}}} \left({}^{\phi}_{\xi}\Theta^M_{u^+} wf^{\frac{1}{\mathbb{q}}} g^{\frac{1}{\mathbb{p}}}\right)(x;\sigma) \leqslant \left({}^{\phi}_{\xi}\Theta^M_{u^+} S\left(\frac{\mathscr{G}_1^* f}{\mathscr{F}_1^* g}\right) wf^{\frac{1}{\mathbb{p}}} g^{\frac{1}{\mathbb{q}}}\right)(x;\sigma), \tag{65}$$

where $S(\cdot)$ denotes the Specht's ratio, $\mathscr{F}_1^* = \big({}^{\phi}_{\xi}\Theta^M_{u^+} wf\big)(x;\sigma)$ and $\mathscr{G}_1^* = \big({}^{\phi}_{\xi}\Theta^M_{u^+} wg\big)(x;\sigma)$.

Proof. It follows from (56) that

$$\big({}^{\phi}_{\xi}\Theta^M_{u^+} wf\big)^{\frac{1}{\mathbb{p}}}(x;\sigma)\big({}^{\phi}_{\xi}\Theta^M_{u^+} wg\big)^{\frac{1}{\mathbb{q}}}(x;\sigma) \leqslant \left({}^{\phi}_{\xi}\Theta^M_{u^+} S\left(\frac{\mathscr{G}_1 f}{\mathscr{F}_1 g}\right) wf^{\frac{1}{\mathbb{p}}} g^{\frac{1}{\mathbb{q}}}\right)(x;\sigma). \tag{66}$$

Since $0 < \mathbb{m} \leqslant f(t)/g(t) \leqslant \mathbb{M}$, then we observe

$$\frac{1}{\mathbb{M}^{\frac{1}{\mathbb{p}}}} f^{\frac{1}{\mathbb{p}}}(t) g^{\frac{1}{\mathbb{q}}}(t) \leqslant g^{\frac{1}{\mathbb{p}}+\frac{1}{\mathbb{q}}}(t) = g(t) \text{ and } \mathbb{m}^{\frac{1}{\mathbb{q}}} f^{\frac{1}{\mathbb{p}}}(t) g^{\frac{1}{\mathbb{q}}}(t) \leqslant f^{\frac{1}{\mathbb{p}}+\frac{1}{\mathbb{q}}}(t) = f(t), \tag{67}$$

$$\frac{1}{\mathbb{M}^{\frac{1}{\mathbb{q}}}} f^{\frac{1}{\mathbb{q}}}(t) g^{\frac{1}{\mathbb{p}}}(t) \leqslant g^{\frac{1}{\mathbb{p}}+\frac{1}{\mathbb{q}}}(t) = g(t) \text{ and } \mathbb{m}^{\frac{1}{\mathbb{p}}} f^{\frac{1}{\mathbb{q}}}(t) g^{\frac{1}{\mathbb{p}}}(t) \leqslant f^{\frac{1}{\mathbb{p}}+\frac{1}{\mathbb{q}}}(t) = f(t). \tag{68}$$

Multiplying simultaneously the inequalities (67) and (68) by $\aleph^{-1}(x)\aleph(t)\mathscr{M}_x^t({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu},$ $\xi, \phi)\xi'(t)w(t)$ and integrating the acquired inequalities in regard to t from u to x, we gain based on the operator (38)

$$\begin{aligned} \frac{1}{\mathbb{M}^{\frac{1}{\mathbb{p}}}} \big({}^{\phi}_{\xi}\Theta^M_{u^+} wf^{\frac{1}{\mathbb{p}}} g^{\frac{1}{\mathbb{q}}}\big)(x;\sigma) &\leqslant \big({}^{\phi}_{\xi}\Theta^M_{u^+} wg\big)(x;\sigma), \\ \mathbb{m}^{\frac{1}{\mathbb{q}}} \big({}^{\phi}_{\xi}\Theta^M_{u^+} wf^{\frac{1}{\mathbb{p}}} g^{\frac{1}{\mathbb{q}}}\big)(x;\sigma) &\leqslant \big({}^{\phi}_{\xi}\Theta^M_{u^+} wf\big)(x;\sigma), \end{aligned} \tag{69}$$

$$\begin{aligned}\frac{1}{\mathbb{M}^{\frac{1}{\mathbb{q}}}}\big({}_{\xi}^{\phi}\Theta_{u^+}^{M}wf^{\frac{1}{\mathbb{q}}}g^{\frac{1}{\mathbb{p}}}\big)(x;\sigma)\leqslant\big({}_{\xi}^{\phi}\Theta_{u^+}^{M}wg\big)(x;\sigma),\\ \mathbb{m}^{\frac{1}{\mathbb{p}}}\big({}_{\xi}^{\phi}\Theta_{u^+}^{M}wf^{\frac{1}{\mathbb{q}}}g^{\frac{1}{\mathbb{p}}}\big)(x;\sigma)\leqslant\big({}_{\xi}^{\phi}\Theta_{u^+}^{M}wf\big)(x;\sigma),\end{aligned}\tag{70}$$

Combining (66), (69) and (70) yields the following inequalities

$$\left(\frac{\mathbb{m}}{\mathbb{M}}\right)^{\frac{1}{\mathbb{pq}}}\left({}_{\xi}^{\phi}\Theta_{u^+}^{M}wf^{\frac{1}{\mathbb{p}}}g^{\frac{1}{\mathbb{q}}}\right)(x;\sigma)\leqslant\left({}_{\xi}^{\phi}\Theta_{u^+}^{M}S\left(\frac{\mathscr{G}_1^*f}{\mathscr{F}_1^*g}\right)wf^{\frac{1}{\mathbb{p}}}g^{\frac{1}{\mathbb{q}}}\right)(x;\sigma),\tag{71}$$

$$\frac{\mathbb{m}^{\frac{1}{\mathbb{p}^2}}}{\mathbb{M}^{\frac{1}{\mathbb{q}^2}}}\left({}_{\xi}^{\phi}\Theta_{u^+}^{M}wf^{\frac{1}{\mathbb{q}}}g^{\frac{1}{\mathbb{p}}}\right)(x;\sigma)\leqslant\left({}_{\xi}^{\phi}\Theta_{u^+}^{M}S\left(\frac{\mathscr{G}_1^*f}{\mathscr{F}_1^*g}\right)wf^{\frac{1}{\mathbb{p}}}g^{\frac{1}{\mathbb{q}}}\right)(x;\sigma),\tag{72}$$

which are the expected inequalities (64) and (65). This completes the proof. □

Theorem 5. *Let $\hbar,\ell>0$, $1/\mathbb{p}+1/\mathbb{q}=1$ with $\mathbb{p}>1$. Assume that f,g,w are three continuous positive functions on $[u,v]$ satisfying $0<\mathbb{m}\leqslant f^{\hbar}(t)/g^{\ell}(t)\leqslant\mathbb{M}$ for all $t\in[u,v]$. Then, for $x\in[u,v]$, we achieve the following FIO inequalities*

$$\begin{aligned}&\big({}_{\xi}^{\phi}\Theta_{u^+}^{M}wf^{\frac{\hbar}{\mathbb{p}}}g^{\frac{\ell}{\mathbb{q}}}\big)(x;\sigma)\leqslant\big({}_{\xi}^{\phi}\Theta_{u^+}^{M}wf^{\hbar}\big)^{\frac{1}{\mathbb{p}}}(x;\sigma)\big({}_{\xi}^{\phi}\Theta_{u^+}^{M}wg^{\ell}\big)^{\frac{1}{\mathbb{q}}}(x;\sigma)\\ &\leqslant\left(\frac{\mathbb{M}}{\mathbb{m}}\right)^{\frac{1}{\mathbb{pq}}}\big({}_{\xi}^{\phi}\Theta_{u^+}^{M}wf^{\frac{\hbar}{\mathbb{p}}}g^{\frac{\ell}{\mathbb{q}}}\big)(x;\sigma)\leqslant\left(\frac{\mathbb{M}}{\mathbb{m}}\right)^{\frac{1}{\mathbb{pq}}}\big({}_{\xi}^{\phi}\Theta_{u^+}^{M}wf^{\hbar}\big)^{\frac{1}{\mathbb{p}}}(x;\sigma)\big({}_{\xi}^{\phi}\Theta_{u^+}^{M}wg^{\ell}\big)^{\frac{1}{\mathbb{q}}}(x;\sigma),\end{aligned}\tag{73}$$

$$\begin{aligned}&\big({}_{\xi}^{\phi}\Theta_{u^+}^{M}wf^{\frac{\hbar}{\mathbb{p}}}g^{\frac{\ell}{\mathbb{q}}}\big)(x;\sigma)\leqslant\big({}_{\xi}^{\phi}\Theta_{u^+}^{M}wf^{\hbar}\big)^{\frac{1}{\mathbb{p}}}(x;\sigma)\big({}_{\xi}^{\phi}\Theta_{u^+}^{M}wg^{\ell}\big)^{\frac{1}{\mathbb{q}}}(x;\sigma)\\ &\qquad\leqslant\frac{\mathbb{M}^{\frac{1}{\mathbb{p}^2}}}{\mathbb{m}^{\frac{1}{\mathbb{q}^2}}}\big({}_{\xi}^{\phi}\Theta_{u^+}^{M}wf^{\frac{\hbar}{\mathbb{q}}}g^{\frac{\ell}{\mathbb{p}}}\big)(x;\sigma)\leqslant\frac{\mathbb{M}^{\frac{1}{\mathbb{p}^2}}}{\mathbb{m}^{\frac{1}{\mathbb{q}^2}}}\big({}_{\xi}^{\phi}\Theta_{u^+}^{M}wf^{\hbar}\big)^{\frac{1}{\mathbb{q}}}(x;\sigma)\big({}_{\xi}^{\phi}\Theta_{u^+}^{M}wg^{\ell}\big)^{\frac{1}{\mathbb{p}}}(x;\sigma),\end{aligned}\tag{74}$$

$$\begin{aligned}\big({}_{\xi}^{\phi}\Theta_{u^+}^{M}wf^{\frac{\hbar}{\mathbb{p}}}\big)^{\mathbb{p}}(x;\sigma)\big({}_{\xi}^{\phi}\Theta_{u^+}^{M}wg^{\frac{\ell}{\mathbb{q}}}\big)^{\mathbb{q}}(x;\sigma)\leqslant\frac{\mathbb{M}}{\mathbb{m}}\big({}_{\xi}^{\phi}\Theta_{u^+}^{M}w(f^{\hbar}g^{\ell})^{\frac{1}{2\mathbb{p}}}\big)^{\mathbb{p}}(x;\sigma)\\ \cdot\big({}_{\xi}^{\phi}\Theta_{u^+}^{M}w(f^{\hbar}g^{\ell})^{\frac{1}{2\mathbb{q}}}\big)^{\mathbb{q}}(x;\sigma).\end{aligned}\tag{75}$$

Proof. It follows from Remark 2 that

$$\big({}_{\xi}^{\phi}\Theta_{u^+}^{M}wf^{\frac{\hbar}{\mathbb{p}}}g^{\frac{\ell}{\mathbb{q}}}\big)(x;\sigma)\leqslant\big({}_{\xi}^{\phi}\Theta_{u^+}^{M}wf^{\hbar}\big)^{\frac{1}{\mathbb{p}}}(x;\sigma)\big({}_{\xi}^{\phi}\Theta_{u^+}^{M}wg^{\ell}\big)^{\frac{1}{\mathbb{q}}}(x;\sigma).\tag{76}$$

Since $0<\mathbb{m}\leqslant f^{\hbar}(t)/g^{\ell}(t)\leqslant\mathbb{M}$, then we observe

$$f^{\hbar}(t)=f^{\frac{\hbar}{\mathbb{p}}+\frac{\hbar}{\mathbb{q}}}(t)=\mathbb{M}^{\frac{1}{\mathbb{q}}}f^{\frac{\hbar}{\mathbb{p}}}(t)g^{\frac{\ell}{\mathbb{q}}}(t),\quad g^{\ell}(t)=g^{\frac{\ell}{\mathbb{p}}+\frac{\ell}{\mathbb{q}}}(t)\leqslant\mathbb{m}^{-\frac{1}{\mathbb{p}}}f^{\frac{\hbar}{\mathbb{p}}}(t)g^{\frac{\ell}{\mathbb{q}}}(t),\tag{77}$$

$$f^{\hbar}(t)=f^{\frac{\hbar}{\mathbb{p}}+\frac{\hbar}{\mathbb{q}}}(t)=\mathbb{M}^{\frac{1}{\mathbb{p}}}f^{\frac{\hbar}{\mathbb{q}}}(t)g^{\frac{\ell}{\mathbb{p}}}(t),\quad g^{\ell}(t)=g^{\frac{\ell}{\mathbb{p}}+\frac{\ell}{\mathbb{q}}}(t)\leqslant\mathbb{m}^{-\frac{1}{\mathbb{q}}}f^{\frac{\hbar}{\mathbb{q}}}(t)g^{\frac{\ell}{\mathbb{p}}}(t).\tag{78}$$

Substituting (77) and (78) into (76) and using the Hölder-type inequalities, we obtain the desired inequalities (73) and (74). Also since $0<\mathbb{m}\leqslant f^{\hbar}(t)/g^{\ell}(t)\leqslant\mathbb{M}$, then we have the following inequalities

$$\mathbb{m}+1\leqslant\frac{f^{\hbar}(t)+g^{\ell}(t)}{g^{\ell}(t)}\leqslant\mathbb{M}+1,\quad\frac{\mathbb{M}+1}{\mathbb{M}}\leqslant\frac{f^{\hbar}(t)+g^{\ell}(t)}{f^{\hbar}(t)}\leqslant\frac{\mathbb{m}+1}{\mathbb{m}}.\tag{79}$$

That is,

$$f^{\hbar}(t)\leqslant\frac{\mathbb{M}}{\mathbb{M}+1}(f^{\hbar}(t)+g^{\ell}(t)),\quad g^{\ell}(t)\leqslant\frac{1}{\mathbb{m}+1}(f^{\hbar}(t)+g^{\ell}(t)),\tag{80}$$

$$(f^{\hbar}(t)+g^{\ell}(t))^2 \leqslant \frac{(\mathbb{m}+1)(\mathbb{M}+1)}{\mathbb{m}} f^{\hbar}(t)g^{\ell}(t)$$
$$\Rightarrow f^{\hbar}(t)+g^{\ell}(t) \leqslant \left(\frac{(\mathbb{m}+1)(\mathbb{M}+1)}{\mathbb{m}}\right)^{\frac{1}{2}} (f^{\hbar}(t)g^{\ell}(t))^{\frac{1}{2}}. \quad (81)$$

It follows from (80) and (81) that based on the operator (38)

$$\begin{aligned}
&\big({}^{\phi}_{\xi}\Theta^{M}_{u^+} w f^{\frac{\hbar}{\mathbb{p}}}\big)^{\mathbb{p}}(x;\sigma)\big({}^{\phi}_{\xi}\Theta^{M}_{u^+} w g^{\frac{\ell}{\mathbb{q}}}\big)^{\mathbb{q}}(x;\sigma)\\
&\leqslant \frac{\mathbb{M}}{\mathbb{M}+1}\big({}^{\phi}_{\xi}\Theta^{M}_{u^+} w (f^{\hbar}+g^{\ell})^{\frac{1}{\mathbb{p}}}\big)^{\mathbb{p}}(x;\sigma)\frac{1}{\mathbb{m}+1}\big({}^{\phi}_{\xi}\Theta^{M}_{u^+} w (f^{\hbar}+g^{\ell})^{\frac{1}{\mathbb{q}}}\big)^{\mathbb{q}}(x;\sigma)\\
&\leqslant \frac{\mathbb{M}}{\mathbb{m}}\big({}^{\phi}_{\xi}\Theta^{M}_{u^+} w (f^{\hbar}g^{\ell})^{\frac{1}{2\mathbb{p}}}\big)^{\mathbb{p}}(x;\sigma)\big({}^{\phi}_{\xi}\Theta^{M}_{u^+} w (f^{\hbar}g^{\ell})^{\frac{1}{2\mathbb{q}}}\big)^{\mathbb{q}}(x;\sigma), \quad (82)
\end{aligned}$$

which implies the desired inequality (75). The proof of Theorem 5 is completed. □

When $\hbar=\ell=1$, from Theorem 5, we have following corollary.

Corollary 2. *For $1/\mathbb{p}+1/\mathbb{q}=1$ with $\mathbb{p}>1$. Suppose that f,g,w are three continuous positive functions on $[u,v]$ satisfying $0<\mathbb{m}\leqslant f(t)/g(t)\leqslant\mathbb{M}$ for all $t\in[u,v]$. Then, for $x\in[u,v]$, we have the following FIO inequalities*

$$\begin{aligned}
\big({}^{\phi}_{\xi}\Theta^{M}_{u^+} w f\big)^{\frac{1}{\mathbb{p}}}(x;\sigma)\big({}^{\phi}_{\xi}\Theta^{M}_{u^+} w g\big)^{\frac{1}{\mathbb{q}}}(x;\sigma) &\leqslant \left(\frac{\mathbb{M}}{\mathbb{m}}\right)^{\frac{1}{\mathbb{p}\mathbb{q}}}\big({}^{\phi}_{\xi}\Theta^{M}_{u^+} w f^{\frac{1}{\mathbb{p}}} g^{\frac{1}{\mathbb{q}}}\big)(x;\sigma)\\
&\leqslant \left(\frac{\mathbb{M}}{\mathbb{m}}\right)^{\frac{1}{\mathbb{p}\mathbb{q}}}\big({}^{\phi}_{\xi}\Theta^{M}_{u^+} w f\big)^{\frac{1}{\mathbb{p}}}(x;\sigma)\big({}^{\phi}_{\xi}\Theta^{M}_{u^+} w g\big)^{\frac{1}{\mathbb{q}}}(x;\sigma), \quad (83)
\end{aligned}$$

$$\begin{aligned}
\big({}^{\phi}_{\xi}\Theta^{M}_{u^+} w f\big)^{\frac{1}{\mathbb{p}}}(x;\sigma)\big({}^{\phi}_{\xi}\Theta^{M}_{u^+} w g\big)^{\frac{1}{\mathbb{q}}}(x;\sigma) &\leqslant \frac{\mathbb{M}^{\frac{1}{\mathbb{p}^2}}}{\mathbb{m}^{\frac{1}{\mathbb{q}^2}}}\big({}^{\phi}_{\xi}\Theta^{M}_{u^+} w f^{\frac{1}{\mathbb{q}}} g^{\frac{1}{\mathbb{p}}}\big)(x;\sigma)\\
&\leqslant \frac{\mathbb{M}^{\frac{1}{\mathbb{p}^2}}}{\mathbb{m}^{\frac{1}{\mathbb{q}^2}}}\big({}^{\phi}_{\xi}\Theta^{M}_{u^+} w f\big)^{\frac{1}{\mathbb{q}}}(x;\sigma)\big({}^{\phi}_{\xi}\Theta^{M}_{u^+} w g\big)^{\frac{1}{\mathbb{p}}}(x;\sigma), \quad (84)
\end{aligned}$$

$$\begin{aligned}
\big({}^{\phi}_{\xi}\Theta^{M}_{u^+} w f^{\frac{1}{\mathbb{p}}}\big)^{\mathbb{p}}(x;\sigma)\big({}^{\phi}_{\xi}\Theta^{M}_{u^+} w g^{\frac{1}{\mathbb{q}}}\big)^{\mathbb{q}}(x;\sigma) &\leqslant \frac{\mathbb{M}}{\mathbb{m}}\big({}^{\phi}_{\xi}\Theta^{M}_{u^+} w (fg)^{\frac{1}{2\mathbb{p}}}\big)^{\mathbb{p}}(x;\sigma)\\
&\cdot\big({}^{\phi}_{\xi}\Theta^{M}_{u^+} w (fg)^{\frac{1}{2\mathbb{q}}}\big)^{\mathbb{q}}(x;\sigma). \quad (85)
\end{aligned}$$

When $\hbar=\mathbb{p}$ and $\ell=\mathbb{q}$, from Theorem 5, we gain following corollary.

Corollary 3. *For $1/\mathbb{p}+1/\mathbb{q}=1$ with $\mathbb{p}>1$. Suppose that f,g,w are three continuous positive functions on $[u,v]$ satisfying $0<\mathbb{m}\leqslant f^{\mathbb{p}}(t)/g^{\mathbb{q}}(t)\leqslant\mathbb{M}$ for all $t\in[u,v]$. Then, for $x\in[u,v]$, we have the following FIO inequalities*

$$\begin{aligned}
\big({}^{\phi}_{\xi}\Theta^{M}_{u^+} w f^{\mathbb{p}}\big)^{\frac{1}{\mathbb{p}}}(x;\sigma)\big({}^{\phi}_{\xi}\Theta^{M}_{u^+} w g^{\mathbb{q}}\big)^{\frac{1}{\mathbb{q}}}(x;\sigma) &\leqslant \left(\frac{\mathbb{M}}{\mathbb{m}}\right)^{\frac{1}{\mathbb{p}\mathbb{q}}}\big({}^{\phi}_{\xi}\Theta^{M}_{u^+} w fg\big)(x;\sigma)\\
&\leqslant \left(\frac{\mathbb{M}}{\mathbb{m}}\right)^{\frac{1}{\mathbb{p}\mathbb{q}}}\big({}^{\phi}_{\xi}\Theta^{M}_{u^+} w f^{\mathbb{p}}\big)^{\frac{1}{\mathbb{p}}}(x;\sigma)\big({}^{\phi}_{\xi}\Theta^{M}_{u^+} w g^{\mathbb{q}}\big)^{\frac{1}{\mathbb{q}}}(x;\sigma), \quad (86)
\end{aligned}$$

$$\begin{aligned}
\big({}^{\phi}_{\xi}\Theta^{M}_{u^+} w f^{\mathbb{p}}\big)^{\frac{1}{\mathbb{p}}}(x;\sigma)\big({}^{\phi}_{\xi}\Theta^{M}_{u^+} w g^{\mathbb{q}}\big)^{\frac{1}{\mathbb{q}}}(x;\sigma) &\leqslant \frac{\mathbb{M}^{\frac{1}{\mathbb{p}^2}}}{\mathbb{m}^{\frac{1}{\mathbb{q}^2}}}\big({}^{\phi}_{\xi}\Theta^{M}_{u^+} w fg\big)(x;\sigma)\\
&\leqslant \frac{\mathbb{M}^{\frac{1}{\mathbb{p}^2}}}{\mathbb{m}^{\frac{1}{\mathbb{q}^2}}}\big({}^{\phi}_{\xi}\Theta^{M}_{u^+} w f^{\mathbb{p}}\big)^{\frac{1}{\mathbb{q}}}(x;\sigma)\big({}^{\phi}_{\xi}\Theta^{M}_{u^+} w g^{\mathbb{q}}\big)^{\frac{1}{\mathbb{p}}}(x;\sigma), \quad (87)
\end{aligned}$$

$$\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wf\right)^{\mathbb{p}}(x;\sigma)\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wg\right)^{\mathbb{q}}(x;\sigma) \leqslant \frac{\mathbb{M}}{\mathbb{m}}\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w(f^{\mathbb{p}}g^{\mathbb{q}})^{\frac{1}{2\mathbb{p}}}\right)^{\mathbb{p}}(x;\sigma)$$
$$\cdot\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w(f^{\mathbb{p}}g^{\mathbb{q}})^{\frac{1}{2\mathbb{q}}}\right)^{\mathbb{q}}(x;\sigma). \quad (88)$$

Theorem 6. *For $\kappa, \vartheta, \mathbb{p}, \mathbb{q}, \mathbb{p}', \mathbb{q}' > 0$. Suppose that f, g, w are three continuous positive functions on $[u,v]$ satisfying $0 < \kappa < \mathbb{m} \leqslant \vartheta f(t)/g(t) \leqslant \mathbb{M}$ for all $t \in [u,v]$. Then, for $x \in [u,v]$, we have the following FIO inequality*

$$\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wf^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma)\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wg^{\mathbb{q}}\right)^{\frac{1}{\mathbb{q}}}(x;\sigma) \leqslant \frac{\mathbb{M}}{\mathbb{m}}\left(\frac{\vartheta}{\mathbb{m}}\right)^{\frac{2\mathbb{p}'}{\mathbb{p}'+\mathbb{q}'}}(\mathbb{m}+\kappa)^{\frac{\mathbb{p}'-\mathbb{q}'}{\mathbb{p}'+\mathbb{q}'}}(\mathbb{M}+\kappa)^{\frac{\mathbb{q}'-\mathbb{p}'}{\mathbb{p}'+\mathbb{q}'}}$$
$$\cdot\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w(f^{\mathbb{p}'}g^{\mathbb{q}'})^{\frac{\mathbb{p}}{\mathbb{p}'+\mathbb{q}'}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma)\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w(f^{\mathbb{p}'}g^{\mathbb{q}'})^{\frac{\mathbb{q}}{\mathbb{p}'+\mathbb{q}'}}\right)^{\frac{1}{\mathbb{q}}}(x;\sigma). \quad (89)$$

Proof. Since $0 < \kappa < \mathbb{m} \leqslant \vartheta f(t)/g(t) \leqslant \mathbb{M}$ for all $t \in [a,b]$, we have

$$\mathbb{m}+\kappa \leqslant \frac{\vartheta f(t)+\kappa g(t)}{g(t)} \leqslant \mathbb{M}+\kappa, \quad \frac{\mathbb{M}+\kappa}{\mathbb{M}} \leqslant \frac{\vartheta f(t)+\kappa g(t)}{\vartheta f(t)} \leqslant \frac{\mathbb{m}+\kappa}{\mathbb{m}}. \quad (90)$$

From the left inequalities of (90), we can obtain

$$\vartheta\left(\frac{\mathbb{M}+\kappa}{\mathbb{M}}\right)\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wf^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma) \leqslant \left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w(\vartheta f+\kappa g)^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma), \quad (91)$$
$$(\mathbb{m}+\kappa)\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wg^{\mathbb{q}}\right)^{\frac{1}{\mathbb{q}}}(x;\sigma) \leqslant \left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w(\vartheta f+\kappa g)^{\mathbb{q}}\right)^{\frac{1}{\mathbb{q}}}(x;\sigma). \quad (92)$$

Multiplying these inequalities (91) and (92), we obtain

$$\frac{\vartheta}{\mathbb{M}}(\mathbb{m}+\kappa)(\mathbb{M}+\kappa)\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wf^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma)\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wg^{\mathbb{q}}\right)^{\frac{1}{\mathbb{q}}}(x;\sigma)$$
$$\leqslant \left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w(\vartheta f+\kappa g)^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma)\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w(\vartheta f+\kappa g)^{\mathbb{q}}\right)^{\frac{1}{\mathbb{q}}}(x;\sigma). \quad (93)$$

On the other hand, from the right inequalities of (90), we have

$$(\vartheta f(t)+\kappa g(t))^{\mathbb{p}'} \leqslant \left(\frac{\vartheta}{\mathbb{m}}(\mathbb{m}+\kappa)\right)^{\mathbb{p}'} f^{\mathbb{p}'}(t), \quad (\vartheta f(t)+\kappa g(t))^{\mathbb{q}'} \leqslant (\mathbb{M}+\kappa)^{\mathbb{q}'} g^{\mathbb{q}'}(t). \quad (94)$$

By multiplying the inequalities (94) and raising the resulting inequality to power $1/(\mathbb{p}'+\mathbb{q}')$, we achieve

$$\vartheta f(t)+\kappa g(t) \leqslant \left(\frac{\vartheta}{\mathbb{m}}(\mathbb{m}+\kappa)\right)^{\frac{\mathbb{p}'}{\mathbb{p}'+\mathbb{q}'}}(\mathbb{M}+\kappa)^{\frac{\mathbb{q}'}{\mathbb{p}'+\mathbb{q}'}}\left(f^{\mathbb{p}'}(t)g^{\mathbb{q}'}(t)\right)^{\frac{1}{\mathbb{p}'+\mathbb{q}'}}. \quad (95)$$

Based on the operator (38) and inequality (95), we derive

$$\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w(\vartheta f+\kappa g)^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma) \leqslant \left(\frac{\vartheta}{\mathbb{m}}(\mathbb{m}+\kappa)\right)^{\frac{\mathbb{p}'}{\mathbb{p}'+\mathbb{q}'}}(\mathbb{M}+\kappa)^{\frac{\mathbb{q}'}{\mathbb{p}'+\mathbb{q}'}}$$
$$\cdot\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w(f^{\mathbb{p}'}g^{\mathbb{q}'})^{\frac{\mathbb{p}}{\mathbb{p}'+\mathbb{q}'}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma), \quad (96)$$

$$\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w(\vartheta f+\kappa g)^{\mathbb{q}}\right)^{\frac{1}{\mathbb{q}}}(x;\sigma) \leqslant \left(\frac{\vartheta}{\mathbb{m}}(\mathbb{m}+\kappa)\right)^{\frac{\mathbb{p}'}{\mathbb{p}'+\mathbb{q}'}}(\mathbb{M}+\kappa)^{\frac{\mathbb{q}'}{\mathbb{p}'+\mathbb{q}'}}$$
$$\cdot\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w(f^{\mathbb{p}'}g^{\mathbb{q}'})^{\frac{\mathbb{q}}{\mathbb{p}'+\mathbb{q}'}}\right)^{\frac{1}{\mathbb{q}}}(x;\sigma). \quad (97)$$

Multiplying these inequalities (96) and (97), we obtain

$$\big({}^{\phi}_{\xi}\Theta^{M}_{u^+}w(\vartheta f+\kappa g)^{\mathbb{p}}\big)^{\frac{1}{\mathbb{p}}}(x;\sigma)\big({}^{\phi}_{\xi}\Theta^{M}_{u^+}w(\vartheta f+\kappa g)^{\mathbb{p}}\big)^{\frac{1}{\mathbb{p}}}(x;\sigma)\leqslant\left(\frac{\vartheta}{\mathbb{m}}(\mathbb{m}+\kappa)\right)^{\frac{2\mathbb{p}'}{\mathbb{p}'+\mathbb{q}'}}$$
$$\cdot(\mathbb{M}+\kappa)^{\frac{2\mathbb{q}'}{\mathbb{p}'+\mathbb{q}'}}\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}w(f^{\mathbb{p}'}g^{\mathbb{q}'})^{\frac{\mathbb{p}}{\mathbb{p}'+\mathbb{q}'}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma)\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}w(f^{\mathbb{p}'}g^{\mathbb{q}'})^{\frac{\mathbb{q}}{\mathbb{p}'+\mathbb{q}'}}\right)^{\frac{1}{\mathbb{q}}}(x;\sigma),\quad(98)$$

which implies the anticipated inequality (89). The proof of Theorem 6 is completed. □

Theorem 7. *For $\hbar,\ell>0$, $1/\mathbb{p}+1/\mathbb{q}=1/\mathbb{p}'+1/\mathbb{q}'=1$ with $\mathbb{p}\geqslant\mathbb{p}'>1$. Let f,g and w be three continuous positive functions on $[u,v]$ satisfying $0<\mathbb{m}\leqslant f^{\hbar}(t)/g^{\ell}(t)\leqslant\mathbb{M}$ for any $t\in[u,v]$. Then, for $x\in[u,v]$, we have the following FIO inequality*

$$\big({}^{\phi}_{\xi}\Theta^{M}_{u^+}wf^{\frac{\mathbb{p}'\hbar}{\mathbb{p}}}\big)^{\frac{1}{\mathbb{p}'}}(x;\sigma)\big({}^{\phi}_{\xi}\Theta^{M}_{u^+}wg^{\frac{\mathbb{q}\ell}{\mathbb{q}'}}\big)^{\frac{1}{\mathbb{q}}}(x;\sigma)\leqslant\frac{\mathbb{M}^{\frac{1}{\mathbb{p}\mathbb{q}}}}{\mathbb{m}^{\frac{1}{\mathbb{p}'\mathbb{q}'}}}\big({}^{\phi}_{\xi}\Theta^{M}_{u^+}w\big)^{\frac{2}{\mathbb{p}'}-\frac{2}{\mathbb{p}}}(x;\sigma)$$
$$\cdot\big({}^{\phi}_{\xi}\Theta^{M}_{u^+}wf^{\frac{\hbar}{\mathbb{p}}}g^{\frac{\ell}{\mathbb{q}}}\big)^{\frac{1}{\mathbb{p}}}(x;\sigma)\big({}^{\phi}_{\xi}\Theta^{M}_{u^+}wf^{\frac{\hbar}{\mathbb{p}'}}g^{\frac{\ell}{\mathbb{q}'}}\big)^{\frac{1}{\mathbb{q}'}}(x;\sigma),\quad(99)$$

$$\big({}^{\phi}_{\xi}\Theta^{M}_{u^+}wf^{\frac{\mathbb{p}'\hbar}{\mathbb{p}}}\big)^{\frac{1}{\mathbb{p}'}}(x;\sigma)\big({}^{\phi}_{\xi}\Theta^{M}_{u^+}wg^{\frac{\mathbb{q}\ell}{\mathbb{q}'}}\big)^{\frac{1}{\mathbb{q}}}(x;\sigma)\leqslant\frac{\mathbb{M}^{\frac{1}{\mathbb{p}^2}}}{\mathbb{m}^{\frac{1}{\mathbb{q}'^2}}}\big({}^{\phi}_{\xi}\Theta^{M}_{u^+}w\big)^{\frac{2}{\mathbb{p}'}-\frac{2}{\mathbb{p}}}(x;\sigma)$$
$$\cdot\big({}^{\phi}_{\xi}\Theta^{M}_{u^+}wf^{\frac{\hbar}{\mathbb{q}}}g^{\frac{\ell}{\mathbb{p}}}\big)^{\frac{1}{\mathbb{p}}}(x;\sigma)\big({}^{\phi}_{\xi}\Theta^{M}_{u^+}wf^{\frac{\hbar}{\mathbb{q}'}}g^{\frac{\ell}{\mathbb{p}'}}\big)^{\frac{1}{\mathbb{q}'}}(x;\sigma).\quad(100)$$

Proof. Since $1/\mathbb{p}+1/\mathbb{q}=1/\mathbb{p}'+1/\mathbb{q}'=1$ with $\mathbb{p}\geqslant\mathbb{p}'>1$, then $\mathbb{q}'\geqslant\mathbb{q}>1$, $0<\mathbb{p}'/\mathbb{p}\leqslant1$ and $0<\mathbb{q}/\mathbb{q}'\leqslant1$. Form Remark 2, we have

$$\big({}^{\phi}_{\xi}\Theta^{M}_{u^+}wf^{\mathbb{p}}\big)(x;\sigma)=\big({}^{\phi}_{\xi}\Theta^{M}_{u^+}(w^{\frac{\mathbb{p}'-\mathbb{p}}{\mathbb{p}'}})(w^{\frac{\mathbb{p}}{\mathbb{p}'}}f^{\mathbb{p}})\big)(x;\sigma)$$
$$\geqslant\big({}^{\phi}_{\xi}\Theta^{M}_{u^+}w\big)^{\frac{\mathbb{p}'-\mathbb{p}}{\mathbb{p}'}}(x;\sigma)\big({}^{\phi}_{\xi}\Theta^{M}_{u^+}wf^{\mathbb{p}'}\big)^{\frac{\mathbb{p}}{\mathbb{p}'}}(x;\sigma),\quad(101)$$

$$\big({}^{\phi}_{\xi}\Theta^{M}_{u^+}wf^{\mathbb{q}'}\big)(x;\sigma)=\big({}^{\phi}_{\xi}\Theta^{M}_{u^+}(w^{\frac{\mathbb{q}-\mathbb{q}'}{\mathbb{q}}})(w^{\frac{\mathbb{q}'}{\mathbb{q}}}f^{\mathbb{q}'})\big)(x;\sigma)$$
$$\geqslant\big({}^{\phi}_{\xi}\Theta^{M}_{u^+}w\big)^{\frac{\mathbb{q}-\mathbb{q}'}{\mathbb{q}}}(x;\sigma)\big({}^{\phi}_{\xi}\Theta^{M}_{u^+}wf^{\mathbb{q}}\big)^{\frac{\mathbb{q}'}{\mathbb{q}}}(x;\sigma).\quad(102)$$

It follows from the hypothesis $0<\mathbb{m}\leqslant f^{\hbar}(t)/g^{\ell}(t)\leqslant\mathbb{M}$ that

$$f^{\hbar}(t)\leqslant\mathbb{M}g^{\ell}(t)\Rightarrow f^{\frac{\hbar}{\mathbb{q}}}(t)\leqslant\mathbb{M}^{\frac{1}{\mathbb{q}}}g^{\frac{\ell}{\mathbb{q}}}(t)\Rightarrow f^{\hbar}(t)\leqslant\mathbb{M}^{\frac{1}{\mathbb{q}}}f^{\frac{\hbar}{\mathbb{p}}}(t)g^{\frac{\ell}{\mathbb{q}}}(t).\quad(103)$$

Multiplying simultaneously the inequalities (103) by $\aleph^{-1}(x)\aleph(t)\mathscr{M}^{t}_{x}({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu},\xi,\phi)$ $\xi'(t)w(t)$ and integrating the acquired inequalities with regard to t from u to x, we obtain, based on the operator (38)

$$\big({}^{\phi}_{\xi}\Theta^{M}_{u^+}wf^{\hbar}\big)^{\frac{1}{\mathbb{p}}}(x;\sigma)\leqslant\mathbb{M}^{\frac{1}{\mathbb{p}\mathbb{q}}}\big({}^{\phi}_{\xi}\Theta^{M}_{u^+}wf^{\frac{\hbar}{\mathbb{p}}}g^{\frac{\ell}{\mathbb{q}}}\big)^{\frac{1}{\mathbb{p}}}(x;\sigma).\quad(104)$$

Replacing f with $f^{\frac{\hbar}{\mathbb{p}}}$ in (101), we deduce

$$\big({}^{\phi}_{\xi}\Theta^{M}_{u^+}wf^{\hbar}\big)(x;\sigma)\geqslant\big({}^{\phi}_{\xi}\Theta^{M}_{u^+}w\big)^{\frac{\mathbb{p}'-\mathbb{p}}{\mathbb{p}'}}(x;\sigma)\big({}^{\phi}_{\xi}\Theta^{M}_{u^+}wf^{\frac{\mathbb{p}'\hbar}{\mathbb{p}}}\big)^{\frac{\mathbb{p}}{\mathbb{p}'}}(x;\sigma),\quad(105)$$

that is,

$$\big({}^{\phi}_{\xi}\Theta^{M}_{u^+}wf^{\frac{\mathbb{p}'\hbar}{\mathbb{p}}}\big)^{\frac{1}{\mathbb{p}'}}(x;\sigma)\leqslant\big({}^{\phi}_{\xi}\Theta^{M}_{u^+}w\big)^{\frac{\mathbb{p}-\mathbb{p}'}{\mathbb{p}\mathbb{p}'}}(x;\sigma)\big({}^{\phi}_{\xi}\Theta^{M}_{u^+}wf^{\hbar}\big)^{\frac{1}{\mathbb{p}}}(x;\sigma).\quad(106)$$

Combining (104) and (106) yields

$$\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wf^{\frac{\mathbb{p}'\hbar}{\mathbb{p}}}\right)^{\frac{1}{\mathbb{p}'}}(x;\sigma) \leqslant \mathbb{M}^{\frac{1}{\mathbb{p}\mathbb{q}}}\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w\right)^{\frac{\mathbb{p}-\mathbb{p}'}{\mathbb{p}\mathbb{p}'}}(x;\sigma)\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wf^{\frac{\hbar}{\mathbb{p}}}g^{\frac{\ell}{\mathbb{q}}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma). \tag{107}$$

On the other hand, from the hypothesis $0 < \mathbb{m} \leqslant f^{\hbar}(t)/g^{\ell}(t) \leqslant \mathbb{M}$, we achieve

$$g^{\ell}(t) \leqslant \frac{1}{\mathbb{m}} f^{\hbar}(t) \Rightarrow g^{\frac{\ell}{\mathbb{p}'}}(t) \leqslant \frac{1}{\mathbb{m}^{\frac{1}{\mathbb{p}'}}} f^{\frac{\hbar}{\mathbb{p}'}}(t) \Rightarrow g^{\ell}(t) \leqslant \frac{1}{\mathbb{m}^{\frac{1}{\mathbb{p}'}}} f^{\frac{\hbar}{\mathbb{p}'}}(t) g^{\frac{\ell}{\mathbb{q}'}}(t). \tag{108}$$

Multiplying simultaneously the inequalities (108) by $\aleph^{-1}(x)\aleph(t)\mathscr{M}_x^t({}_{\sigma_1,\sigma_2}^{\varrho_1,\varrho_2}M_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n},\xi,\phi)$ $\xi'(t)w(t)$ and integrating the obtained inequalities in regard to t from u to x, we achieve based on the operator (38)

$$\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wg^{\ell}\right)^{\frac{1}{\mathbb{q}'}}(x;\sigma) \leqslant \frac{1}{\mathbb{m}^{\frac{1}{\mathbb{p}'\mathbb{q}'}}}\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wf^{\frac{\hbar}{\mathbb{p}'}}g^{\frac{\ell}{\mathbb{q}'}}\right)^{\frac{1}{\mathbb{q}'}}(x;\sigma). \tag{109}$$

Replacing f with $g^{\frac{\ell}{\mathbb{q}'}}$ in (102), we deduce

$$\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wg^{\ell}\right)(x;\sigma) \geqslant \left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w\right)^{\frac{\mathbb{q}-\mathbb{q}'}{\mathbb{q}}}(x;\sigma)\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wg^{\frac{\mathbb{q}\ell}{\mathbb{q}'}}\right)^{\frac{\mathbb{q}'}{\mathbb{q}}}(x;\sigma), \tag{110}$$

that is,

$$\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wg^{\frac{\mathbb{q}\ell}{\mathbb{q}'}}\right)^{\frac{1}{\mathbb{q}}}(x;\sigma) \leqslant \left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w\right)^{\frac{\mathbb{q}'-\mathbb{q}}{\mathbb{q}\mathbb{q}'}}(x;\sigma)\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wg^{\ell}\right)^{\frac{1}{\mathbb{q}'}}(x;\sigma). \tag{111}$$

Combining (108) and (110) yields

$$\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wg^{\frac{\mathbb{q}\ell}{\mathbb{q}'}}\right)^{\frac{1}{\mathbb{q}}}(x;\sigma) \leqslant \frac{1}{\mathbb{m}^{\frac{1}{\mathbb{p}'\mathbb{q}'}}}\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w\right)^{\frac{\mathbb{q}'-\mathbb{q}}{\mathbb{q}\mathbb{q}'}}(x;\sigma)\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wf^{\frac{\hbar}{\mathbb{p}'}}g^{\frac{\ell}{\mathbb{q}'}}\right)^{\frac{1}{\mathbb{q}'}}(x;\sigma). \tag{112}$$

By multiplying the inequalities (107) and (111), then we achieve the desired inequality (99). Similar to the proof of inequality (99), we also deduce inequality (100). The proof of Theorem 7 is completed. □

Remark 3. *If $\mathbb{p} = \mathbb{p}'$, it is easy to see that the inequalities (99) and (100) reduce to the second inequalities of (73) and (74), respectively.*

4. Reverse Minkowski Type Inequalities

In this section, we will consider some reverse Minkowski-type inequalities for modified unified generalized FIOs with extended unified MLFs.

Theorem 8. *Suppose that f, g, w are three continuous positive functions on $[u, v]$ satisfying $0 < \mathbb{m} \leqslant f(t)/g(t) \leqslant \mathbb{M}$ for any $t \in [u, v]$ and $\mathbb{p} \geqslant 1$. Then, for $x \in [u, v]$, the following FIO inequalities hold*

$$\begin{aligned}\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w(f+g)^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma) &\leqslant \left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wf^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma) + \left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wg^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma)\\ &\leqslant \mathbb{c}_1\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w(f+g)^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma),\end{aligned} \tag{113}$$

where $\mathbb{c}_1$ is defined in (18).

Proof. When $\mathbb{p} = 1$, the first inequality of (113) becomes an equation. When $\mathbb{p} > 1$, by taking advantage of the Hölder's inequality in Remark 2, we can obtain for $1/\mathbb{p} + 1/\mathbb{q} = 1$

$$
\begin{aligned}
&\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w(f+g)^{\mathbb{p}}\right)(x;\sigma) = \left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w(f+g)(f+g)^{\mathbb{p}-1}\right)(x;\sigma) \\
&\quad = \left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wf(f+g)^{\mathbb{p}-1}\right)(x;\sigma) + \left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wg(f+g)^{\mathbb{p}-1}\right)(x;\sigma) \\
&\quad \leqslant \left(\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wf^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma) + \left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wg^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma)\right)\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w(f+g)^{\mathbb{q}(\mathbb{p}-1)}\right)^{\frac{1}{\mathbb{q}}}(x;\sigma). \quad (114)
\end{aligned}
$$

Since $1/\mathbb{p} + 1/\mathbb{q} = 1$, then $\mathbb{q}(\mathbb{p}-1) = \mathbb{p}$. Multiplying the inequality (114) by $\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w(f+g)^{\mathbb{p}}\right)^{-1/\mathbb{q}}(x;\sigma)$, we can acquire the first inequality of (113).

Since $0 < \mathbb{m} \leqslant f(t)/g(t) \leqslant \mathbb{M}$, then we can observe

$$
(\mathbb{M}+1)f(t) \leqslant \mathbb{M}(f(t)+g(t)) \Rightarrow (\mathbb{M}+1)^{\mathbb{p}} f^{\mathbb{p}}(t) \leqslant \mathbb{M}^{\mathbb{p}}(f(t)+g(t))^{\mathbb{p}}. \quad (115)
$$

Multiplying simultaneously the inequality (115) by $\aleph^{-1}(x)\aleph(t)\mathscr{M}_x^t({}_{\sigma_1,\sigma_2}^{\varrho_1,\varrho_2}M_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}, \xi, \phi)$ $\xi'(t)w(t)$ and integrating the acquired inequality in regard to t from u to x, we gain based on the operator (38)

$$
\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wf^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma) \leqslant \frac{\mathbb{M}}{\mathbb{M}+1}\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w(f+g)^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma). \quad (116)
$$

Also since $0 < \mathbb{m} \leqslant f(t)/g(t) \leqslant \mathbb{M}$, then we can write

$$
(\mathbb{m}+1)g(t) \leqslant f(t)+g(t) \Rightarrow (\mathbb{m}+1)^{\mathbb{p}} g^{\mathbb{p}}(t) \leqslant (f(t)+g(t))^{\mathbb{p}}. \quad (117)
$$

Multiplying simultaneously the inequality (117) by $\aleph^{-1}(x)\aleph(t)\mathscr{M}_x^t({}_{\sigma_1,\sigma_2}^{\varrho_1,\varrho_2}M_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}, \xi, \phi)$ $\xi'(t)w(t)$ and integrating the obtained inequality with regard to t from u to x, we acquire based on the operator (38)

$$
\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wg^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma) \leqslant \frac{1}{\mathbb{m}+1}\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w(f+g)^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma). \quad (118)
$$

Adding (116) and (118) yields the second inequality of (113). This completes the proof. □

Theorem 9. *Assume that f, g, w are three continuous positive functions on $[u, v]$ satisfying $0 < \mathbb{m} \leqslant f(t)/g(t) \leqslant \mathbb{M}$ for all $t \in [u, v]$ and $\mathbb{p} \geqslant 1$. Then, for $x \in [u, v]$, we have the following fractional integral inequality*

$$
\begin{aligned}
&\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wf^{\mathbb{p}}\right)^{\frac{2}{\mathbb{p}}}(x;\sigma) + \left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wg^{\mathbb{p}}\right)^{\frac{2}{\mathbb{p}}}(x;\sigma) \\
&\qquad \geqslant \mathbb{c}_2\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wf^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma)\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wg^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma), \quad (119)
\end{aligned}
$$

where $\mathbb{c}_2$ is defined in (18).

Proof. Combining (116) and (118) yields the following inequality

$$
\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wf^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma)\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wg^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma) \leqslant \frac{\mathbb{M}\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w(f+g)^{\mathbb{p}}\right)^{\frac{2}{\mathbb{p}}}(x;\sigma)}{(\mathbb{m}+1)(\mathbb{M}+1)}. \quad (120)
$$

Applying Minkowski's inequality to the right side of (120), we have

$$
\frac{\mathbb{M}\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w(f+g)^{\mathbb{p}}\right)^{\frac{2}{\mathbb{p}}}(x;\sigma)}{(\mathbb{m}+1)(\mathbb{M}+1)} \leqslant \left(\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wf^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma) + \left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wg^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma)\right)^2. \quad (121)
$$

According to the inequalities (120) and (121), we have the desired inequality (119). This completes the proof. □

Theorem 10. *Let* $1/\mathbb{p} + 1/\mathbb{q} = 1$ *with* $\mathbb{p} > 1$*. Let* f, g *and* w *be three continuous positive functions on* $[u, v]$ *satisfying* $0 < \mathbb{m} \leqslant f(t)/g(t) \leqslant \mathbb{M}$ *for all* $t \in [u, v]$*. Then, for* $x \in [u, v]$*, we have the following fractional integral inequality*

$$\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} wfg\right)(x;\sigma) \leqslant \mathbb{c}_3\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w(f^{\mathbb{p}} + g^{\mathbb{p}})\right)(x;\sigma) + \mathbb{c}_4\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w(f^{\mathbb{q}} + g^{\mathbb{q}})\right)(x;\sigma), \quad (122)$$

where $\mathbb{c}_3 = 2^{\mathbb{p}-1}\mathbb{M}^{\mathbb{p}}/(\mathbb{p}(\mathbb{M}+1)^{\mathbb{p}})$ *and* $\mathbb{c}_4 = 2^{\mathbb{q}-1}/(\mathbb{q}(\mathbb{m}+1)^{\mathbb{q}})$*.*

Proof. Similar to (116) and (118), we can easily obtain

$$\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} wf^{\mathbb{p}}\right)(x;\sigma) \leqslant \frac{\mathbb{M}^{\mathbb{p}}}{(\mathbb{M}+1)^{\mathbb{p}}}\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w(f+g)^{\mathbb{p}}\right)(x;\sigma), \quad (123)$$

$$\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} wg^{\mathbb{q}}\right)(x;\sigma) \leqslant \frac{1}{(\mathbb{m}+1)^{\mathbb{q}}}\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w(f+g)^{\mathbb{q}}\right)(x;\sigma). \quad (124)$$

It follows from Young inequality $\mathcal{A}^{1/\mathbb{p}}\mathcal{B}^{1/\mathbb{q}} \leqslant \mathcal{A}/\mathbb{p} + \mathcal{B}/\mathbb{q}$ with $\mathcal{A} = f^{\mathbb{p}}(t)$ and $\mathcal{B} = g^{\mathbb{q}}(t)$ that we have the following FIO inequality

$$\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} wfg\right)(x;\sigma) \leqslant \frac{1}{\mathbb{p}}\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} wf^{\mathbb{p}}\right)(x;\sigma) + \frac{1}{\mathbb{q}}\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} wg^{\mathbb{q}}\right)(x;\sigma). \quad (125)$$

Substituting (123) and (124) into (125) yields

$$\begin{aligned}\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} wfg\right)(x;\sigma) \leqslant{}& \frac{\mathbb{M}^{\mathbb{p}}}{\mathbb{p}(\mathbb{M}+1)^{\mathbb{p}}}\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w(f+g)^{\mathbb{p}}\right)(x;\sigma)\\ &+ \frac{1}{\mathbb{q}(\mathbb{m}+1)^{\mathbb{q}}}\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w(f+g)^{\mathbb{q}}\right)(x;\sigma). \quad (126)\end{aligned}$$

Applying the inequality $(\mathcal{A}+\mathcal{B})^{\mathbb{j}} \leqslant 2^{\mathbb{j}-1}(\mathcal{A}^{\mathbb{j}} + \mathcal{B}^{\mathbb{j}})$, $\mathbb{j} > 1$, $\mathcal{A}, \mathcal{B}$, to the right part of (126), we have the following inequalities

$$\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w(f+g)^{\mathbb{p}}\right)(x;\sigma) \leqslant 2^{\mathbb{p}-1}\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w(f^{\mathbb{p}} + g^{\mathbb{p}})\right)(x;\sigma), \quad (127)$$

$$\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w(f+g)^{\mathbb{q}}\right)(x;\sigma) \leqslant 2^{\mathbb{q}-1}\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w(f^{\mathbb{q}} + g^{\mathbb{q}})\right)(x;\sigma). \quad (128)$$

Substituting (127) and (128) into (126) yields the desired inequality (122). This completes the proof. □

Theorem 11. *Suppose that* f, g, w *are three continuous positive functions on* $[u, v]$ *satisfying* $0 < \mathbb{m}_1 \leqslant f(t) \leqslant \mathbb{M}_1$ *and* $0 < \mathbb{m}_2 \leqslant g(t) \leqslant \mathbb{M}_2$ *for all* $t \in [u, v]$ *and* $\mathbb{p} \geqslant 1$*. Then, for* $x \in [u, v]$*, the following FIO inequalities hold*

$$\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} wf^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma) + \left({}^{\phi}_{\xi}\Theta^{M}_{u^+} wg^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma) \leqslant \mathbb{c}_5\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w(f+g)^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma), \quad (129)$$

where $\mathbb{c}_5 = \left(\mathbb{M}_1(\mathbb{m}_1 + \mathbb{M}_2) + \mathbb{M}_2(\mathbb{m}_2 + \mathbb{M}_1)\right)/\left((\mathbb{m}_1 + \mathbb{M}_2)(\mathbb{m}_2 + \mathbb{M}_1)\right)$*.*

Proof. It follows from the hypothesis $0 < \mathbb{m}_2 \leqslant g(t) \leqslant \mathbb{M}_2$ that

$$\frac{1}{\mathbb{M}_2} \leqslant \frac{1}{g(t)} \leqslant \frac{1}{\mathbb{m}_2}. \quad (130)$$

Carrying the product between (130) and $0 < \mathbb{m}_1 \leqslant f(t) \leqslant \mathbb{M}_1$, we can observe

$$\frac{\mathbb{m}_1}{\mathbb{M}_2} \leqslant \frac{f(t)}{g(t)} \leqslant \frac{\mathbb{M}_1}{\mathbb{m}_2}. \tag{131}$$

According to the second inequality of (113), we can gain the desired inequality (129). The proof of Theorem 11 is completed. □

Theorem 12. *Assume that f, g, w are three continuous positive functions on $[u, v]$ satisfying $0 < \mathbb{m} \leqslant f(t)/g(t) \leqslant \mathbb{M}$ for all $t \in [u, v]$. Then, for $x \in [u, v]$, the following FIO inequalities hold*

$$\frac{1}{\mathbb{M}}\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wfg\right)(x;\sigma) \leqslant \frac{1}{(\mathbb{m}+1)(\mathbb{M}+1)}\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}w(f+g)^2\right)(x;\sigma)$$
$$\leqslant \frac{1}{\mathbb{m}}\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wfg\right)(x;\sigma). \tag{132}$$

Proof. Since $0 < \mathbb{m} \leqslant f(t)/g(t) \leqslant \mathbb{M}$, we have

$$(\mathbb{m}+1)g(t) \leqslant f(t) + g(t) \leqslant (\mathbb{M}+1)g(t). \tag{133}$$

It follows from $0 < \mathbb{m} \leqslant f(t)/g(t) \leqslant \mathbb{M}$ that $(1/\mathbb{M}) \leqslant g(t)/f(t) \leqslant (1/\mathbb{m})$, which implies

$$\left(\frac{\mathbb{M}+1}{\mathbb{M}}\right)f(t) \leqslant f(t) + g(t) \leqslant \left(\frac{\mathbb{m}+1}{\mathbb{m}}\right)f(t). \tag{134}$$

Realizing the product between (133) and (134) yields

$$\frac{1}{\mathbb{M}}f(t)g(t) \leqslant \frac{1}{(\mathbb{m}+1)(\mathbb{M}+1)}(f(t)+g(t))^2 \leqslant \frac{1}{\mathbb{m}}f(t)g(t). \tag{135}$$

Multiplying simultaneously the inequality (135) by $\aleph^{-1}(x)\aleph(t)\mathscr{M}^{t}_{x}\left({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}, \xi, \phi\right)$ $\xi'(t)w(t)$ and integrating the obtained inequality in regard to t from u to x, we deduce based on the operator (38)

$$\frac{1}{\mathbb{M}}\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wfg\right)(x;\sigma) \leqslant \frac{1}{(\mathbb{m}+1)(\mathbb{M}+1)}\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}w(f+g)^2\right)(x;\sigma)$$
$$\leqslant \frac{1}{\mathbb{m}}\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wfg\right)(x;\sigma), \tag{136}$$

which is the desired inequality (132). This completes the proof. □

Theorem 13. *Suppose that f, g, w are three continuous positive functions on $[u, v]$ satisfying $0 < \kappa < \mathbb{m} \leqslant f(t)/g(t) \leqslant \mathbb{M}$ for all $t \in [u, v]$ and $\mathbb{p} \geqslant 1$. Then, for $x \in [u, v]$, we obtain*

$$\frac{\mathbb{M}+1}{\mathbb{M}-\kappa}\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}w(f-\kappa g)^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma) \leqslant \left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wf^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma) + \left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wg^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma)$$
$$\leqslant \frac{\mathbb{m}+1}{\mathbb{m}-\kappa}\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}w(f-\kappa g)^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma). \tag{137}$$

Proof. Taking $0 < \kappa < \mathbb{m} \leqslant f(t)/g(t) \leqslant \mathbb{M}$, we have

$$\mathbb{m} - \kappa \leqslant \frac{f(t) - \kappa g(t)}{g(t)} \leqslant \mathbb{M} - \kappa \Rightarrow \frac{1}{\mathbb{M}-\kappa} \leqslant \frac{g(t)}{f(t) - \kappa g(t)} \leqslant \frac{1}{\mathbb{m}-\kappa}, \tag{138}$$

which demonstrates

$$\left(\frac{1}{\mathbb{M}-\kappa}\right)^{\mathbb{p}}(f(t)-\kappa g(t))^{\mathbb{p}} \leqslant g^{\mathbb{p}}(t) \leqslant \left(\frac{1}{\mathbb{m}-\kappa}\right)^{\mathbb{p}}(f(t)-\kappa g(t))^{\mathbb{p}}. \tag{139}$$

Multiplying simultaneously the inequality (139) by $\aleph^{-1}(x)\aleph(t)\mathscr{M}_x^t({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu},\xi,\phi)$ $\xi'(t)w(t)$ and integrating the resulting inequality with regard to t from u to x, we achieve based on the operator (38)

$$\frac{1}{\mathbb{M}-\kappa}\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}w(f-\kappa g)^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma)\leqslant\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wg^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma)\leqslant\frac{1}{\mathbb{m}-\kappa}\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}w(f-\kappa g)^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma). \quad (140)$$

It follows also from $0<\kappa<\mathbb{m}\leqslant f(t)/g(t)\leqslant\mathbb{M}$ that

$$\frac{1}{\mathbb{M}}\leqslant\frac{g(t)}{f(t)}\leqslant\frac{1}{\mathbb{m}}\Rightarrow\frac{\mathbb{m}-\kappa}{\mathbb{m}}\leqslant\frac{f(t)-\kappa g(t)}{f(t)}\leqslant\frac{\mathbb{M}-\kappa}{\mathbb{M}}, \quad (141)$$

which implies

$$\left(\frac{\mathbb{M}}{\mathbb{M}-\kappa}\right)^{\mathbb{p}}(f(t)-\kappa g(t))^{\mathbb{p}}\leqslant f^{\mathbb{p}}(t)\leqslant\left(\frac{\mathbb{m}}{\mathbb{m}-\kappa}\right)^{\mathbb{p}}(f(t)-\kappa g(t))^{\mathbb{p}}. \quad (142)$$

Multiplying simultaneously the inequality (142) by $\aleph^{-1}(x)\aleph(t)\mathscr{M}_x^t({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu},\xi,\phi)$ $\xi'(t)w(t)$ and integrating the obtained result with regard to t from u to x, we gain based on the operator (38)

$$\frac{\mathbb{M}}{\mathbb{M}-\kappa}\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}w(f-\kappa g)^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma)\leqslant\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wf^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma)\leqslant\frac{\mathbb{m}}{\mathbb{m}-\kappa}\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}w(f-\kappa g)^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma). \quad (143)$$

Adding (140) and (143) yields the desired inequality (137). This completes the proof. □

Theorem 14. *Let f,g and w be three continuous positive functions on $[u,v]$ satisfying $0<\mathbb{m}\leqslant f(t)/g(t)\leqslant\mathbb{M}$ for all $t\in[u,v]$ and $\mathbb{p}\geqslant 1$. Then, for $x\in[u,v]$, we have the following inequality*

$$\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wf^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma)+\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wg^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma)\leqslant 2\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wh^{\mathbb{p}}(f,g)\right)^{\frac{1}{\mathbb{p}}}(x;\sigma), \quad (144)$$

where $h(f(t),g(t))=\max\{(\mathbb{M}/\mathbb{m}+1)f(t)-\mathbb{M}g(t),((\mathbb{m}+\mathbb{M})g(t)-f(t))/\mathbb{m}\}$.

Proof. Since $0<\mathbb{m}\leqslant f(t)/g(t)\leqslant\mathbb{M}$, we have

$$0\leqslant\mathbb{m}\leqslant\mathbb{m}+\mathbb{M}-\frac{f(t)}{g(t)}\leqslant\mathbb{M}. \quad (145)$$

It follows from (145) that

$$g(t)\leqslant\frac{(\mathbb{m}+\mathbb{M})g(t)-f(t)}{\mathbb{m}}\leqslant h(f(t),g(t)). \quad (146)$$

Multiplying simultaneously the inequality (146) by $\aleph^{-1}(x)\aleph(t)\mathscr{M}_x^t({}^{\varrho_1,\varrho_2}_{\sigma_1,\sigma_2}M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu},\xi,\phi)$ $\xi'(t)w(t)$ and integrating the obtained result in regard to t from u to x, we achieve based on the operator (38)

$$\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wg^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma)\leqslant\left({}^{\phi}_{\xi}\Theta^{M}_{u^+}wh^{\mathbb{p}}(f,g)\right)^{\frac{1}{\mathbb{p}}}(x;\sigma). \quad (147)$$

Also since $0 < \mathbb{m} \leqslant f(t)/g(t) \leqslant \mathbb{M}$, then $(1/\mathbb{M}) \leqslant g(t)/f(t) \leqslant (1/\mathbb{m})$, which implies

$$\frac{1}{\mathbb{M}} \leqslant \frac{1}{\mathbb{M}} + \frac{1}{\mathbb{m}} - \frac{g(t)}{f(t)} \leqslant \frac{1}{\mathbb{m}}. \tag{148}$$

It follows from (148) that

$$f(t) \leqslant \left(\frac{\mathbb{M}}{\mathbb{m}} + 1\right) f(t) - \mathbb{M} g(t) \leqslant h(f(t), g(t)). \tag{149}$$

Multiplying simultaneously the inequality (149) by $\aleph^{-1}(x)\aleph(t)\mathscr{M}_x^t({}_{\sigma_1,\sigma_2}^{\varrho_1,\varrho_2}M_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}, \xi, \phi)$ $\xi'(t)w(t)$ and integrating the resulting inequality in regard to t from u to x, we achieve based on the operator (38)

$$\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w f^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma) \leqslant \left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w h^{\mathbb{p}}(f,g)\right)^{\frac{1}{\mathbb{p}}}(x;\sigma). \tag{150}$$

Adding (147) and (150) yields the desired inequality (144). The proof of Theorem 14 is completed. □

Theorem 15. *Assume that f, g, w are three continuous positive functions on $[u, v]$ satisfying $0 < \mathbb{m} \leqslant f(t)/g(t) + g(t)/f(t) \leqslant \mathbb{M}$ for all $t \in [u, v]$ and $\mathbb{p} \geqslant 1$. Then, for $x \in [u, v]$, we have*

$$\mathbb{m}\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w f^{\mathbb{p}} g^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma) \leqslant \left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w (f^2+g^2)^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma) \leqslant \mathbb{M}\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w f^{\mathbb{p}} g^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma), \tag{151}$$

$$\begin{aligned} \frac{1}{\mathbb{M}}\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w (f^2+g^2)^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma) &\leqslant \left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w f^{\mathbb{p}} g^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma) \\ &\leqslant \frac{1}{\mathbb{m}}\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w (f^2+g^2)^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma). \end{aligned} \tag{152}$$

Proof. Since $0 < \mathbb{m} \leqslant f(t)/g(t) + g(t)/f(t) \leqslant \mathbb{M}$, we have

$$\mathbb{m} \leqslant \frac{f^2(t)+g^2(t)}{f(t)g(t)} \leqslant \mathbb{M} \Rightarrow \mathbb{m} f^{\mathbb{p}}(t) g^{\mathbb{p}}(t) \leqslant (f^2(t)+g^2(t))^{\mathbb{p}} \leqslant \mathbb{M} f^{\mathbb{p}}(t) g^{\mathbb{p}}(t). \tag{153}$$

$$\begin{aligned} \frac{1}{\mathbb{M}} \leqslant \frac{f(t)g(t)}{f^2(t)+g^2(t)} \leqslant \frac{1}{\mathbb{m}} \Rightarrow \frac{1}{\mathbb{M}}(f^2(t)+g^2(t))^{\mathbb{p}} &\leqslant f^{\mathbb{p}}(t) g^{\mathbb{p}}(t) \\ &\leqslant \frac{1}{\mathbb{m}}(f^2(t)+g^2(t))^{\mathbb{p}}. \end{aligned} \tag{154}$$

Multiplying simultaneously the inequalities (153) and (154) by $\mathscr{M}_x^t({}_{\sigma_1,\sigma_2}^{\varrho_1,\varrho_2}M_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}, \xi, \phi)$ $\aleph^{-1}(x)\aleph(t)\xi'(t)w(t)$ and integrating the obtained results with regard to t from u to x, we gain based on the operator (38)

$$\mathbb{m}\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w f^{\mathbb{p}} g^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma) \leqslant \left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w (f^2+g^2)^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma) \leqslant \mathbb{M}\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w f^{\mathbb{p}} g^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma), \tag{155}$$

$$\begin{aligned} \frac{1}{\mathbb{M}}\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w (f^2+g^2)^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma) &\leqslant \left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w f^{\mathbb{p}} g^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma) \\ &\leqslant \frac{1}{\mathbb{m}}\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w (f^2+g^2)^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}}}(x;\sigma), \end{aligned} \tag{156}$$

which are the anticipated inequalities (151) and (152). This completes the proof. □

Remark 4. *By using the different settings of the parameters and functions in (38), Theorems 8 and 9 can reduce to the reverse Minkowski-type Riemann–Liouville FIO inequalities [20], the reverse Minkowski-type Hadamard FIO inequalities [20,21], the reverse Minkowski-type generalized k-*

FIO inequalities [24], the reverse Minkowski-type Katugampola FIO inequalities [25], the reverse weighted Minkowski-type inequalities for generalized FIOs with the Wright function [39] and the reverse weighted Minkowski-type inequalities for weighted FIOs with a monotonically increasing function [40], respectively.

5. Some Applications

In this section, by utilizing the reverse Hölder- and Minkowski-type inequalities obtained in the front, we will present some other inequalities for modified unified generalized FIOs with extended unified MLFs.

Theorem 16 (Jensen's inequality). *Let f and w be three continuous positive functions on $[u,v]$ and $0<\ell<\hbar$. Then, for $x\in[u,v]$, we have FIO inequalities*

$$\frac{\big({}_{\xi}^{\phi}\Theta_{u^+}^{M}wf^{\ell}\big)^{\frac{1}{\ell}}(x;\sigma)}{\big({}_{\xi}^{\phi}\Theta_{u^+}^{M}w\big)^{\frac{1}{\ell}}(x;\sigma)}\leqslant\frac{\big({}_{\xi}^{\phi}\Theta_{u^+}^{M}wf^{\hbar}\big)^{\frac{1}{\hbar}}(x;\sigma)}{\big({}_{\xi}^{\phi}\Theta_{u^+}^{M}w\big)^{\frac{1}{\hbar}}(x;\sigma)}\leqslant\frac{\Big({}_{\xi}^{\phi}\Theta_{u^+}^{M}S\big(\frac{\mathscr{G}_3f^{\hbar}}{\mathscr{F}_3}\big)wf^{\ell}\Big)^{\frac{1}{\ell}}(x;\sigma)}{\big({}_{\xi}^{\phi}\Theta_{u^+}^{M}w\big)^{\frac{1}{\ell}}(x;\sigma)},\tag{157}$$

where $S(\cdot)$ denotes the Specht's ratio, $\mathscr{F}_3=\big({}_{\xi}^{\phi}\Theta_{u^+}^{M}wf^{\hbar}\big)(x;\sigma)$ and $\mathscr{G}_3=\big({}_{\xi}^{\phi}\Theta_{u^+}^{M}w\big)(x;\sigma)$.

Proof. Since $0<\ell<\hbar$, then $\mathbb{p}=\hbar/\ell>1$ and $\mathbb{q}=\hbar/(\hbar-\ell)>1$. By employing the Hölder's inequality in Remark 2, we have

$$\begin{aligned}\big({}_{\xi}^{\phi}\Theta_{u^+}^{M}wf^{\ell}\big)^{\frac{1}{\ell}}(x;\sigma)&=\big({}_{\xi}^{\phi}\Theta_{u^+}^{M}(w^{\ell/\hbar}f^{\ell})w^{1-\ell/\hbar}\big)^{\frac{1}{\ell}}(x;\sigma)\\&\leqslant\Big(\Big({}_{\xi}^{\phi}\Theta_{u^+}^{M}(w^{\ell/\hbar}f^{\ell})^{\hbar/\ell}\Big)^{\frac{\ell}{\hbar}}(x;\sigma)\Big({}_{\xi}^{\phi}\Theta_{u^+}^{M}(w^{1-\ell/\hbar})^{\hbar/(\hbar-\ell)}\Big)^{\frac{\hbar-\ell}{\hbar}}(x;\sigma)\Big)^{\frac{1}{\ell}}\\&=\big({}_{\xi}^{\phi}\Theta_{u^+}^{M}wf^{\hbar}\big)^{\frac{1}{\hbar}}(x;\sigma)\big({}_{\xi}^{\phi}\Theta_{u^+}^{M}w\big)^{\frac{1}{\ell}-\frac{1}{\hbar}}(x;\sigma),\end{aligned}\tag{158}$$

which implies the left inequality of (157). From the hypotheses, we obtain

$$\begin{aligned}&\Big({}_{\xi}^{\phi}\Theta_{u^+}^{M}S\Big(\frac{\mathscr{G}_3f^{\hbar}}{\mathscr{F}_3}\Big)wf^{\ell}\Big)^{\frac{1}{\ell}}(x;\sigma)\\&\qquad=\Big({}_{\xi}^{\phi}\Theta_{u^+}^{M}S\Big(\frac{\mathscr{G}_3(w^{\ell/\hbar}f^{\ell})^{\hbar/\ell}}{\mathscr{F}_3(w^{1-\ell/\hbar})^{\hbar/(\hbar-\ell)}}\Big)(w^{\ell/\hbar}f^{\ell})w^{1-\ell/\hbar}\Big)^{\frac{1}{\ell}}(x;\sigma).\end{aligned}\tag{159}$$

By employing the third inequality of (41) to the right-hand part of (159), we observe

$$\begin{aligned}&\Big({}_{\xi}^{\phi}\Theta_{u^+}^{M}S\Big(\frac{\mathscr{G}_3(w^{\ell/\hbar}f^{\ell})^{\hbar/\ell}}{\mathscr{F}_3(w^{1-\ell/\hbar})^{\hbar/(\hbar-\ell)}}\Big)(w^{\ell/\hbar}f^{\ell})w^{1-\ell/\hbar}\Big)^{\frac{1}{\ell}}(x;\sigma)\\&\geqslant\Big(\Big({}_{\xi}^{\phi}\Theta_{u^+}^{M}(w^{\ell/\hbar}f^{\ell})^{\hbar/\ell}\Big)^{\frac{\ell}{\hbar}}(x;\sigma)\Big({}_{\xi}^{\phi}\Theta_{u^+}^{M}(w^{1-\ell/\hbar})^{\hbar/(\hbar-\ell)}\Big)^{\frac{\hbar-\ell}{\hbar}}(x;\sigma)\Big)^{\frac{1}{\ell}}\\&=\big({}_{\xi}^{\phi}\Theta_{u^+}^{M}wf^{\hbar}\big)^{\frac{1}{\hbar}}(x;\sigma)\big({}_{\xi}^{\phi}\Theta_{u^+}^{M}w\big)^{\frac{1}{\ell}-\frac{1}{\hbar}}(x;\sigma),\end{aligned}\tag{160}$$

which implies the right-hand desired inequality (157). The proof of Theorem 16 is completed. □

Theorem 17 (Weighted power mean inequality)**.** *Assume that f, w are two continuous positive functions on $[u,v]$ and $0 < \ell$. Then, for $x \in [u,v]$, we have FIO inequalities*

$$\frac{\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} wf^{\ell}\right)^{\frac{1}{\ell}}(x;\sigma)}{\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w\right)^{\frac{1}{\ell}}(x;\sigma)} \leqslant \frac{\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} wf^{2\ell}\right)^{\frac{1}{2\ell}}(x;\sigma)}{\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w\right)^{\frac{1}{2\ell}}(x;\sigma)} \leqslant \frac{\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} S\left(\frac{\mathscr{F}_3^*}{\mathscr{G}_3 f^{2\ell}}\right) wf^{\ell}\right)^{\frac{1}{\ell}}(x;\sigma)}{\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w\right)^{\frac{1}{\ell}}(x;\sigma)}, \tag{161}$$

where $S(\cdot)$ denotes the Specht's ratio, $\mathscr{F}_3^* = \left({}^{\phi}_{\xi}\Theta^{M}_{u^+} wf^{2\ell}\right)(x;\sigma)$ and $\mathscr{G}_3 = \left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w\right)(x;\sigma)$.

Proof. From the left-hand inequality of (157) with $\hbar = 2\ell$, we know the left-hand inequality of (161) holds. From the hypotheses, we obtain

$$\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} S\left(\frac{\mathscr{F}_3^*}{\mathscr{G}_3 f^{2\ell}}\right) wf^{\ell}\right)^{\frac{1}{\ell}}(x;\sigma) = \left({}^{\phi}_{\xi}\Theta^{M}_{u^+} S\left(\frac{\mathscr{F}_3^* w}{\mathscr{G}_3 w f^{2\ell}}\right)(w^{1/2} f^{\ell}) w^{1/2}\right)^{\frac{1}{\ell}}(x;\sigma). \tag{162}$$

By applying the third inequality of (41) with $\mathbb{p} = \mathbb{q} = 1/2$ to the right-hand part of (162), we can acquire

$$\begin{aligned}
&\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} S\left(\frac{\mathscr{F}_3^* w}{\mathscr{G}_3 w f^{2\ell}}\right)(w^{1/2} f^{\ell}) w^{1/2}\right)^{\frac{1}{\ell}}(x;\sigma) \\
&\qquad \geqslant \left(\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} wf^{2\ell}\right)^{\frac{1}{2}}(x;\sigma)\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w\right)^{\frac{1}{2}}(x;\sigma)\right)^{\frac{1}{\ell}} \\
&\qquad = \left({}^{\phi}_{\xi}\Theta^{M}_{u^+} wf^{2\ell}\right)^{\frac{1}{2\ell}}(x;\sigma)\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w\right)^{\frac{1}{\ell}-\frac{1}{2\ell}}(x;\sigma),
\end{aligned} \tag{163}$$

which implies the right-hand anticipated inequality (161). The proof of Theorem 17 is completed. □

Theorem 18 (Radon's inequality)**.** *Suppose that f, g, w be three continuous positive functions on $[u,v]$ and $0 < \ell$, $1 \leqslant \hbar$. Then, for $x \in [u,v]$, we have FIO inequalities*

$$\begin{aligned}
\frac{\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} wfg^{\hbar-1}\right)^{\ell+\hbar}(x;\sigma)}{\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} wg^{\hbar}\right)^{\ell+\hbar-1}(x;\sigma)} &\leqslant \left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w\left(\frac{f^{\ell+\hbar}}{g^{\ell}}\right)\right)(x;\sigma) \\
&\leqslant \frac{\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} S\left(\frac{\mathscr{G}_4 f^{\ell+\hbar}}{\mathscr{F}_4 g^{\ell+\hbar}}\right) wf^{\hbar}\right)^{\frac{\ell+\hbar}{\hbar}}(x;\sigma)}{\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} wg^{\hbar}\right)^{\frac{\ell}{\hbar}}(x;\sigma)},
\end{aligned} \tag{164}$$

where $S(\cdot)$ denotes the Specht's ratio, $\mathscr{F}_4 = \left({}^{\phi}_{\xi}\Theta^{M}_{u^+} wf^{\frac{\ell+\hbar}{\hbar}}/g^{\ell}\right)(x;\sigma)$ and $\mathscr{G}_4 = \left({}^{\phi}_{\xi}\Theta^{M}_{u^+} wg^{\hbar}\right)(x;\sigma)$.

Proof. For the convex $\Lambda(t) = t^{\ell+\hbar}$ on $[0,+\infty)$, then we have following inequality

$$\Lambda\left(\frac{\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w\mathrm{Y}\right)(x;\sigma)}{\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w\right)(x;\sigma)}\right) \leqslant \frac{\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} \Lambda(w\mathrm{Y})\right)(x;\sigma)}{\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w\right)(x;\sigma)} \quad \text{for positive function } \mathrm{Y}. \tag{165}$$

By applying the above inequality (165) with $w = wg$ and $\mathrm{Y} = f/g$, we can obtain

$$\begin{aligned}
\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w\left(\frac{f^{\ell+\hbar}}{g^{\ell}}\right)\right)(x;\sigma) &= \left({}^{\phi}_{\xi}\Theta^{M}_{u^+} wg^{\hbar}\left(\frac{f}{g}\right)^{\ell+\hbar}\right)(x;\sigma) \\
&\geqslant \left({}^{\phi}_{\xi}\Theta^{M}_{u^+} wg^{\hbar}\right)(x;\sigma)\left(\frac{\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} wfg^{\hbar-1}\right)(x;\sigma)}{\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} wg^{\hbar}\right)(x;\sigma)}\right)^{\ell+\hbar} = \frac{\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} wfg^{\hbar-1}\right)^{\ell+\hbar}(x;\sigma)}{\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} wg^{\hbar}\right)^{\ell+\hbar-1}(x;\sigma)},
\end{aligned} \tag{166}$$

which is the left-hand desired inequality (164). For $\mathbb{p} = (\ell + \hbar)/\hbar$ and $\mathbb{q} = (\ell + \hbar)/\ell$, by replacing $f(t)$ and $g(t)$ by $\mathscr{U}(t)$ and $\mathscr{V}(t)$ in (41), respectively, we have

$$\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w\mathscr{U}^{\frac{\ell+\hbar}{\hbar}}\right)^{\frac{\hbar}{\ell+\hbar}}(x;\sigma)\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w\mathscr{V}^{\frac{\ell+\hbar}{\ell}}\right)^{\frac{\ell}{\ell+1}}(x;\sigma) \leqslant \left({}_{\xi}^{\phi}\Theta_{u^+}^{M} S\left(\frac{\widehat{\mathscr{G}}_1\mathscr{U}^{\frac{\ell+\hbar}{\hbar}}}{\widehat{\mathscr{F}}_1\mathscr{V}^{\frac{\ell+\hbar}{\ell}}}\right)w\mathscr{U}\mathscr{V}\right)(x;\sigma), \quad (167)$$

where $\widehat{\mathscr{F}_1} = \left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w\mathscr{U}^{\frac{\ell+\hbar}{\hbar}}\right)(x;\sigma)$ and $\widehat{\mathscr{G}_1} = \left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w\mathscr{V}^{\frac{\ell+\hbar}{\ell}}\right)(x;\sigma)$.

Letting $\mathscr{U} = (\mathscr{X}/\mathscr{Y})^{\hbar/(\ell+\hbar)}$ and $\mathscr{V} = \mathscr{X}^{\ell/(\ell+\hbar)}\mathscr{Y}^{\hbar/(\ell+\hbar)}$ in (167) yields

$$\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w\left(\frac{\mathscr{X}}{\mathscr{Y}}\right)\right)^{\frac{\hbar}{\ell+\hbar}}(x;\sigma)\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w\left(\mathscr{X}\mathscr{Y}^{\frac{\hbar}{\ell}}\right)\right)^{\frac{\ell}{\ell+\hbar}}(x;\sigma) \leqslant \left({}_{\xi}^{\phi}\Theta_{u^+}^{M} S\left(\frac{\overline{\mathscr{G}}_1}{\overline{\mathscr{F}}_1\mathscr{Y}^{\frac{\ell+\hbar}{\ell}}}\right)w\mathscr{X}\right)(x;\sigma), \quad (168)$$

where $\overline{\mathscr{F}}_1 = \left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w\mathscr{X}/\mathscr{Y}\right)(x;\sigma)$ and $\overline{\mathscr{G}}_1 = \left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w\mathscr{X}\mathscr{Y}^{\frac{\hbar}{\ell}}\right)(x;\sigma)$.

Replacing $\mathscr{X}$ and $\mathscr{Y}$ by $f^{\hbar}(t)$ and $(g(t)/f(t))^{\ell}$, we can observe

$$\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w\left(\frac{f^{\ell+\hbar}}{g^{\ell}}\right)\right)^{\frac{\hbar}{\ell+\hbar}}(x;\sigma)\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wg^{\hbar}\right)^{\frac{\ell}{\ell+\hbar}}(x;\sigma) \leqslant \left({}_{\xi}^{\phi}\Theta_{u^+}^{M} S\left(\frac{\mathscr{G}_4 f^{\ell+\hbar}}{\mathscr{F}_4 g^{\ell+\hbar}}\right)wf^{\hbar}\right)(x;\sigma), \quad (169)$$

where $\mathscr{F}_4 = \left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wf^{\frac{\ell+\hbar}{\hbar}}/g^{\ell}\right)(x;\sigma)$ and $\mathscr{G}_4 = \left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wg^{\hbar}\right)(x;\sigma)$. The foregoing inequality (169) can yield the right-hand inequality of (164). This completes the proof. □

Theorem 19. *For $0 < \ell$ and $1 \leqslant \hbar$, assume that f, g, w are three continuous positive functions satisfying $0 < \mathbb{m} \leqslant (f(t)/g(t))^{\ell+\hbar} \leqslant \mathbb{M}$, $t \in [u,v]$. Then, for $x \in [u,v]$, we have*

$$\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w\left(\frac{f^{\ell+\hbar}}{g^{\ell}}\right)\right)^{\hbar}(x;\sigma) \leqslant \left(\frac{\mathbb{M}}{\mathbb{m}}\right)^{\frac{\ell\hbar}{\ell+\hbar}}\frac{\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wf^{\hbar}\right)^{\ell+\hbar}(x;\sigma)}{\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wg^{\hbar}\right)^{\ell}(x;\sigma)}. \quad (170)$$

Proof. For $\mathbb{p} = (\ell+\hbar)/\hbar$ and $\mathbb{q} = (\ell+\hbar)/\ell$, making $f = \mathscr{U}$ and $g = \mathscr{V}$ in the left-hand inequality of (86) yields with $\mathbb{m} \leqslant \mathscr{U}^{\mathbb{p}}/\mathscr{V}^{\mathbb{q}} \leqslant \mathbb{M}$

$$\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w\mathscr{U}^{\frac{\ell+\hbar}{\hbar}}\right)^{\frac{\hbar}{\ell+\hbar}}(x;\sigma)\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w\mathscr{V}^{\frac{\ell+\hbar}{\ell}}\right)^{\frac{\ell}{\ell+\hbar}}(x;\sigma) \leqslant \left(\frac{\mathbb{M}}{\mathbb{m}}\right)^{\frac{\ell\hbar}{(\ell+\hbar)^2}}\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w\mathscr{U}\mathscr{V}\right)(x;\sigma). \quad (171)$$

Let $\mathscr{U} = f^{\hbar}/g^{\ell\hbar/(\ell+\hbar)}$ and $\mathscr{V} = g^{\ell\hbar/(\ell+\hbar)}$, from the condition $0 < \mathbb{m} \leqslant (f(t)/g(t))^{\ell+\hbar} \leqslant \mathbb{M}$, then $\mathbb{m} \leqslant \mathscr{U}^{\mathbb{p}}/\mathscr{V}^{\mathbb{q}} \leqslant \mathbb{M}$. Using the inequality (171), we obtain

$$\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} w\left(\frac{f^{\ell+\hbar}}{g^{\ell}}\right)\right)^{\frac{\hbar}{\ell+\hbar}}(x;\sigma)\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wg^{\hbar}\right)^{\frac{\ell}{\ell+\hbar}}(x;\sigma) \leqslant \left(\frac{\mathbb{M}}{\mathbb{m}}\right)^{\frac{\ell\hbar}{(\ell+\hbar)^2}}\left({}_{\xi}^{\phi}\Theta_{u^+}^{M} wf^{\hbar}\right)(x;\sigma), \quad (172)$$

which is the right-hand anticipated inequality (170) by traightforward calculation. This completes the proof. □

Theorem 20. *Suppose that f, g, w are three continuous positive functions on $[u,v]$ satisfying $0 < \mathbb{m} \leqslant f(t)/g(t) \leqslant \mathbb{M}$ for all $t \in [u,v]$ and $\mathbb{p} \geqslant 1$. Then, for $x \in [u,v]$, the following FIO inequality holds*

$$\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w f^{\mathbb{P}}\right)(x;\sigma) + \left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w g^{\mathbb{P}}\right)(x;\sigma) \leqslant \mathbb{c}_6 \left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w (f+g)^{\mathbb{P}}\right)(x;\sigma) + \mathbb{c}_7 \left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w (f-g)^{\mathbb{P}}\right)(x;\sigma), \quad (173)$$

where $\mathbb{c}_6 = \left((\mathbb{M}+1)^{\mathbb{P}} + \mathbb{M}^{\mathbb{P}}(\mathbb{m}+1)^{\mathbb{P}}\right) / \left(2(\mathbb{m}+1)^{\mathbb{P}}(\mathbb{M}+1)^{\mathbb{P}}\right)$ *and* $\mathbb{c}_7 = (1+\mathbb{m}^{\mathbb{P}}) / \left(2(\mathbb{m}-1)^{\mathbb{P}}\right)$.

Proof. It follows from (116) and (118) with $\kappa = 1$ that

$$\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w f^{\mathbb{P}}\right)(x;\sigma) + \left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w g^{\mathbb{P}}\right)(x;\sigma) \leqslant \left(\left(\frac{1}{\mathbb{m}+1}\right)^{\mathbb{P}} + \left(\frac{\mathbb{M}}{\mathbb{M}+1}\right)^{\mathbb{P}}\right) \cdot \left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w (f+g)^{\mathbb{P}}\right)(x;\sigma). \quad (174)$$

On the other hand, from (140) and (143), we have

$$\left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w g^{\mathbb{P}}\right)(x;\sigma) + \left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w f^{\mathbb{P}}\right)(x;\sigma) \leqslant \left(\left(\frac{1}{\mathbb{m}-1}\right)^{\mathbb{P}} + \left(\frac{\mathbb{m}}{\mathbb{m}-1}\right)^{\mathbb{P}}\right) \cdot \left({}^{\phi}_{\xi}\Theta^{M}_{u^+} w (f-g)^{\mathbb{P}}\right)(x;\sigma). \quad (175)$$

Adding (174) and (175) yields the expected inequality (173). This finishes the proof. □

6. Conclusions

In this paper, we have investigated certain novel reverse Hölder- and Minkowski-type inequalities for modified unified generalized FIOs with extended unified MLFs. A large amount of the existing fractional Hölder- and Minkowski-type integral inequalities in the literature can be seen as the special cases of the main results of this paper. As applications, the reverse analogs of weighted Radon-, Jensen- and power mean-type inequalities for modified unified generalized FIOs with extended unified MLFs have been also presented. Following the main results of this article, we will investigate some Grüss-, Pólya-Szegö-, Beckenbach-, Bellman-type inequalities and related results for modified unified generalized FIOs with extended unified MLFs in future research.

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