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Existence of Positive Solutions to Boundary Value Problems with Mixed Riemann–Liouville and Quantum Fractional Derivatives

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Abstract: In this paper, by using the Leggett–Williams fixed-point theorem, we study the existence of positive solutions to fractional differential equations with mixed Riemann–Liouville and quantum fractional derivatives. To prove the effectiveness of our main result, we investigate an interesting example.

Keywords: cone; boundary value problem; fixed-point theorem; Riemann–Liouville fractional derivative; quantum fractional calculus

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1. Introduction

The q-difference calculus was first introduced by Jackson [1,2]. In [3,4], the authors, for the first time, studied fractional q-difference calculus. For more details on q-difference calculus, we refer readers to [5–7] and the references therein.

Fractional differential equations with q-difference are very interesting, and we refer readers to [8–17] and the references therein. For example, in [8,9], Ferreira studied the existence of positive solutions to q-fractional differential equations by using fixed-point theorem in cones. The authors in [14] established the existence of triple-positive solutions for fractional q-difference equations by using the q-Laplace transform and the fixed-point index theorem. In [15], the authors dealt with the solvability of fractional q-integro-difference systems using Krasnoselskii's, Schauder's and Schaefer's fixed-point theorems. In [10], the authors prove the existence of a unique iterative solution to a fractional q-difference equation using a novel fixed-point theorem. Ülke and Topal [12], by using Schauder's fixed-point theorem, studied the existence of solutions for a differential equation with fractional q-difference on the half-line.

In the literature, there exist papers mixing several kinds of fractional derivatives. For example, in [18–22], the authors investigated the existence of solutions to fractional differential equations by mixing Riemann–Liouville and Caputo fractional derivatives. In [23], the authors studied the existence and uniqueness results for mixed derivatives involving two fractional operators. In [24], the authors studied Hyers–Ulam stability for a class of impulsive coupled fractional differential equations with mixing the Caputo derivatives and ordinary derivative. As for some recent results on fractional calculus and fractional integro-differential equations, we refer to [25–27] and the references therein.

In [9], Ferreira investigated the existence of positive solutions to a class of nonlinear q-fractional boundary value problems. Also, in [28], by using fixed-point index theory, Zhang studied the existence of positive solutions to a class of singular boundary value problems for fractional differential equations with nonlinearity that changes sign. In this paper, we initiate the study of mixing two different fractional calculi by investigating a problem containing both a Riemann–Liouville fractional derivative and a quantum fractional derivative, which, as far as we know, is a new area of research. Thus, by using Leggett–Williams fixed-point theorem, we will extend the results of the papers in [9,28] to a combined boundary value problem with mixed Riemann–Liouville and quantum fractional derivatives.

So, inspired by the above articles, the objective of the present paper is to apply the Leggett–Williams fixed-point theorem to study the existence of multiple positive solutions to the following problem:

$$\begin{cases} D_\rho^\vartheta(D_{0^+}^\gamma \mathcal{Q}(z)) - f(z, \mathcal{Q}(z), \mathcal{Q}^{(1)}(z), \dots, \mathcal{Q}^{(n-2)}(z)) = 0, \quad z \in (0,1), \\ \mathcal{Q}(0) = \mathcal{Q}^{(1)}(0) = \cdots = \mathcal{Q}^{(n-2)}(0) = \mathcal{Q}^{(n-2)}(1) = 0, \\ D_{0^+}^\gamma \mathcal{Q}(0) = D_\rho(D_{0^+}^\gamma \mathcal{Q}(0) = D_\rho(D_{0^+}^\gamma \mathcal{Q})(1) = 0, \end{cases} \tag{1}$$

where D_ρ^ϑ is the quantum fractional derivative of order $2 < \vartheta \leq 3$ and quantum number $\rho \in (0,1)$, $D_{0^+}^\gamma$ is the Riemann–Liouville fractional derivative of order $n-1 < \gamma \leq n$, $n \geq 2$ and $f \in C([0,1] \times \mathbb{R}^{n-1}; [0,+\infty))$. Note that (1) is a problem with mixed Riemann–Liouville and ρ-difference fractional derivatives.

By using suitable changes in variables, we can split the problem (1) into two other problems for which Green's functions and their bounds are known from [9,28]. Then, we prove that problem (1) has at least three positive solutions by applying Legget–Williams fixed-point theorem. The used method is standard, but its configuration in the present problem is new. The obtained results are new and contribute to this new research topic concerning the study of positive solutions of boundary value problems, in which a combination of two fractional calculi is used.

The paper is organized as follows. In Section 2, some preliminary facts are recalled and basic properties are provided, which are needed later. In Section 3, we establish our main results, which concern the existence of at least three positive solutions for problem (1), via Leggett–Williams fixed-point theorem. An example illustrating the result is presented in Section 4.

2. Preliminaries

In this section, we introduce some basic definitions and lemmas.

Let $\rho \in (0,1)$ be a quantum number and define the ρ-Gamma function as

$$\Gamma_\rho(\mathbf{g}) = \frac{(1-\rho)^{(\mathbf{g}-1)}}{(1-\rho)^{\mathbf{g}-1}}, \quad \mathbf{g} \in \mathbb{R} \setminus \{\dots, -2, -1, 0\},$$

and this satisfies $\Gamma_\rho(\mathbf{g}+1) = [\mathbf{g}]_\rho \Gamma_\rho(\mathbf{g})$, where $[\mathbf{g}]_\rho$ is a number in q-calculus defined by

$$[\mathbf{g}]_\rho = \frac{1-\rho^{\mathbf{g}}}{1-\rho},$$

and the quantum power function is defined by

$$(1-\rho)^{(\mathbf{g}-1)} = \prod_{i=0}^{\infty} \frac{1-\rho^{i+1}}{1-\rho^{\mathbf{g}+i}}.$$

The ρ-derivative of a function $\psi : [0,\infty) \to \mathbb{R}$ is defined as

$$(D_\rho \psi)(\varsigma) = \frac{\psi(\varsigma) - \psi(\rho\varsigma)}{(1-\rho)\varsigma}, \quad \varsigma > 0,$$

and $(D_\rho \psi)(0) = \lim_{\varsigma \to 0} (D_\rho \psi)(\varsigma)$.

The ρ-integral in $[0,b]$ is defined as follows:

$$(I_\rho\psi)(\varsigma) = \int_0^\varsigma \psi(z)d_\rho z = \varsigma(1-\rho)\sum_{n=0}^{\infty}\psi(\rho^n\varsigma)\rho^n, \quad \varsigma \in [0,b].$$

We can obtain

$$(D_\rho I_\rho \psi)(\varsigma) = \psi(\varsigma),$$

and if ψ is continuous at $\varsigma = 0$, then

$$(I_\rho D_\rho \psi)(\varsigma) = \psi(\varsigma) - \psi(0).$$

On the properties of the D_ρ and I_ρ, we refer the reader to [29].

Definition 1. *Let $\psi \in L^1(\mathbb{R}^+)$. Then, the fractional ρ-integral of the Riemann–Liouville fractional-type operator of order $\vartheta > 0$ is defined as*

$$(I_\rho^\vartheta \psi)(z) = \frac{1}{\Gamma_\rho(\vartheta)}\int_0^z (z-\rho\tau)^{(\vartheta-1)}\psi(\tau)d_\rho\tau,$$

where the kernel of quantum power function is defined by

$$(z-\rho\tau)^{(\vartheta-1)} = \prod_{i=0}^{\infty}\frac{z-\tau\rho^{i+1}}{z-\tau\rho^{\vartheta+i}}.$$

Definition 2. *The fractional ρ-derivative of the Riemann–Liouville fractional type of order $\vartheta > 0$ is defined by $(D_\rho^0\psi)(z) = \psi(z)$ and*

$$(D_\rho^\vartheta\psi)(z) = \begin{cases} (D_\rho^m I_\rho^{m-\vartheta}\psi)(z), & m-1 < \vartheta \le m, \\ D_\rho^m\psi(z), & \vartheta = m, \end{cases}$$

where $m \in \mathbb{Z}^+$.

If $\rho \to 0$, then we have the definitions of the Riemann–Liouville fractional integral and derivative as

$$(I_{0+}^\vartheta\psi)(z) = \frac{1}{\Gamma(\vartheta)}\int_0^z (z-\tau)^{\vartheta-1}\psi(\tau)d\tau,$$

and

$$(D_{0+}^\vartheta\psi)(z) = \begin{cases} (D^m I^{m-\vartheta}\psi)(z), & m-1 < \vartheta \le m, \\ D^m\psi(z), & \vartheta = m, \end{cases}$$

respectively, where $D^m = d^m/dz^m$.

Lemma 1 ([8])**.** *Let $\vartheta > 0$ and $l \in \mathbb{N}$. Then,*

$$(I_\rho^\vartheta D_\rho^l\psi)(z) = (D_\rho^l I_\rho^\vartheta\psi)(z) - \sum_{k=0}^{l-1}\frac{z^{\vartheta-l+k}}{\Gamma_\rho(\vartheta+k-l+1)}(D_\rho^k\psi)(0).$$

Let $\mathcal{G}(z) = -D^{\gamma}_{0^+}\mathcal{Q}(z)$, so the mixed fractional Riemann–Liouville and quantum boundary value problem

$$\begin{cases} D^{\vartheta}_{\rho}(D^{\gamma}_{0^+}\mathcal{Q}(z)) - f(z, \mathcal{Q}(z), \dots, \mathcal{Q}^{(n-2)}(z)) = 0, & z \in (0,1), \\ D^{\gamma}_{0^+}\mathcal{Q}(0) = D_{\rho}(D^{\gamma}_{0^+}\mathcal{Q})(0) = D_{\rho}(D^{\gamma}_{0^+}\mathcal{Q})(1) = 0, \end{cases}$$

changes into a fractional quantum boundary value problem of the form

$$\begin{cases} D^{\vartheta}_{\rho}(\mathcal{G}(z)) = -f(z, \mathcal{Q}(z), \dots, \mathcal{Q}^{(n-2)}(z)), & z \in (0,1), \\ \mathcal{G}(0) = D_{\rho}(\mathcal{G}(0)) = D_{\rho}(\mathcal{G}(1)) = 0. \end{cases} \tag{2}$$

Lemma 2 ([9]). *Problem (2) has a solution*

$$\mathcal{G}(z) = \frac{1}{\Gamma_{\rho}(\vartheta)} \int_0^1 \Phi(z, \rho\omega) f(\omega, \mathcal{Q}(\omega), \mathcal{Q}^{(1)}(\omega), \dots, \mathcal{Q}^{(n-2)}(\omega)) d_{\rho}\omega, \tag{3}$$

where

$$\Phi(z,\omega) = \begin{cases} z^{\vartheta-1}(1-\omega)^{(\vartheta-2)} - (z-\omega)^{(\vartheta-1)}, & 0 \le \omega \le z \le 1, \\ z^{\vartheta-1}(1-\omega)^{(\vartheta-2)}, & 0 \le z \le \omega \le 1. \end{cases} \tag{4}$$

Lemma 3 ([28]). *Assume that $\mathcal{Q} = I^{n-2}_{0^+}v(z)$, then the fractional Riemann–Liouville boundary value problem*

$$\begin{cases} D^{\gamma}_{0^+}\mathcal{Q}(t) = -\mathcal{G}(z), & z \in (0,1), \\ \mathcal{Q}(0) = \mathcal{Q}^{(1)}(0) = \dots = \mathcal{Q}^{(n-2)}(0) = \mathcal{Q}^{(n-2)}(1) = 0, \end{cases} \tag{5}$$

has a solution

$$v(z) = \int_0^1 \Theta(z,\omega)\mathcal{G}(\omega) d\omega, \tag{6}$$

where

$$\Theta(z,\omega) = \frac{1}{\Gamma(\gamma - n + 2)} \begin{cases} (z(1-\omega))^{\gamma-n+1} - (z-\omega)^{\gamma-n+1}, & 0 \le \omega \le z \le 1, \\ (z(1-\omega))^{\gamma-n+1}, & 0 \le z \le \omega \le 1. \end{cases} \tag{7}$$

Lemma 4 ([9]). *The following relations of kernels containing the quantum power functions hold:*

$$\begin{aligned} &\Phi(z,\rho\omega) \ge 0 \quad \text{and} \quad \Phi(z,\rho\omega) \le \Phi(1,\rho\omega) \quad \text{for all} \quad 0 \le z,\omega \le 1, \\ &\Phi(z,\rho\omega) \ge z^{\vartheta-1}\Phi(1,\rho\omega) \quad \text{for} \quad 0 \le z,\omega \le 1. \end{aligned}$$

Lemma 5 ([28]). *The following relations of kernels containing the usual power functions hold:*

(1) $\Theta(z,\omega) \ge 0$, $\Theta(z,\omega) \le \Theta(\omega,\omega)$, $\forall\, 0 \le z,\omega \le 1$;

(2) $\min_{\frac{1}{4}\le t\le \frac{3}{4}} \Theta(z,\omega) \ge \varrho(s)\Theta(\omega,\omega)$, $\forall\, 0 \le \omega \le 1$, *where*

$$\varrho(\omega) = \begin{cases} \dfrac{\left(\frac{3}{4}(1-\omega)\right)^{\gamma-n+1} - \left(\frac{3}{4}-\omega\right)^{\gamma-n+1}}{(\omega(1-\omega))^{\gamma-n+1}}, & 0 < \omega \le r, \\ \left(\dfrac{1}{4\omega}\right)^{\gamma-n+1}, & r \le \omega < 1, \end{cases} \tag{8}$$

and $\frac{1}{4} < r < \frac{3}{4}$.

Now, we consider the mixed fractional Riemann–Liouville and quantum boundary value problem (1). Assume that $\mathcal{Q} = I_{0+}^{n-2}v(t)$; then, by applying Lemmas 2 and 3, $\mathcal{Q} \in C([0,1])$ is a solution of the following equation:

$$v(z) = \frac{1}{\Gamma_\rho(\vartheta)} \int_0^1 \Theta(z,\omega) \left(\int_0^1 \Phi(\omega,\rho\zeta) f(\omega, I_{0+}^{n-2}v(\zeta), I_{0+}^{n-3}v(\zeta), \dots, I_{0+}^{1}v(\zeta), v(\zeta)) d_\rho\zeta \right) d\omega, \tag{9}$$

if and only if $v \in C([0,1])$ is a solution of (1).

3. Three Positive Solutions

In this section, we will study the existence of positive solutions to problem (1). We now consider the Banach space $X = C([0,1])$ with the usual norm $\|\mathcal{Q}\| = \max_{0\le z\le 1} |\mathcal{Q}(z)|$. The cones $\Lambda \subset X$ can be defined as follows:

$$\Lambda = \left\{ v \in X : \min_{\frac{1}{4}\le z\le \frac{3}{4}} v(z) \ge \left(\frac{1}{4}\right)^{\vartheta-1} \frac{\eta}{3}\|v\|,\ z \in [0,1] \right\},$$

where $\eta = \min_{\frac{1}{4}\le\omega\le\frac{3}{4}} \varrho(\omega)$

Define a operator $T : \Lambda \to X$ by

$$Tv(z) = \frac{1}{\Gamma_\rho(\vartheta)} \int_0^1 \Theta(z,\omega) \left(\int_0^1 \Phi(\omega,\rho\zeta) f(\omega, I_{0+}^{n-2}v(\zeta), I_{0+}^{n-3}v(\zeta), \dots, I_{0+}^{1}v(\zeta), v(\zeta)) d_\rho\zeta \right) d\omega. \tag{10}$$

By similar methods in [28] (Lemma 2.7), we have that if $\mathcal{Q} = I_{0+}^{n-2}v(t)$, $v \in C[0,1]$, then the function $\mathcal{Q}(z) = I_{0+}^{n-2}v(z)$ is a positive solution of (1). Clearly, the existence of a positive solution for (1) is equivalent to the existence of a positive fixed point of T in Λ with $\mathcal{Q}(z) = I_{0+}^{n-2}v(z)$.

Lemma 6. *Assume that*

(H1) $f \in C([0,1]\times\mathbb{R}^{n-1};\mathbb{R}^+)$ and $|f(z,\mathcal{Q}_1,\dots,\mathcal{Q}_{n-1})| \le \Omega$ on $[0,1]$ for a constant $\Omega > 0$.

Then, $T : \Lambda \to \Lambda$ is well defined and completely continuous.

Proof. For every $v \in \Lambda$, clearly, $T\mathcal{Q}(z) \ge 0$ on $[0,1]$. Let $v \in \Lambda$. So, by Lemmas 4 and 5, we obtain

$$\begin{aligned}
\|(Tv)\| &\le \frac{1}{\Gamma_\rho(\vartheta)} \int_0^1 \Theta(z,\omega) \left(\int_0^1 \Phi(\omega,\rho\zeta) f(\omega, I_{0+}^{n-2}v(\zeta), I_{0+}^{n-3}v(\zeta), \dots, I_{0+}^{1}v(\zeta), v(\zeta)) d_\rho\zeta \right) d\omega \\
&\le \frac{1}{\Gamma_\rho(\vartheta)} \left(\int_0^{\frac{1}{4}} + \int_{\frac{1}{4}}^{\frac{3}{4}} + \int_{\frac{3}{4}}^{1} \right) \\
&\quad \times \Theta(z,\omega) \left(\int_0^1 \Phi(1,\rho\zeta) f(\omega, I_{0+}^{n-2}v(\zeta), I_{0+}^{n-3}v(\zeta), \dots, I_{0+}^{1}v(\zeta), v(\zeta)) d_\rho\zeta \right) d\omega \\
&\le \frac{3}{\Gamma_\rho(\vartheta)} \int_{\frac{1}{4}}^{\frac{3}{4}} \Theta(z,\omega) \left(\int_0^1 \Phi(1,\rho\zeta) f(\omega, I_{0+}^{n-2}v(\zeta), I_{0+}^{n-3}v(\zeta), \dots, I_{0+}^{1}v(\zeta), v(\zeta)) d_\rho\zeta \right) d\omega.
\end{aligned}$$

Also, we have

$$\begin{aligned}
(Tv)(t) &\ge \frac{1}{\Gamma_\rho(\vartheta)} \int_{\frac{1}{4}}^{\frac{3}{4}} \Theta(z,\omega) \left(\int_0^1 \Phi(\omega,\rho\zeta) f(\omega, I_{0+}^{n-2}v(\zeta), I_{0+}^{n-3}v(\zeta), \dots, I_{0+}^{1}v(\zeta), v(\zeta)) d_\rho\zeta \right) d\omega \\
&\ge \frac{1}{\Gamma_\rho(\vartheta)} \int_{\frac{1}{4}}^{\frac{3}{4}} \varrho(\omega) \\
&\quad \times \Theta(\omega,\omega) \left(\int_0^1 \omega^{\alpha-1} \Phi(1,\rho\zeta) f(\omega, I_{0+}^{n-2}v(\zeta), I_{0+}^{n-3}v(\zeta), \dots, I_{0+}^{1}v(\zeta), v(\zeta)) d_\rho\zeta \right) d\omega
\end{aligned}$$

$$
\begin{aligned}
&\geq \left(\frac{1}{4}\right)^{\vartheta-1} \frac{\eta}{\Gamma_\rho(\vartheta)} \int_{\frac{1}{4}}^{\frac{3}{4}} \Theta(\omega,\omega) \\
&\quad \times \left(\int_0^1 \Phi(1,\rho\zeta) f(\omega, I_{0^+}^{n-2}v(\zeta), I_{0^+}^{n-3}v(\zeta), \dots, I_{0^+}^{1}v(\zeta), v(\zeta)) d_\rho\zeta\right) d\omega \\
&\geq \left(\frac{1}{4}\right)^{\vartheta-1} \frac{\eta}{3}\|Tv\|.
\end{aligned}
$$

Hence, $T : \Lambda \to \Lambda$ is well defined. Now, we claim that T maps bounded sets into bounded sets. To this end, in view of (H1), and Lemmas 4 and 5, we obtain

$$
|(Tv)(t)| \leq \frac{\Omega}{\Gamma_\rho(\vartheta)} \int_0^1 \Theta(\omega,\omega)\left(\int_0^1 \Phi(1,\rho\zeta) d_\rho\zeta\right) d\omega < +\infty.
$$

Define $\Theta_1(z,\omega) = (z(1-\omega))^{\gamma-n+1} - (z-\omega)^{\gamma-n+1}$ and $\Theta_2(z,\omega) = (z(1-\omega))^{\gamma-n+1}$. By (H1), and Lemmas 4 and 5, for any $v \in X$ and $z_1, z_2 \in [0,1]$ with $z_1 < z_2$, we have

$$
\begin{aligned}
& |(Tv)(z_2) - (Tv)(z_1)| \\
\leq\ & \frac{\Omega}{\Gamma_\rho(\vartheta)} \left(\int_0^{z_1} + \int_{z_1}^{z_2} + \int_{z_2}^{1}\right) |\Theta(z_2,\omega) - \Theta(z_1,\omega)| \left(\int_0^1 \Phi(1,\rho\zeta) d_\rho\zeta\right) d\omega \\
=\ & \frac{\Omega}{\Gamma_\rho(\vartheta)\Gamma(\gamma-n+2)} \left(\int_0^1 \Phi(1,\rho\zeta) d_\rho\zeta\right) \left\{ \int_0^{z_1} |\Theta_1(z_2,\omega) - \Theta_1(z_1,\omega)| d\omega \right. \\
& \left. + \int_{z_1}^{z_2} |\Theta_1(z_2,\omega) - \Theta_2(z_1,\omega)| d\omega + \int_{z_2}^{1} |\Theta_2(z_2,\omega) - \Theta_2(z_1,\omega)| d\omega \right\}.
\end{aligned} \tag{11}
$$

Since $\Theta_1(z,\omega)$ and $\Theta_2(z,\omega)$ are uniformly continuous on $[0,1]$, then we obtain

$$
\int_0^{z_1} |\Theta_1(z_2,\omega) - \Theta_1(z_1,\omega)| d\omega, \int_{z_2}^{1} |\Theta_2(z_2,\omega) - \Theta_2(z_1,\omega)| d\omega \to 0, \quad \text{as } z_2 \to z_1. \tag{12}
$$

Also,

$$
\begin{aligned}
& \int_{z_1}^{z_2} |\Theta_1(z_2,\omega) - \Theta_2(z_1,\omega)| d\omega \\
=\ & \frac{1}{\gamma-n+2} \left\{ z_2^{\gamma-n+1}(1-z_1)^{\gamma-n+2} - z_2^{\gamma-n+1}(1-z_2)^{\gamma-n+2} \right. \\
& \left. -(z_2-z_1)^{\gamma-n+2} + z_1^{\gamma-n+1}(1-z_2)^{\gamma-n+2} - z_1^{\gamma-n+1}(1-z_1)^{\gamma-n+2} \right\} \\
\to\ & 0, \quad \text{as } z_2 \to z_1.
\end{aligned} \tag{13}
$$

So, (11)–(13) yield that T is equicontinuous on $[0,1]$. Therefore, by Arzelá–Ascoli theorem, we have the conclusion. □

We now present the Leggett–Williams fixed-point theorem [30]. Let X be a Banach space, Λ be a cone in X and $\Upsilon : \Lambda \to [0,+\infty)$ be a concave non-negative continuous functional on Λ.

Let $0 < \phi_1 < \phi_2$. Define Λ_δ and $\Lambda(\Upsilon, \phi_1, \phi_2)$ as

$$
\Lambda_\delta = \{\mathcal{Q} \in \Lambda | \|\mathcal{Q}\| < \delta\},
$$

and

$$
\Lambda(\Upsilon, \phi_1, \phi_2) = \{\mathcal{Q} \in \Lambda : \ \phi_1 \leq \Upsilon(\mathcal{Q}), \|\mathcal{Q}\| \leq \phi_2\}.
$$

Theorem 1. ([30]). *Let* Υ *be a concave non-negative continuous functional on* Λ *with* $\Upsilon(\mathcal{Q}) \leq \|\mathcal{Q}\|$ *for every* $\mathcal{Q} \in \overline{\Lambda}_{\phi_3}$ *and* $T : \overline{\Lambda}_{\phi_3} \to \overline{\Lambda}_{\phi_3}$ *be a completely continuous operator. Assume that there exist* $0 < \phi_1 < \phi_2 < d \leq \phi_3$ *such that*

(D1) $\mathcal{Q} \in \Lambda(\Upsilon, \phi_2, d) : \ \Upsilon(\mathcal{Q}) > \phi_2 \neq \emptyset$, *and* $\Upsilon(T\mathcal{Q}) > \phi_2$ *for* $\mathcal{Q} \in \Lambda(\Upsilon, \phi_2, d)$;
(D2) $\|T\mathcal{Q}\| < \phi_1$ *for* $\|\mathcal{Q}\| \leq \phi_1$;
(D3) $\Upsilon(T\mathcal{Q}) > b$ *for* $\mathcal{Q} \in \Lambda(\Upsilon, \phi_2, \phi_3)$ *with* $\|T\mathcal{Q}\| > d$.

Then, T has at least three fixed points, $\mathcal{Q}_1$, $\mathcal{Q}_2$ *and* $\mathcal{Q}_3$ *with* $\|\mathcal{Q}_1\| < \phi_1, \phi_2 < \Upsilon(\mathcal{Q}_2)$ *and* $\|\mathcal{Q}_3\| > \phi_1$, *with* $\Upsilon(\mathcal{Q}_3) < \phi_2$.

We define Υ as

$$\Upsilon(v) = \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} v(t).$$

Clearly, Υ is a concave non-negative continuous functional on X and for every $v \in X, \Upsilon(v) \leq \|v\|$.

Let

$$\begin{aligned} \varpi &= \frac{1}{\Gamma_\rho(\vartheta)} \int_0^1 \Theta(\omega, \omega) \left(\int_0^1 \Phi(1, \rho\zeta) d_\rho \zeta \right) d\omega, \\ R &= \left(\frac{1}{4}\right)^{\vartheta - 1} \frac{\eta}{\Gamma_\rho(\vartheta)} \int_{\frac{1}{4}}^{\frac{3}{4}} \Theta(\omega, \omega) \left(\int_0^1 \Phi(1, \rho\zeta) d_\rho \zeta \right) d\omega. \end{aligned}$$

We now state and prove our main result.

Theorem 2. *Suppose that (H1) holds and there exist positive constants* ϕ_1, ϕ_2, ϕ_3 *such that* $0 < \phi_1 < \phi_2 \leq \phi_3$ *and*

(F1) $f(z, \zeta_1, \ldots, \zeta_{n-1}) \leq \dfrac{\phi_3}{\varpi}, \forall\, (z, \zeta_1, \ldots, \zeta_{n-1}) \in [0,1] \times [0, \phi_3]^{n-1}$;

(F2) $f(z, \zeta_1, \ldots, \zeta_{n-1}) \leq \dfrac{\phi_1}{\varpi}, \forall\, (z, \zeta_1, \ldots, \zeta_{n-1}) \in [0,1] \times [0, \phi_1]^{n-1}$;

(F3) $f(z, \zeta_1, \ldots, \zeta_{n-1}) > \dfrac{\phi_2}{R}, \forall\, (z, \zeta_1, \ldots, \zeta_{n-1}) \in \left[\frac{1}{4}, \frac{3}{4}\right] \times [\phi_2, \phi_3]^{n-1}$.

Then, (1) has at least three positive solutions

$$\mathcal{Q}_1(z) = I_{0^+}^{n-2} v_1(z), \ \ \mathcal{Q}_2(z) = I_{0^+}^{n-2} v_2(z), \ \ \mathcal{Q}_3(z) = I_{0^+}^{n-2} v_3(z)$$

such that $\|v_1\| < \phi_1, \phi_2 < \Upsilon(v_2(z))$, *and* $\|v_3\| > \phi_1$, *with* $\Upsilon(v_3(z)) < \phi_2$.

Proof. We apply Theorem 1 to prove this theorem. For any $v \in \overline{P}_{\phi_3}$, by (F1), we obtain

$$\begin{aligned} \|Tv\| &= \max_{z \in [0,1]} \left| \frac{1}{\Gamma_\rho(\vartheta)} \int_0^1 \Theta(z, \omega) \right. \\ &\quad \left. \times \left(\int_0^1 \Phi(\omega, \rho\zeta) f(\omega, I_{0^+}^{n-2} v(\zeta), I_{0^+}^{n-3} v(\zeta), \ldots, I_{0^+}^{1} v(\zeta), v(\zeta)) d_\rho \zeta \right) d\omega \right| \\ &\leq \frac{1}{\Gamma_\rho(\vartheta)} \frac{\phi_3}{\varpi} \int_0^1 \Theta(\omega, \omega) \left(\int_0^1 \Phi(1, \rho\zeta) d_\rho \zeta \right) d\omega = \phi_3. \end{aligned}$$

So, $T : \overline{\Lambda}_{\phi_3} \to \overline{\Lambda}_{\phi_3}$ and T is completely continuous (see Lemma 6). Also, by (F2) and the above argument, we have that $\|Tv\| < \phi_1$ if $v \in \overline{\Lambda}_{\phi_1}$. Hence, (D2) of Theorem 1 holds.

Set

$$\lambda(z) = \frac{\phi_2 + \phi_3}{2}, \ \ \forall\, z \in [0,1].$$

Clearly, $\lambda(z) \in \Lambda$ and $\|\lambda\| = \frac{\phi_2+\phi_3}{2}$, and then

$$\Upsilon(\lambda) = \frac{\phi_2 + \phi_3}{2} > \phi_2.$$

Consequently, $\lambda \in \{v \in \Lambda(\Upsilon, \phi_2, d) : \vartheta(v) > \phi_2\} \neq \emptyset$. Furthermore, for any $v \in \Lambda(\Upsilon, \phi_2, d)$, we obtain

$$\phi_2 \leq v(z) \leq \phi_3, \ \ \forall\, z \in \left[\frac{1}{4}, \frac{3}{4}\right].$$

So, (F3) yields that

$$f(z, \zeta_1, \ldots, \zeta_{n-1}) > \frac{\phi_2}{R} \ \ \forall\, z \in \left[\frac{1}{4}, \frac{3}{4}\right], \ \ \phi_2 \leq \|v = (\zeta_1, \ldots, \zeta_{n-1})\| \leq \phi_3.$$

Then, by Lemmas 4 and 5, we can obtain

$$\begin{aligned}
\vartheta((Tv)(z)) &= \min_{\frac{1}{4} \leq z \leq \frac{3}{4}} ((Tv)(z)) \\
&= \frac{1}{\Gamma_\rho(\vartheta)} \int_0^1 \min_{\frac{1}{4} \leq z \leq \frac{3}{4}} \Theta(z, \omega) \\
&\quad \times \left(\int_0^1 \Phi(\omega, \rho\zeta) f(\omega, I_{0^+}^{n-2} v(\zeta), I_{0^+}^{n-3} v(\zeta), \ldots, I_{0^+}^{1} v(\zeta), v(\zeta)) d_\rho \zeta \right) d\omega \\
&\geq \frac{\phi_2}{R} \left(\frac{1}{4}\right)^{\vartheta - 1} \frac{\eta}{\Gamma_\rho(\vartheta)} \int_{\frac{1}{4}}^{\frac{3}{4}} \Theta(\omega, \omega) \left(\int_0^1 \Phi(1, \rho\zeta) d_\rho \zeta \right) d\omega \\
&= \phi_2.
\end{aligned}$$

Therefore, (D1) of Theorem 1 holds.

Finally, we claim that (D3) of Theorem 1 holds. If $v \in \Lambda(\Upsilon, \phi_2, \phi_3)$ and $\|Tv\| > d$, then $\Upsilon((Tv)(z)) > \phi_2$. So, (D3) of Theorem 1 holds.

Therefore, Theorem 1 implies that (1) has three positive solutions $\mathcal{Q}_1(z) = I_{0^+}^{n-2} v_1(z)$, $\mathcal{Q}_2(z) = I_{0^+}^{n-2} v_2(z)$, $\mathcal{Q}_3(z) = I_{0^+}^{n-2} v_3(z)$ such that $\|v_1\| < \phi_1, \phi_2 < \Upsilon(v_2(z))$, and $\|v_3\| > \phi_1$, with $\Upsilon(v_3(z)) < \phi_2$. The proof is completed. $\square$

4. Example

Consider the following mixed fractional Riemann–Liouville and quantum boundary value problem of the form

$$\begin{cases} D_{e^{-\pi}}^{2.5} (D_{0^+}^{1.5} \mathcal{Q}) - f(z, \mathcal{Q}(z)) = 0, & z \in (0,1), \\ \mathcal{Q}(0) = \mathcal{Q}(1) = 0, & \\ D_{0^+}^{1.5} \mathcal{Q}(0) = D_{e^{-\pi}} (D_{0^+}^{1.5} \mathcal{Q}(0) = D_{e^{-\pi}} (D_{0^+}^{1.5} \mathcal{Q})(1) = 0, & \end{cases} \tag{14}$$

where $\vartheta = 2.5$, $\gamma = 1.5$, $\rho = e^{-\pi}$, $n = 2$ and

$$f(z, \mathcal{Q}) = \begin{cases} \frac{z}{100} + \cos^2(\pi(1 - \mathcal{Q})), \quad z \in [0,1], \ \ 0 \leq \mathcal{Q} < 1 \\ \frac{z}{100} + \cos^2(\pi(1 - \mathcal{Q})) + 250 \arctan(\mathcal{Q} - 1), \ \ z \in [0,1], \ \ 1 \leq \mathcal{Q} \leq 2, \\ \frac{z}{100} + \cos^2(\pi(1 - \mathcal{Q})) + 250 \arctan(\mathcal{Q} - 1) + \cos^2\left(\frac{\pi}{2}(\mathcal{Q} - 1)\right), \ \ z \in [0,1], \ \ \mathcal{Q} \geq 4. \end{cases}$$

So, through direct calculations, we obtain

$$\varpi = 0.31293072, \qquad R = 0.0130532975.$$

By choosing $\phi_1 = 1, \phi_2 = 2, \phi_3 = 71$, we obtain

$$f(z, \mathcal{Q}) \leq 206.51 \leq \frac{\phi_3}{\varpi} = 223.69219, \quad \text{for all} \quad 0 \leq z \leq 1,\ 0 \leq \mathcal{Q} \leq 100;$$
$$f(z, \mathcal{Q}) \leq 1.01 \leq \frac{\phi_2}{\varpi} = 3.195595, \quad \text{for all} \quad 0 \leq z \leq 1,\ 0 \leq \mathcal{Q} \leq 1;$$
$$f(z, \mathcal{Q}) \geq 160.51 > \frac{\phi_2}{R} = 153.217, \quad \text{for all} \quad \frac{1}{4} \leq z \leq \frac{3}{4},\ 2 \leq \mathcal{Q} \leq 71.$$

Hence, f satisfies the conditions (H1) and (F1)–(F3). So, Theorem 2 yields that the problem (14) has at least three positive solutions.

5. Conclusions

In this paper, we considered a fractional differential equation involving fractional quantum differences and the Riemann–Liouville fractional derivatives. We studied the existence of at least three positive solutions by using the Leggett–Williams fixed-point theorem. Finally, we investigated the consistency of our theoretical findings by demonstrating an example. In future works, we can extend this problem to more fractional derivatives, such as the Hadamard fractional derivative, ψ-Hilfer and discrete fractional differential equations.

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