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Multiple Solutions to a Non-Local Problem of Schrödinger–Kirchhoff Type in $\mathbb{R}^N$

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Abstract: The main purpose of this paper is to show the existence of a sequence of infinitely many small energy solutions to the nonlinear elliptic equations of Kirchhoff–Schrödinger type involving the fractional p-Laplacian by employing the dual fountain theorem as a key tool. Because of the presence of a non-local Kirchhoff coefficient, under conditions on the nonlinear term given in the present paper, we cannot obtain the same results concerning the existence of solutions in similar ways as in the previous related works. For this reason, we consider a class of Kirchhoff coefficients that are different from before to provide our multiplicity result. In addition, the behavior of nonlinear terms near zero is slightly different from previous studies.

Keywords: fractional p-Laplacian; Kirchhoff function; weak solution; dual fountain theorem

MSC: 35D30; 35J20; 35J60; 35J66

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1. Introduction

In the last two decades, an increasing amount of attention has been devoted to the study of fractional Sobolev spaces and the corresponding non-local equations because they can be substantiated as a model for many physical phenomena that arise in the research of optimization; fractional quantum mechanics; the thin obstacle problem; anomalous diffusion in plasma; frames propagation; geophysical fluid dynamics; and American options in finances, image process, game theory, and Lévy processes (see [1–6] and references therein for more details).

In this direction, the present paper is devoted to a non-local problem of Schrödinger–Kirchhoff type as follows:

$$M\big([w]_{s,p}\big)\mathfrak{L}_{\mathfrak{K}}w(y)+\mathcal{V}(y)|w|^{p-2}w=\lambda g(y,w)\quad\text{in}\quad\mathbb{R}^N,\tag{1}$$

where $[w]_{s,p}:=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|w(y)-w(x)|^p\mathfrak{K}(y,x)\,dy\,dx$, $1<p<\frac{N}{s}$, $M\in C(\mathbb{R}^+)$ is a function of Kirchhoff type, $\mathcal{V}:\mathbb{R}^N\to(0,\infty)$ is continuous potential function, and a Carathéodory function $g:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ satisfies the subcritical and p-superlinear nonlinearity. Here, $\mathfrak{L}_{\mathfrak{K}}$ is non-local operator defined pointwise as:

$$\mathfrak{L}_{\mathfrak{K}}w(y)=2\int_{\mathbb{R}^N}|w(y)-w(x)|^{p-2}(w(y)-w(x))\mathfrak{K}(y,x)dx\quad\text{for all } y\in\mathbb{R}^N,$$

where $\mathfrak{K}:\mathbb{R}^N\times\mathbb{R}^N\to(0,+\infty)$ is a kernel function with the following properties:

($\mathcal{L}1$) $a\mathfrak{K}\in L^1(\mathbb{R}^N\times\mathbb{R}^N)$, where $a(y,x)=\min\{|y-x|^p,1\}$;
($\mathcal{L}2$) there is a constant $\beta_0>0$ such that $\mathfrak{K}(y,x)|y-x|^{N+sp}\geq\beta_0$ for almost all $(y,x)\in\mathbb{R}^N\times\mathbb{R}^N$ and $y\neq x$, where $0<s<1$;

($\mathcal{L}3$) $\mathfrak{K}(y,x) = \mathfrak{K}(x,y)$ for all $(y,x) \in \mathbb{R}^N \times \mathbb{R}^N$.

When$\mathfrak{K}(y,x) = |y-x|^{-(N+sp)}$, the operator $\mathfrak{L}_{\mathfrak{K}}$ becomes the fractional p-Laplacian operator $(-\Delta)_p^s$ defined as:

$$(-\Delta)_p^s w(y) = 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(y)} \frac{|w(y)-w(x)|^{p-2}(w(y)-w(x))}{|y-x|^{N+sp}}\, dx, \quad y \in \mathbb{R}^N,$$

where $s \in (0,1)$ and $B_\varepsilon(y) := \{x \in \mathbb{R}^N : |y-x| \le \varepsilon\}$. Moreover, the Kirchhoff coefficient $M : [0,\infty) \to \mathbb{R}^+$ fulfils the following requirements:

($\mathcal{M}1$) $M \in C(\mathbb{R}^+)$ fulfils $\inf_{\zeta \in \mathbb{R}^+} M(\zeta) \ge \tau_0$ for a positive constant τ_0;

($\mathcal{M}2$) there exists a constant $\vartheta \ge 1$ and a non-negative constant K such that $\vartheta \mathcal{M}(\zeta) = \vartheta \int_0^\zeta M(\eta) d\eta \ge M(\zeta)\zeta$ and:

$$\widehat{\mathcal{M}}(t\zeta) \le \widehat{\mathcal{M}}(\zeta) + K$$

for $\zeta \ge 0$ and $t \in [0,1]$, where $\widehat{\mathcal{M}}(\zeta) = \vartheta \mathcal{M}(\zeta) - M(\zeta)\zeta$.

The study on elliptic problems with the non-local Kirchhoff term was initially introduced by Kirchhoff [7] to investigate an expansion of the classical D'Alembert's wave equation by taking the changes in the length of the strings during the vibrations into account. The variational problems of Kirchhoff type have a powerful background in various applications in physics and have been concentrically explored by many researchers in recent decades; as an illustration, see [8–24] and the references therein. A detailed discussion about the physical implications based on the fractional Kirchhoff model was first proposed by the work of Fiscella and Valdinoci [25]. In their paper, they obtained the existence of nontrivial solutions by exploiting the mountain pass theorem and a truncation argument on the non-local Kirchhoff term. In particular, the condition imposed on the non-degenerate Kirchhoff function $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is an increasing and continuous function with ($\mathcal{M}1$); see also [26] and references therein. However, this increasing condition eliminates the case that is not monotone, for instance:

$$M(\zeta) = (1+\zeta)^{-1} + (1+\zeta)^k \text{ with } 0 < k < 1$$

for all $\zeta \in \mathbb{R}_0^+$. For this purpose, the authors in [22] discussed the existence of multiple solutions to a class of fractional p-Laplacian equations of Schrödinger–Kirchhoff type, where the Kirchhoff function M is continuous, and satisfies ($\mathcal{M}1$) and the condition:

($\mathcal{M}3$) for $0 < s < 1$, there is $\vartheta \in [1, \frac{N}{N-sp})$ such that $\vartheta \mathcal{M}(\zeta) \ge M(\zeta)\zeta$ for any $t \ge 0$.

We also recommend the papers [22,27–34] for some recent results in this direction.

Very recently, the existence result of a positive ground state solution for the elliptic problem of Kirchhoff type with critical exponential growth was studied by Huang-Deng [35] under the following condition:

($\mathcal{M}4$) there exists $\vartheta > 1$ such that $\frac{M(\zeta)}{\zeta^{\theta-1}}$ is non-increasing for $\zeta > 0$.

From this condition, it is immediate that $\widehat{\mathcal{M}}(\zeta)$ is non-decreasing for all $\zeta \ge 0$ and, thus, we have the following condition:

($\mathcal{M}3$)$'$ there exists $\vartheta > 1$ such that $\vartheta \mathcal{M}(\zeta) \ge M(\zeta)\zeta$ for any $\zeta \ge 0$.

This is weaker than ($\mathcal{M}4$). A typical model for M satisfying ($\mathcal{M}1$) and ($\mathcal{M}3$)$'$ is given by $M(\zeta) = 1 + a\zeta^\vartheta$ with $a \ge 0$ for all $\zeta \ge 0$; hence, the condition ($\mathcal{M}3$)$'$ includes the above classical example, as well as the case that is not monotone. For this reason, the nonlinear elliptic equations with the Kirchhoff coefficient satisfying ($\mathcal{M}3$)$'$ (or ($\mathcal{M}3$)) have recently been extensively researched by many authors; see [11,15,18,22,28,32–34,36–38].

Remark 1. *Let us consider:*

$$M(\zeta) = \left(1 + \frac{\zeta^r}{\sqrt{1+\zeta^{2r}}}\right)\zeta^{r-1} + (1+\zeta)^{-\alpha}$$

with its primitive function:

$$\mathcal{M}(\zeta) = \frac{1}{r}\left(\zeta^r + \sqrt{1+\zeta^{2r}} - 1\right) + \frac{1}{1-\alpha}(1+\zeta)^{1-\alpha} - \frac{1}{1-\alpha}$$

for all $\zeta \geq 0$. Then, it is clear that:

$$\widehat{\mathcal{M}}(\zeta) = \left(\frac{\vartheta}{r} - 1\right)\zeta^r + \left(\frac{\vartheta}{r} - \frac{\zeta^{2r}}{1+\zeta^{2r}}\right)\sqrt{1+\zeta^{2r}} + \left(\frac{\vartheta}{1-\alpha}(1+\zeta) - \zeta\right)(1+\zeta)^{-\alpha} - \frac{\vartheta}{1-\alpha} - \frac{\vartheta}{r}.$$

If $r = 2$ and $N = 4$ in $(\mathcal{M}3)'$, then we cannot find a constant $\vartheta \in [1,2)$ satisfying $\widehat{\mathcal{M}}(\zeta) \geq 0$ for any $\zeta \geq 0$, by being $\lim_{\zeta\to\infty} \widehat{\mathcal{M}}(\zeta) = -\infty$. If $1 < \alpha \leq r$, there exists a constant $\vartheta \geq r$ such that $\widehat{\mathcal{M}}(\zeta) \geq 0$ holds for all $\zeta \geq 0$.

Additionally, if we set $r = \vartheta = 1.5$ and $1 < \alpha \leq r$, we then have:

$$\widehat{\mathcal{M}}(\zeta) = \frac{1}{\sqrt{1+\zeta^3}} + \left(\frac{3}{2-2\alpha}(1+\zeta) - \zeta\right)(1+\zeta)^{-\alpha} - \frac{3}{2-2\alpha} - 1$$

which is not non-decreasing and $\widehat{\mathcal{M}}(\zeta) \geq 0$ for all $\zeta \geq 0$ from the simple computation. Hence, this example does not satisfy the condition $(\mathcal{M}4)$. This implies that:

$$\widehat{\mathcal{M}}(\zeta) - \widehat{\mathcal{M}}(t\zeta) \geq 0$$

does not hold. However, we can choose a positive constant K satisfying our condition $(\mathcal{M}2)$. Of course, since $(\mathcal{M}4)$ implies the condition $(\mathcal{M}2)$, our condition is a generalization of the condition $(\mathcal{M}4)$.

Inspired by the fact illustrated in the remark above, the primary aim of the present paper is devoted to deriving the multiplicity result of solutions to the Kirchhoff–Schrödinger type problem with the fractional p-Laplacian on a class of a non-local Kirchhoff coefficient M that differs slightly from the previous related studies [11,15,18,22,28,32–34,36–38]. In particular, for the superlinear p-Laplacian problem:

$$-\mathrm{div}(|\nabla w|^{p-2}\nabla w) + \mathcal{V}(y)|w|^{p-2}w = g(y,w) \text{ in } \mathbb{R}^N,$$

the existence result of a ground-state solution is dealt with in the paper [39]. Here, the potential function $\mathcal{V} \in C(\mathbb{R}^N)$ fulfils appropriate conditions and the Carathéodory function $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ fulfils the conditions (F1)–(F2) and the conditions as follows:

(g1) there is a constant $\mu \geq 1$, such that:

$$\mu\left[g(y,\zeta)\zeta - p\vartheta G(y,\zeta)\right] \geq g(y,t\zeta)(t\zeta) - p\vartheta G(y,t\zeta)$$

for $(y,\zeta) \in \mathbb{R}^N \times \mathbb{R}$ and $t \in [0,1]$, where ϑ was given in $(\mathcal{M}2)$ and $G(y,\zeta) = \int_0^\zeta g(y,t)\,dt$;

(g2) $G(y,\zeta) = o(|\zeta|^p)$ as $\zeta \to 0$ uniformly for all $y \in \mathbb{R}^N$.

The condition (g1) was initially provided by the work of Jeanjean [40]. Afterward, there has been considerable research dealing with the p-Laplacian problem; see [39,41] and see also [18,42–44] for variable exponents $p(\cdot)$. However, because of the presence of a non-local Kirchhoff coefficient M, the same results concerning the existence of solutions cannot be obtained, even if we follow the similar ways as in [18,39,41,42,44]. More precisely, under conditions (F1) and (g1)–(g2), we cannot verify the compactness condition of the

Palais–Smale type for an energy functional corresponding to (1) when assumptions ($\mathcal{M}1$) and ($\mathcal{M}3$) are satisfied. In particular, to guarantee this compactness condition of an energy functional corresponding to problems of the elliptic type with the nonlinear term satisfying (F2), it is crucial that $\widehat{\mathcal{M}}(\zeta)$ is non-decreasing for all $\zeta \geq 0$. Because of this reason, when ($\mathcal{M}3$) is satisfied, many researchers have considered some conditions of the nonlinear term which differ from (F2); see [11,15,18,22,28,31–34,36–38,45]. From this perspective, one of the novelties of the present paper is to accomplish the existence of a sequence of small energy solutions to (1) without the monotonicity of $\widehat{\mathcal{M}}$ when (F2) is assumed. The other one is to provide our main result without assuming the condition (g2), which is essential in obtaining the compactness condition of the Palais–Smale type and ensuring assumptions in the dual fountain theorem. These arguments are based on the recent work [46].

To this end, by utilizing the dual fountain theorem as the key tool, we provide the result of the existence of multiple solutions on classes of the Kirchhoff coefficient M and the nonlinear function g, which differ from the previous related studies [11,15,18,22,28,31–34,36–38,45]. The basic idea of our proof for the existence of a sequence of small energy solutions to problem (1) comes from recent studies [45–47]. This multiplicity result for nonlinear problems of the elliptic type is specifically inspired by contributions from recent studies [3,18,21,39,41,42,48–51].

The structure of the present paper is as follows. Section 2 presents some requisite preliminary knowledge of the function space to be utilized throughout this paper. In Section 3, we present the variational framework associated with problem (1), and then we illustrate the result of the existence of a sequence of nontrivial small energy solutions to the fractional p-Laplacian equations under suitable assumptions.

2. Preliminaries

In this section, we present some useful definitions and fundamental properties of the fractional Sobolev spaces that will be used in the present paper. Let $0 < s < 1 < q \in (1, +\infty)$ and q_s^* be the fractional critical Sobolev exponent, that is:

$$q_s^* := \begin{cases} \frac{Nq}{N-sq} & \text{if } sq < N, \\ +\infty & \text{if } sq \geq N. \end{cases}$$

We define the fractional Sobolev space $W^{s,q}(\mathbb{R}^N)$ as follows:

$$W^{s,q}(\mathbb{R}^N) := \left\{ w \in L^q(\mathbb{R}^N) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w(y) - w(x)|^q}{|y-x|^{N+qs}} \, dydx < +\infty \right\},$$

endowed with the norm:

$$\|w\|_{W^{s,q}(\mathbb{R}^N)} := \left(\|w\|_{L^q(\mathbb{R}^N)}^q + |w|_{W^{s,q}(\mathbb{R}^N)}^q \right)^{\frac{1}{q}},$$

where:

$$\|w\|_{L^q(\mathbb{R}^N)}^q := \int_{\mathbb{R}^N} |w|^q \, dx \quad \text{and} \quad |w|_{W^{s,q}(\mathbb{R}^N)}^q := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w(y) - w(x)|^q}{|y-x|^{N+qs}} \, dydx.$$

Then, $W^{s,q}(\mathbb{R}^N)$ is a separable and reflexive Banach space. Additionally, the space $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{s,q}(\mathbb{R}^N)$, namely, $W_0^{s,q}(\mathbb{R}^N) = W^{s,q}(\mathbb{R}^N)$ (see [5,52]).

Lemma 1 ([5,53])**.** *Let $0 < s < 1 < p < +\infty$ with $ps < N$. Then, there is a constant $C > 0$ depending on s, p, and N, such that:*

$$\|w\|_{L^{p_s^*}(\mathbb{R}^N)} \leq C \, |w|_{W^{s,p}(\mathbb{R}^N)}$$

for all $w \in W^{s,p}(\mathbb{R}^N)$. Additionally, the space $W^{s,p}(\mathbb{R}^N)$ is continuously embedded in $L^t(\mathbb{R}^N)$ for any $t \in [p, p_s^*]$. Moreover, the embedding:

$$W^{s,p}(\mathbb{R}^N) \hookrightarrow L^t_{loc}(\mathbb{R}^N)$$

is compact for $t \in [p, p_s^*)$.

Now, let us consider the space $W^{s,p}_{\mathfrak{K}}(\mathbb{R}^N)$ defined as follows:

$$W^{s,p}_{\mathfrak{K}}(\mathbb{R}^N) := \left\{ w \in L^p(\mathbb{R}^N) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |w(y) - w(x)|^p \mathfrak{K}(y,x)\, dydx < +\infty \right\},$$

where $\mathfrak{K} : \mathbb{R}^N \times \mathbb{R}^N \setminus \{(0,0)\} \to (0, +\infty)$ is a kernel function satisfying the properties $(\mathcal{L}1)$–$(\mathcal{L}3)$. By the condition $(\mathcal{L}1)$, the function:

$$(y,x) \mapsto (w(y) - w(x))\mathfrak{K}^{\frac{1}{p}}(y,x) \in L^p(\mathbb{R}^{2N})$$

for all $w \in C_0^\infty(\mathbb{R}^N)$. Let us denote by $W^{s,p}_{\mathfrak{K}}(\mathbb{R}^N)$ the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm:

$$\|w\|_{W^{s,p}_{\mathfrak{K}}(\mathbb{R}^N)} := \left(\|w\|^p_{L^p(\mathbb{R}^N)} + |w|^p_{W^{s,p}_{\mathfrak{K}}(\mathbb{R}^N)} \right)^{\frac{1}{p}},$$

where:

$$|w|^p_{W^{s,p}_{\mathfrak{K}}(\mathbb{R}^N)} := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |w(y) - w(x)|^p \mathfrak{K}(y,x)\, dydx.$$

Lemma 2 ([33])**.** *Let $\mathfrak{K} : \mathbb{R}^N \times \mathbb{R}^N \setminus \{(0,0)\} \to (0,\infty)$ be a kernel function with the conditions $(\mathcal{L}1)$–$(\mathcal{L}3)$. If $w \in W^{s,p}_{\mathfrak{K}}(\mathbb{R}^N)$, then $w \in W^{s,p}(\mathbb{R}^N)$. Moreover:*

$$\|w\|_{W^{s,p}(\mathbb{R}^N)} \le \max\{1, \beta_0^{-\frac{1}{p}}\} \|w\|_{W^{s,p}_{\mathfrak{K}}(\mathbb{R}^N)}.$$

From Lemmas 1 and 2, we can obtain the following assertion immediately.

Lemma 3 ([33])**.** *Let $\mathfrak{K} : \mathbb{R}^N \times \mathbb{R}^N \setminus \{(0,0)\} \to (0,\infty)$ satisfy the conditions $(\mathcal{L}1)$–$(\mathcal{L}3)$. Then, there exists a constant $C_0 > 0$ depending on s, p, and N, such that for any $w \in W^{s,p}_{\mathfrak{K}}(\mathbb{R}^N)$ and $p \le q \le p_s^*$:*

$$\begin{aligned} \|w\|^p_{L^q(\mathbb{R}^N)} &\le C_0 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w(y) - w(x)|^p}{|y - x|^{N+ps}} dydx \\ &\le \frac{C_0}{\beta_0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |w(y) - w(x)|^p \mathfrak{K}(y,x)\, dydx. \end{aligned}$$

Additionally, the space $W^{s,p}_{\mathfrak{K}}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ for any $q \in [p, p_s^*]$. Furthermore, the embedding:

$$W^{s,p}_{\mathfrak{K}}(\mathbb{R}^N) \hookrightarrow L^q_{loc}(\mathbb{R}^N)$$

is compact for $q \in [p, p_s^*)$.

Next, we suppose that the potential function $\mathcal{V}$ satisfies:

(V) $\mathcal{V} \in C(\mathbb{R}^N)$, $\inf_{y \in \mathbb{R}^N} \mathcal{V}(y) > 0$, and $\operatorname{meas}\{y \in \mathbb{R}^N : \mathcal{V}(y) \le \mathcal{V}_0\} < +\infty$ for all $\mathcal{V}_0 \in \mathbb{R}$.

On the linear subspace:

$$W^{s,p}_{\mathfrak{K},\mathcal{V}}(\mathbb{R}^N) := \left\{ w \in L^p_{\mathcal{V}}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |w(y) - w(x)|^p \mathfrak{K}(y,x)\, dydx < +\infty \right\},$$

we equip the norm:

$$\|w\|_{W^{s,p}_{\mathfrak{K},\mathcal{V}}(\mathbb{R}^N)} := \left(\|w\|^p_{L^p_{\mathcal{V}}(\mathbb{R}^N)} + |w|^p_{W^{s,p}_{\mathfrak{K}}(\mathbb{R}^N)}\right)^{\frac{1}{p}},$$

where:

$$\|w\|^p_{L^p_{\mathcal{V}}(\mathbb{R}^N)} := \int_{\mathbb{R}^N} \mathcal{V}(y)|w|^p\,dy.$$

Then, $W^{s,p}_{\mathfrak{K},\mathcal{V}}(\mathbb{R}^N)$ is continuously embedded into $W^{s,p}(\mathbb{R}^N)$ as a closed subspace. Therefore, $(W^{s,p}_{\mathfrak{K},\mathcal{V}}(\mathbb{R}^N), \|\cdot\|_{W^{s,p}_{\mathfrak{K},\mathcal{V}}(\mathbb{R}^N)})$ is also a separable reflexive Banach space.

In view of Lemma 3, the following assertion was carried out by the same scheme as that in [54].

Lemma 4. *Assume that the conditions* $(\mathcal{L}1)$–$(\mathcal{L}3)$ *and* (V) *hold. Let* $0 < s < 1 < p < +\infty$ *with* $ps < N$*. Then, there is compact embedding* $W^{s,p}_{\mathfrak{K},\mathcal{V}}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ *for* $q \in [p, p^*_s)$*.*

Throughout this paper, we denote $E := W^{s,p}_{\mathfrak{K},\mathcal{V}}(\mathbb{R}^N)$, and the kernel function $\mathfrak{K} : \mathbb{R}^N \times \mathbb{R}^N \setminus \{(0,0)\} \to (0,\infty)$ ensures the assumptions $(\mathcal{L}1)$–$(\mathcal{L}3)$. Additionally, let the Kirchhoff function M satisfy the conditions $(\mathcal{M}1)$–$(\mathcal{M}2)$ and the potential $\mathcal{V}$ fulfil the condition (V). $\langle\cdot,\cdot\rangle$ denotes the pairing of E and its dual E^*.

3. Variational Framework and Main Result

This section provides the existence result of infinitely many nontrivial small energy solutions to (1) by utilizing the dual fountain theorem under appropriate assumptions.

Definition 1. *We say that* $w \in E$ *is a weak solution of* (1) *if:*

$$M([w]_{s,p}) \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |w(y)-w(x)|^{p-2}(w(y)-w(x))(z(y)-z(x))\mathfrak{K}(y,x)\,dydx$$
$$+ \int_{\mathbb{R}^N} \mathcal{V}(y)|w|^{p-2}wz\,dy = \lambda \int_{\mathbb{R}^N} g(y,w)z\,dy$$

for any $z \in E$*, where:*

$$[w]_{s,p} := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |w(y)-w(x)|^p \mathfrak{K}(y,x)\;dy\,dx.$$

Let us define the functional $A : E \to \mathbb{R}$ by:

$$A(w) = \frac{1}{p}\mathcal{M}([w]_{s,p}) + \int_{\mathbb{R}^N} \frac{\mathcal{V}(y)}{p}|w|^p\,dy.$$

Let $G(y,\zeta) = \int_0^\zeta g(y,t)\,dt$. Suppose that:

(F1) $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is the Carathéodory function and there are $\rho_2 > 0$ and $\kappa \in [p, p^*_s)$, $0 \le \rho_1 \in L^{\kappa'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that:

$$|g(y,\zeta)| \le \rho_1(y) + \rho_2|\zeta|^{\ell-1}$$

for all $(y,\zeta) \in \mathbb{R}^N \times \mathbb{R}$ where $1 < p < \ell < p^*_s$.

(F2) There is $\theta \ge 1$ such that:

$$\theta\mathcal{G}(y,\zeta) \ge \mathcal{G}(y,t\zeta)$$

for $(y,\zeta) \in \mathbb{R}^N \times \mathbb{R}$ and $t \in [0,1]$, where $\mathcal{G}(y,\zeta) = g(y,\zeta)\zeta - p\vartheta G(y,\zeta)$, and ϑ is given in $(\mathcal{M}2)$.

(F3) There are real numbers $C > 0$, $1 < m < p$, $\tau > 1$ with $p \le \tau' m \le p^*$ and a positive function $\nu \in L^\infty(\mathbb{R}^N) \cap L^\tau(\mathbb{R}^N)$ such that:

$$\liminf_{|\zeta|\to 0} \frac{g(y,\zeta)}{\nu(y)|\zeta|^{m-2}\zeta} \ge C$$

uniformly for almost all $y \in \mathbb{R}^N$.

(F4) $\lim_{|\zeta|\to\infty} \frac{G(y,\zeta)}{|\zeta|^{\vartheta p}} = \infty$ uniformly for almost all $y \in \mathbb{R}^N$.

Remark 2. *Let us consider the function:*

$$g(y,\zeta) = \sigma(y)\left(\nu(y)|\zeta|^{m-2}\zeta + |\zeta|^{\ell-2}\zeta \ln(1+|\zeta|) + \frac{|\zeta|^{\ell-1}\zeta}{1+|\zeta|}\right)$$

with its primitive function:

$$G(y,\zeta) = \sigma(y)\left(\frac{\nu(y)}{m}|\zeta|^m + \frac{1}{\ell}|\zeta|^\ell \ln(1+|\zeta|)\right)$$

for all $\zeta \in \mathbb{R}$, where $m < p < \ell$ and a continuous function $\sigma : \mathbb{R}^N \to \mathbb{R}$ holds $0 < \inf_{y\in\mathbb{R}^N} \sigma(y) \le \sup_{y\in\mathbb{R}^N} \sigma(y) < \infty$. Then, when $\vartheta = 1$, this example fulfils the assumptions (F1)–(F4)*, but not* (g2)*.*

Let the functional $B_\lambda : E \to \mathbb{R}$ be defined as:

$$B_\lambda(v) = \lambda \int_{\mathbb{R}^N} G(y,v)\, dy$$

for any $v \in E$. Then, it is trivial that B_λ is of class $C^1(E,\mathbb{R})$ with:

$$\left\langle B_\lambda'(v), z\right\rangle = \lambda \int_{\mathbb{R}^N} g(y,v) z\, dy$$

for any $v, z \in E$. Additionally, the functional $I_\lambda : E \to \mathbb{R}$ is defined as:

$$I_\lambda(v) = A(v) - B_\lambda(v)$$

for any $v \in E$. Then, taking into account Lemma 3.2 in [43], we get that the functional $I_\lambda \in C^1(E,\mathbb{R})$, and its Fréchet derivative is:

$$\begin{aligned}\left\langle I_\lambda'(w), z\right\rangle =& M([w]_{s,p}) \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |w(y)-w(x)|^{p-2}(w(y)-w(x))(z(y)-z(x))\mathfrak{K}(y,x)\, dy\, dx \\ &+ \int_{\mathbb{R}^N} \mathcal{V}(y)|w|^{p-2} wz\, dy - \lambda \int_{\mathbb{R}^N} g(y,w) z\, dy\end{aligned}$$

for any $w, z \in E$.

Definition 2. *We say that I_λ satisfies the Cerami condition at level c ($(C)_c$-condition, for short) if any $(C)_c$-sequence $\{w_n\}_n \subset E$ for any $c \in \mathbb{R}$, i.e., $I_\lambda(w_n) \to c$ and $\|I_\lambda'(w_n)\|_{E^*}(1 + \|w_n\|_E) \to 0$ as $n \to \infty$ has a convergent subsequence in E.*

The following assertion is crucial to establish the existence of multiple solutions to our problem. The fundamental idea of proofs of this consequence follows similar arguments as in [46]; see also [18].

Lemma 5. *It is assumed that* (F1), (F2), *and* (F4) *hold. Then, the functional I_λ assures the $(C)_c$-condition for any $\lambda > 0$.*

Proof. For any $c \in \mathbb{R}$, let $\{w_n\}$ be a $(C)_c$-sequence in E, i.e.,

$$I_\lambda(w_n) \to c \text{ and } \langle I'_\lambda(w_n), w_n\rangle = o(1) \to 0, \text{ as } n \to \infty. \tag{2}$$

From the same arguments as in [22], we know I'_λ is of type (S_+) and so it suffices to show the boundedness of $\{w_n\}$ in E since E is reflexive. To do this, suppose to the contrary that $\|w_n\|_E > 1$ and $\|w_n\|_E \to \infty$ as $n \to \infty$, and a sequence $\{z_n\}$ is defined by $z_n = w_n/\|w_n\|_E$. Then, up to a subsequence, still denoted by $\{z_n\}$, we obtain $z_n \rightharpoonup z$ in E as $n \to \infty$, and by Lemma 4:

$$z_n(y) \to z(y) \text{ a.e. in } \mathbb{R}^N,\ z_n \to z \text{ in } L^q(\mathbb{R}^N), \text{ and } z_n \to z \text{ in } L^p(\mathbb{R}^N) \tag{3}$$

as $n \to \infty$, where $p < q < {p_s}^*$.

Due to $(\mathcal{M}1)$–$(\mathcal{M}2)$ and the relation (2), we have:

$$\begin{aligned} I_\lambda(w_n) &= \frac{1}{p}\mathcal{M}([w_n]_{s,p}) + \frac{1}{p}\int_{\mathbb{R}^N} \mathcal{V}(y)|w_n|^p\,dy - \lambda\int_{\mathbb{R}^N} G(y,w_n)\,dy \\ &\geq \frac{\tau_0}{\vartheta p}[w_n]_{s,p} + \frac{1}{p}\int_{\mathbb{R}^N} \mathcal{V}(y)|w_n|^p dy - \lambda\int_{\mathbb{R}^N} G(y,w_n)dy \\ &\geq \frac{\min\{\tau_0,\vartheta\}}{\vartheta p}\Big([w_n]_{s,p} + \int_{\mathbb{R}^N} \mathcal{V}(y)|w_n|^p dy\Big) - \lambda\int_{\mathbb{R}^N} G(y,w_n)dy \\ &\geq \frac{\min\{\tau_0,\vartheta\}}{\vartheta p}\|w_n\|_E^p - \lambda\int_{\mathbb{R}^N} G(y,w_n)\,dy. \end{aligned} \tag{4}$$

Since $\|w_n\|_E \to \infty$ as $n \to \infty$, we assert by (4) that:

$$\lambda\int_{\mathbb{R}^N} G(y,w_n)\,dy \geq \frac{\min\{\tau_0,\vartheta\}}{\vartheta p}\|w_n\|_E^p - I_\lambda(w_n) \to \infty \quad \text{as} \quad n \to \infty. \tag{5}$$

Moreover, the assumption $(\mathcal{M}2)$ and the relation (5) imply that:

$$\begin{aligned} I_\lambda(w_n) &= \frac{1}{p}\mathcal{M}([w_n]_{s,p}) + \frac{1}{p}\int_{\mathbb{R}^N} \mathcal{V}(y)|w_n|^p dy - \lambda\int_{\mathbb{R}^N} G(y,w_n)dy \\ &\leq \frac{\mathcal{M}(1)}{p}\Big(1 + ([w_n]_{s,p})^\vartheta\Big) + \frac{1}{p}\int_{\mathbb{R}^N} \mathcal{V}(y)|w_n|^p dy - \lambda\int_{\mathbb{R}^N} G(y,w_n)dy \\ &\leq C_1\frac{\max\{\mathcal{M}(1),1\}}{p}\Big(1 + [w_n]_{s,p} + \int_{\mathbb{R}^N} \mathcal{V}(y)|w_n|^p dy\Big)^\vartheta - \lambda\int_{\mathbb{R}^N} G(y,w_n)dy \\ &\leq 2^\vartheta C_2\|w_n\|_E^{\vartheta p} - \lambda\int_{\mathbb{R}^N} G(y,w_n)dy \end{aligned} \tag{6}$$

for some positive constants C_1, C_2. Then, we obtain by the relation (6) that:

$$2^\vartheta C_2 \geq \frac{1}{\|w_n\|_E^{\vartheta p}}\Big(\lambda\int_{\mathbb{R}^N} G(y,w_n)\,dy + I_\lambda(w_n)\Big). \tag{7}$$

It follows from (F4) that there is a positive constant $\zeta_0 > 1$, such that $G(y,\zeta) > |\zeta|^{\vartheta p}$ for all $y \in \mathbb{R}^N$ and for any $|\zeta| > \zeta_0$. Owing to (F1), we infer $|G(y,\zeta)| \leq \mathcal{T}$ for all $(y,\zeta) \in \mathbb{R}^N \times [-\zeta_0,\zeta_0]$ and for some positive constant $\mathcal{T}$. We can choose a $\mathcal{T}_0 \in \mathbb{R}$ such that $G(y,\zeta) \geq \mathcal{T}_0$ for all $(y,\zeta) \in \mathbb{R}^N \times \mathbb{R}$, and, thus:

$$\frac{G(y,w_n) - \mathcal{T}_0}{\|w_n\|_E^{\vartheta p}} \geq 0, \tag{8}$$

for all $n \in \mathbb{N}$ and for all $y \in \mathbb{R}^N$. Put $\Gamma_1 = \{y \in \mathbb{R}^N : z(y) \neq 0\}$. Assume that $\text{meas}(\Gamma_1) \neq 0$. In accordance with (3), we arrive at the conclusion that $|w_n(y)| = |z_n(y)|\|w_n\|_E \to \infty$ as $n \to \infty$ for all $y \in \Gamma_1$. Additionally, on account of (F4), we get:

$$\lim_{n\to\infty} \frac{G(y,w_n)}{\|w_n\|_E^{\vartheta p}} = \lim_{n\to\infty} \frac{G(y,w_n)}{|w_n|^{\vartheta p}} |z_n|^{\vartheta p} = \infty \tag{9}$$

for all $y \in \Gamma_1$. According to the Fatou lemma and (5)–(9), we infer that:

$$\begin{aligned}
2^{\vartheta} C_2 &= \liminf_{n\to\infty} \frac{2^{\vartheta} C_2 \int_{\mathbb{R}^N} G(y,w_n)\,dy}{\int_{\mathbb{R}^N} G(y,w_n)\,dy + I_\lambda(w_n)} \\
&\geq \liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{G(y,w_n)}{\|w_n\|_E^{\vartheta p}}\,dy \\
&= \liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{G(y,w_n)}{\|w_n\|_E^{\vartheta p}}\,dy - \limsup_{n\to\infty} \int_{\mathbb{R}^N} \frac{\mathcal{T}_0}{\|w_n\|_E^{\vartheta p}}\,dy \\
&\geq \liminf_{n\to\infty} \int_{\Gamma_1} \frac{G(y,w_n) - \mathcal{T}_0}{\|w_n\|_E^{\vartheta p}}\,dy \\
&\geq \int_{\Gamma_1} \liminf_{n\to\infty} \frac{G(y,w_n) - \mathcal{T}_0}{\|w_n\|_E^{\vartheta p}}\,dy \\
&= \int_{\Gamma_1} \liminf_{n\to\infty} \frac{G(y,w_n)}{|w_n|^{\vartheta p}} |z_n|^{\vartheta p}\,dy - \int_{\Gamma_1} \limsup_{n\to\infty} \frac{\mathcal{T}_0}{\|w_n\|_E^{\vartheta p}}\,dy = \infty,
\end{aligned}$$

which is a contradiction. Thus, $z(y) = 0$ for almost all $y \in \mathbb{R}^N$. Since $I_\lambda(tw_n)$ is continuous at every $t \in [0,1]$, for each $n \in \mathbb{N}$, we can choose $t_n \in [0,1]$ satisfying:

$$I_\lambda(t_n w_n) := \max_{t\in[0,1]} I_\lambda(tw_n).$$

Let $\{\zeta_k\}$ be a positive sequence of real numbers with $\lim_{k\to\infty} \zeta_k = \infty$ and $\zeta_k > 1$ for any k. Then, it is immediate that $\|\zeta_k z_n\|_E = \zeta_k > 1$ for any k and n. Let k be fixed. As $z_n \to 0$ strongly in $L^q(\mathbb{R}^N)$ as $n \to \infty$, due to the continuity of the Nemytskii operator, we infer $G(y,\zeta_k z_n) \to 0$ in $L^1(\mathbb{R}^N)$ as $n \to \infty$; hence,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} G(y,\zeta_k z_n)\,dy = 0. \tag{10}$$

Because $\|w_n\|_E \to \infty$ as $n \to \infty$, we have $\|w_n\|_E > \zeta_k$ for n large enough. Hence, according to $(\mathcal{M}1)$–$(\mathcal{M}2)$ and the relation (10), we have:

$$\begin{aligned}
I_\lambda(t_n w_n) &\geq I_\lambda\left(\frac{\zeta_k}{\|w_n\|_E} w_n\right) = I_\lambda(\zeta_k z_n) \\
&= \frac{1}{p}\mathcal{M}([\zeta_k z_n]_{s,p}) + \frac{1}{p}\int_{\mathbb{R}^N} \mathcal{V}(y)|\zeta_k z_n|^p\,dy - \lambda \int_{\mathbb{R}^N} G(y,\zeta_k z_n)\,dy \\
&\geq \frac{\min\{\tau_0,\vartheta\}}{\vartheta p}\left([\zeta_k z_n]_{s,p} + \int_{\mathbb{R}^N} \mathcal{V}(y)|\zeta_k z_n|^p\,dy\right) - \lambda \int_{\mathbb{R}^N} G(y,\zeta_k z_n)\,dy \\
&\geq \frac{\min\{\tau_0,\vartheta\}}{\vartheta p}\|\zeta_k z_n\|_E^p - \lambda \int_{\mathbb{R}^N} G(y,\zeta_k z_n)\,dy \\
&\geq \frac{\min\{\tau_0,\vartheta\}}{\vartheta p}\zeta_k^p
\end{aligned}$$

for sufficiently large n. Then, letting n and k approach infinity, one has:

$$\lim_{n\to\infty} I_\lambda(t_n w_n) = \infty. \tag{11}$$

Since $I_\lambda(0) = 0$ and $I_\lambda(w_n) \to c$ as $n \to \infty$, it is obvious that $t_n \in (0,1)$ and $\langle I'_\lambda(t_n w_n), t_n w_n \rangle = 0$. Therefore, by the assumptions $(\mathcal{M}2)$, (F2), as well as the relation (2), for all n large enough, we have:

$$
\begin{aligned}
\frac{1}{\theta}I_\lambda(t_nw_n) &= \frac{1}{\theta}I_\lambda(t_nw_n) - \frac{1}{p\theta\vartheta}\langle I'_\lambda(t_nw_n), t_nw_n\rangle \\
&= \frac{1}{p\theta\vartheta}\Big[\vartheta\mathcal{M}([t_nw_n]_{s,p}) - M([t_nw_n]_{s,p})[t_nw_n]_{s,p}\Big] \\
&\quad + \frac{1}{p\theta}\int_{\mathbb{R}^N}\left(\mathcal{V}(y)|t_nw_n|^p - \frac{\mathcal{V}(y)}{\vartheta}|t_nw_n|^p\right)dy \\
&\quad + \frac{\lambda}{p\theta\vartheta}\int_{\mathbb{R}^N}(g(y,t_nw_n)t_nw_n - p\vartheta G(y,t_nw_n))\,dy \\
&= \frac{1}{p\theta\vartheta}\Big[\vartheta\mathcal{M}\left(t_n^p[w_n]_{s,p}\right) - t_n^pM\left(t_n^p[w_n]_{s,p}\right)[w_n]_{s,p}\Big] \\
&\quad + \frac{1}{p\theta}\int_{\mathbb{R}^N}\left(\mathcal{V}(y)|t_nw_n|^p - \frac{\mathcal{V}(y)}{\vartheta}|t_nw_n|^p\right)dy \\
&\quad + \frac{\lambda}{p\theta\vartheta}\int_{\mathbb{R}^N}\mathcal{G}(y,t_nw_n)\,dy \\
&= \frac{1}{p\theta\vartheta}\widehat{\mathcal{M}}\left(t_n^p[w_n]_{s,p}\right) + \frac{t_n^p}{p\theta}\int_{\mathbb{R}^N}\left(\mathcal{V}(y)|w_n|^p - \frac{\mathcal{V}(y)}{\vartheta}|w_n|^p\right)dy \\
&\quad + \frac{\lambda}{p\theta\vartheta}\int_{\mathbb{R}^N}\mathcal{G}(y,t_nw_n)\,dy \\
&\le \frac{1}{p\theta\vartheta}\Big[\widehat{\mathcal{M}}([w_n]_{s,p}) + K\Big] + \frac{1}{p\theta}\int_{\mathbb{R}^N}\left(\mathcal{V}(y)|w_n|^p - \frac{\mathcal{V}(y)}{\vartheta}|w_n|^p\right)dy \\
&\quad + \frac{\lambda}{p\vartheta}\int_{\mathbb{R}^N}\mathcal{G}(y,w_n)\,dy \\
&\le \frac{1}{p\vartheta}\Big[\vartheta\mathcal{M}([w_n]_{s,p}) - M([w_n]_{s,p})[w_n]_{s,p}\Big] \\
&\quad + \frac{1}{p}\int_{\mathbb{R}^N}\left(\mathcal{V}(y)|w_n|^p - \frac{\mathcal{V}(y)}{\vartheta}|w_n|^p\right)dy \\
&\quad + \frac{\lambda}{p\vartheta}\int_{\mathbb{R}^N}\mathcal{G}(y,w_n)\,dy + \frac{1}{p\vartheta}K \\
&\le \mathcal{M}([w_n]_{s,p}) + \int_{\mathbb{R}^N}\mathcal{V}(y)|w_n|^p\,dy - \lambda\int_{\mathbb{R}^N}G(y,w_n)\,dy \\
&\quad - \frac{1}{p\vartheta}M([w_n]_{s,p})[w_n]_{s,p} - \frac{1}{p\vartheta}\int_{\mathbb{R}^N}\mathcal{V}(y)|w_n|^p\,dy \\
&\quad + \frac{\lambda}{p\vartheta}\int_{\mathbb{R}^N}g(y,w_n)w_n\,dy + \frac{1}{p\vartheta}K \\
&= I_\lambda(w_n) - \frac{1}{p\vartheta}\langle I'_\lambda(w_n), w_n\rangle + \frac{1}{p\vartheta}K \to c + \frac{1}{p\vartheta}K, \quad \text{as} \quad n\to\infty,
\end{aligned}
$$

which contradicts (11). This completes the proof. □

Let $\mathfrak{G}$ be a separable and reflexive Banach space. Then, we know (see [55,56]) that we choose $\{f_k\} \subseteq \mathfrak{G}$ and $\{g_k^*\} \subseteq \mathfrak{G}^*$ such that:

$$\mathfrak{G} = \overline{\operatorname{span}\{f_k : k = 1,2,\cdots\}}, \quad \mathfrak{G}^* = \overline{\operatorname{span}\{g_k^* : k = 1,2,\cdots\}},$$

and:

$$\langle g_i^*, f_j\rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Let us denote $\mathfrak{G}_k = \operatorname{span}\{f_k\}$, $\mathfrak{Y}_n = \bigoplus_{k=1}^{n}\mathfrak{G}_k$, and $\mathfrak{Z}_n = \overline{\bigoplus_{k=n}^{\infty}\mathfrak{G}_k}$.

Definition 3. Assume that $(\mathfrak{G}, \|\cdot\|)$ is a real reflexive and separable Banach space, $\mathcal{T} \in C^1(\mathfrak{G}, \mathbb{R})$, $c \in \mathbb{R}$. We say that $\mathcal{T}$ fulfills the $(C)_c^*$-condition (with respect to $\mathfrak{Y}_k$) if any sequence $\{w_k\}_{k\in\mathbb{N}} \subset \mathfrak{G}$ for which $w_k \in \mathfrak{Y}_k$, for any $k \in \mathbb{N}$,

$$\mathcal{T}(w_k) \to c \quad \text{and} \quad \|(\mathcal{T}|_{\mathfrak{Y}_k})'(w_k)\|_{\mathfrak{G}^*}(1 + \|w_k\|) \to 0 \text{ as } k \to \infty,$$

has a subsequence converging to a critical point of $\mathcal{T}$.

Proposition 1 ([49]). *Suppose that $(\mathfrak{G}, \|\cdot\|_{\mathfrak{G}})$ is a Banach space, $\mathcal{T} \in C^1(\mathfrak{G}, \mathbb{R})$ is an even functional. If there is $n_0 > 0$, so that for each $n \geq n_0$, there exists $\delta_n > \gamma_n > 0$, such that:*

(D1) $\inf\{\mathcal{T}(w) : w \in \mathfrak{Z}_n, \|w\|_{\mathfrak{G}} = \delta_n\} \geq 0$;

(D2) $\varphi_n := \max\{\mathcal{T}(w) : w \in \mathfrak{Y}_n, \|w\|_{\mathfrak{G}} = \gamma_n\} < 0$;

(D3) $\psi_n := \inf\{\mathcal{T}(w) : w \in \mathfrak{Z}_n, \|w\|_{\mathfrak{G}} \leq \delta_n\} \to 0$ *as* $n \to \infty$;

(D4) $\mathcal{T}$ *fulfills the* $(C)_c^*$*-condition for every* $c \in [\psi_{n_0}, 0)$,

then, $\mathcal{T}$ *has a sequence of negative critical values* $c_k < 0$ *satisfying* $c_k \to 0$ *as* $k \to \infty$.

Lemma 6. *Assume that* (F1)–(F2) *and* (F4) *hold. Then,* I_λ *satisfies the* $(C)_c^*$*-condition.*

Proof. Let $c \in \mathbb{R}$ and let the sequence $\{w_k\}$ in E be such that $w_k \in \mathfrak{Y}_k$, for any $k \in \mathbb{N}$,

$$I_\lambda(w_k) \to c \quad \text{and} \quad \|(I_\lambda|_{\mathfrak{Y}_k})'(w_k)\|_{E^*}(1 + \|w_k\|) \to 0 \text{ as } k \to \infty.$$

Therefore, we get $c = I_\lambda(w_k) + o_k(1)$ and $\langle I'_\lambda(w_k), w_k\rangle = o_k(1)$, where $o_k(1) \to 0$ as $k \to \infty$. Repeating the argument from the proof of Lemma 5, we derive the boundedness of $\{w_k\}$ in E. Then the idea of the rest of the proof is fundamentally the same as that in ([49], Lemma 3.12). □

Our main consequence is formulated as follows. The basic idea of proof of this theorem comes from the paper [46]. For the sake of convenience of the readers, we give the proof.

Theorem 1. *Suppose that* (F1)–(F4) *hold and* $g(y, -\zeta) = -g(y, \zeta)$ *for all* $(y, \zeta) \in \mathbb{R}^N \times \mathbb{R}$. *Then, the problem* (1) *possesses a sequence of nontrivial solutions* $\{w_k\}$ *in* E *satisfying* $I_\lambda(w_k) \to 0$ *as* $k \to \infty$ *for all* $\lambda > 0$.

Proof. By means of Lemma 6 and the oddness of g, the functional I_λ ensures the $(C)_c^*$-condition for every $c \in \mathbb{R}$, and that it is even. Thus, we will show the conditions (D1)–(D3) of Proposition 1.

(D1): Let us denote:

$$\eta_{1,n} = \sup_{\|w\|_E=1, w\in\mathfrak{Z}_n} \|w\|_{L^\kappa(\mathbb{R}^N)} \quad \text{and} \quad \eta_{2,n} = \sup_{\|w\|_E=1, w\in\mathfrak{Z}_n} \|w\|_{L^\ell(\mathbb{R}^N)}.$$

Then, it is clear to derive that $\eta_{1,n} \to 0$ and $\eta_{2,n} \to 0$ as $n \to \infty$ (see [49]). Denote $\eta_n = \max\{\eta_{1,n}, \eta_{2,n}\}$. Let $\eta_n < 1$ for n large enough. Then, it follows from $(\mathcal{M}1)$, $(\mathcal{M}2)$, and (F1) that:

$$\begin{aligned}
I_\lambda(w) &= \frac{1}{p}\mathcal{M}([w]_{s,p}) + \int_{\mathbb{R}^N} \frac{\mathcal{V}(y)}{p}|w|^p\,dy - \lambda\int_{\mathbb{R}^N} g(y,w)\,dy \\
&\geq \frac{\min\{\tau_0, \vartheta\}}{\vartheta p}\|w\|_E^p - \lambda\int_{\mathbb{R}^N} g(y,w)\,dy \\
&\geq \frac{\min\{\tau_0, \vartheta\}}{\vartheta p}\|w\|_E^p - \lambda\|\rho_1\|_{L^{\kappa'}(\mathbb{R}^N)}\|w\|_{L^\kappa(\mathbb{R}^N)} - \frac{\lambda\rho_2}{\ell}\|w\|_{L^\ell(\mathbb{R}^N)}^\ell \\
&\geq \frac{\min\{\tau_0, \vartheta\}}{\vartheta p}\|w\|_E^p - \lambda\|\rho_1\|_{L^{\kappa'}(\mathbb{R}^N)}\eta_n\|w\|_E - \frac{\lambda\rho_2}{\ell}\eta_n^\ell\|w\|_E^\ell \\
&\geq \frac{\min\{\tau_0, \vartheta\}}{\vartheta p}\|w\|_E^p - \lambda\|\rho_1\|_{L^{\kappa'}(\mathbb{R}^N)}\eta_n\|w\|_E - \frac{\lambda\rho_2}{\ell}\eta_n^\ell\|w\|_E^{2\ell}
\end{aligned}$$

for n large enough and $\|w\|_E \geq 1$. Choose:

$$\delta_n = \left[\frac{2\rho_2\lambda\vartheta p\eta_n^\ell}{\ell\min\{\tau_0,\vartheta\}}\right]^{\frac{1}{p-2\ell}}. \tag{12}$$

Let $w \in \mathfrak{Z}_n$ with $\|w\|_E = \delta_n > 1$ for sufficiently large n. Then, there exists $n_0 \in \mathbb{N}$ such that:

$$\begin{aligned} I_\lambda(w) &\geq \frac{\min\{\tau_0,\vartheta\}}{\vartheta p}\|w\|_E^p - \frac{\lambda\rho_2}{\ell}\eta_n^\ell\|w\|_E^{2\ell} - \lambda\|\rho_1\|_{L^{\kappa'}(\mathbb{R}^N)}\eta_n\|w\|_E \\ &\geq \frac{\min\{\tau_0,\vartheta\}}{2\vartheta p}\delta_n^p - \lambda\eta_n^{\frac{p-\ell}{p-2\ell}}\|\rho_1\|_{L^{\kappa'}(\mathbb{R}^N)}\left[\frac{2\rho_2\lambda\vartheta p}{\ell\min\{\tau_0,\vartheta\}}\right]^{\frac{1}{p-2\ell}} \\ &\geq 0 \end{aligned}$$

for all $n \in \mathbb{N}$ with $n \geq n_0$, since $\lim_{n\to\infty}\delta_n = \infty$. Consequently, we assert:

$$\inf\{I_\lambda(w) : w \in \mathfrak{Z}_n, \|w\|_E = \delta_n\} \geq 0.$$

(D2): Because $\mathfrak{Y}_n$ is finite-dimensional and so all the norms are equivalent, there exists $\varsigma_{1,n} > 0$ and $\varsigma_{2,n} > 0$, such that for any $w \in \mathfrak{Y}_n$:

$$\varsigma_{1,n}\|w\|_E \leq \|w\|_{L^m(\nu,\mathbb{R}^N)} \text{ and } \|w\|_{L^\ell(\mathbb{R}^N)} \leq \varsigma_{2,n}\|w\|_E.$$

For any $w \in \mathfrak{Y}_n$ with $\|w\|_E \leq 1$, in accordance with (F1) and (F3), there are $\mathcal{C}_1, \mathcal{C}_2 > 0$, such that:

$$G(y,\zeta) \geq \mathcal{C}_1\nu(y)|\zeta|^m - \mathcal{C}_2|\zeta|^\ell$$

for almost all $(y,\zeta) \in \mathbb{R}^N \times \mathbb{R}$. Note that:

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{1}{p}|w(y)-w(x)|^p\mathfrak{K}(y,x)\,dy\,dx \leq \mathcal{C}_3$$

for some positive constant $\mathcal{C}_3$. Then, we have:

$$\begin{aligned} I_\lambda(w) &= \frac{1}{p}\mathcal{M}\big([w]_{s,p}\big) + \int_{\mathbb{R}^N}\frac{\mathcal{V}(y)}{p}|w|^p dy - \lambda\int_{\mathbb{R}^N} g(y,w)\,dy \\ &\leq \left(\sup_{0\leq\xi\leq\mathcal{C}_3} M(\xi)\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{1}{p}|w(y)-w(x)|^p\mathfrak{K}(y,x)\,dy\,dx \\ &\quad + \frac{1}{p}\int_{\mathbb{R}^N}\mathcal{V}(y)|w|^p dy - \mathcal{C}_1\int_{\mathbb{R}^N}\nu(y)|w|^m dy + \mathcal{C}_2\int_{\mathbb{R}^N}|w|^\ell\,dy \\ &\leq C_4\|w\|_E^p - \mathcal{C}_1\|z\|_{L^m(\nu,\mathbb{R}^N)}^m + \mathcal{C}_2\|w\|_{L^\ell(\mathbb{R}^N)}^\ell \\ &\leq C_4\|w\|_E^p - \mathcal{C}_1\varsigma_{1,n}^m\|w\|_E^m + \mathcal{C}_2\varsigma_{2,n}^\ell\|w\|_E^\ell \end{aligned}$$

for a positive constant C_4. Let $h(t) = C_4t^p - \mathcal{C}_1\varsigma_{1,n}^m t^m + \mathcal{C}_2\varsigma_{2,n}^\ell t^\ell$. Since $m < p < \ell$, there is a $t_0 \in (0,1)$ which is sufficiently small such that $h(t) < 0$ for all $t \in (0,t_0)$. Thus $I_\lambda(w) < 0$ for all $w \in \mathfrak{Y}_n$ with $\|w\|_E = t_0$. If we choose $\gamma_n = t_0$ for all $n \in \mathbb{N}$, we get:

$$\varphi_n := \max\{I_\lambda(w) : w \in \mathfrak{Y}_n, \|w\|_E = \gamma_n\} < 0.$$

If necessary, we can replace n_0 with a large value, so that $\delta_n > \gamma_n > 0$ for all $n \geq n_0$.

(D3): Because $\mathfrak{Y}_n \cap \mathfrak{Z}_n \neq \phi$ and $0 < \gamma_n < \delta_n$, we have $\psi_n \leq \varphi_n < 0$ for all $n \geq n_0$. For any $w \in \mathfrak{Z}_n$ with $\|w\|_E = 1$ and $0 < t < \delta_n$, one has:

$$\begin{aligned}
I_\lambda(tw) &= \frac{1}{p}\mathcal{M}\left([tw]_{s,p}\right) + \int_{\mathbb{R}^N} \frac{\mathcal{V}(y)}{p}|tw|^p dy - \lambda \int_{\mathbb{R}^N} G(y,tw)\,dy \\
&\geq \frac{\min\{\tau_0,\vartheta\}}{\vartheta p}\|tw\|_E^p - \lambda\|\rho_1\|_{L^{\kappa'}(\mathbb{R}^N)}\|tw\|_{L^{\kappa}(\mathbb{R}^N)} - \frac{\lambda\rho_2}{\ell}\|tw\|_{L^\ell(\mathbb{R}^N)}^\ell \\
&\geq -\lambda\|\rho_1\|_{L^{\kappa'}(\mathbb{R}^N)}\delta_n\eta_n - \frac{\lambda\rho_2}{\ell}\delta_n^\ell\eta_n^\ell
\end{aligned}$$

for large enough n. Hence, it follows from the definition of δ_n that:

$$\begin{aligned}
0 > \psi_n &\geq -\lambda\|\rho_1\|_{L^{\kappa'}(\mathbb{R}^N)}\delta_n\eta_n - \frac{\lambda\rho_2}{\ell}\delta_n^\ell\eta_n^\ell \\
&= -\lambda\|\rho_1\|_{L^{\kappa'}(\mathbb{R}^N)}\left[\frac{2\rho_2\lambda\vartheta p}{\ell\min\{\tau_0,\vartheta\}}\right]^{\frac{1}{p-2\ell}}\eta_n^{\frac{p-\ell}{p-2\ell}} \\
&\quad - \frac{\lambda\rho_2}{\ell}\left[\frac{2\rho_2\lambda\vartheta p}{\ell\min\{\tau_0,\vartheta\}}\right]^{\frac{\ell}{p-2\ell}}\eta_n^{\frac{\ell(p-\ell)}{p-2\ell}}. \qquad (13)
\end{aligned}$$

Because $p < \ell$ and $\eta_n \to 0$ as $n \to \infty$, we assert that $\lim_{n\to\infty}\psi_n = 0$.

As a consequence, all assumptions of Proposition 1 are ensured, and, thus, the problem (1) possesses a sequence of nontrivial solutions $\{w_k\}$ in E satisfying $I_\lambda(w_k) \to 0$ as $k \to \infty$ for any $\lambda > 0$. □

4. Conclusions

In the present paper, by employing the dual fountain theorem as the key tool, we give the multiplicity result of small energy solutions on classes of the Kirchhoff coefficient M and the nonlinear term h, which differ from the previous related works. In particular, we provide our main result when we do not assume the monotonicity of $\widehat{\mathcal{M}}$ in ($\mathcal{M}$2), and the condition (g2), which are essential in showing the compactness condition of the Palais–Smale type and verifying all hypotheses in the dual fountain theorem. These are novelties of this paper.

Moreover, a new research direction in a strong relationship is the study of the fractional $p(\cdot)$-Laplacian:

$$M\Big([w]_{s,p(\cdot,\cdot)}\Big)\mathcal{L}w(y) + \mathcal{V}(y)|w|^{p(y)-2}w = \lambda g(y,w) \quad \text{in} \quad \mathbb{R}^N,$$

where:

$$[w]_{s,p(\cdot,\cdot)} := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w(y)-w(x)|^{p(y,x)}}{p(y,x)|y-x|^{N+sp(y,x)}}\,dy\,dx$$

and the operator $\mathcal{L}$ is defined by:

$$\mathcal{L}w(y) = 2\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^N\setminus B_\varepsilon(y)} \frac{|w(y)-w(x)|^{p(y,x)-2}(w(y)-w(x))}{|y-x|^{N+sp(y,x)}}\,dy, \quad y \in \mathbb{R}^N,$$

where $s \in (0,1)$ and $B_\varepsilon(y) := \{x \in \mathbb{R}^N : |x-y| \leq \varepsilon\}$. Let us consider the condition:

(f2) there is a positive constant $\theta \geq 1$, such that:

$$\mathcal{G}(y,t\zeta) \leq \theta\mathcal{G}(y,\zeta)$$

for $t \in [0,1]$ and $(y,\zeta) \in \mathbb{R}^N \times \mathbb{R}$, where:

$$\mathcal{G}(y,\zeta) = g(y,\zeta)\zeta - \vartheta \sup_{(y,x)\in\mathbb{R}^N\times\mathbb{R}^N} p(y,x)G(y,\zeta).$$

Very recently, the author in [30] provided a multiplicity result of solutions to a nonlocal problem of the Schrödinger–Kirchhoff type with a variable exponent under a suitable condition, which is different from $(f2)$. As far as we know, there is no such existence result of solutions to the nonlinear elliptic problems involving the fractional $p(\cdot)$-Laplacian when $(f2)$ is assumed.

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