

Entropy Area Law in Quantum Field Theories and Spin Systems [†]

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Abstract: The entanglement entropy measures quantum correlations and it can be seen as the uncertainty on a quantum state. In one spatial dimension, the entanglement entropy scales as the boundary that divides two subsystems, so an area law has been proposed. However, the entanglement entropy diverges logarithmically at conformally invariant critical points, so the area law does not hold. The purpose of the work is to find a way to get more information about a critical state. The ground state of the Heisenberg XXZ model at criticality is analyzed by means of critical Ising eigenstates. Two ways of analysis are followed: a basis made of Ising eigenstates is built up and used to represent the XXZ ground state, then the Shannon entropy in the new basis is computed; the adiabatic evolution from the Ising ground state to the XXZ ground state. The result is that the Shannon entropy in the Ising basis scales linearly with the length of the system, while a phase transition is encountered during the adiabatic evolution. The conclusion is that there is no net gain in information after the procedure and possibly it is related to the fact the two systems stand in different phases.

Keywords: entanglement; entropy; area law; spin chain

1. Introduction

The Bekenstein-Hawking entropy of black holes is surprisingly proportional to the horizon area, namely, the surface of the system instead of the volume. In quantum systems, a similar area law holds for the entanglement entropy. In fact, first attempts to reproduce the entropy area law were focused on seeking a quantum source for the entropy of black holes [1]. However, the entanglement entropy area law does not hold under critical conditions.

The entanglement is typical of quantum mechanics and separates this from classical mechanics. Once two quantum systems have interacted, the overall state is not a product state and it is entangled. The entanglement entropy measures the entanglement, namely, quantum correlations that arise within correlation length. It is defined as the von Neumann entropy of the reduced density matrix of a subsystem ρ [2]

$$S = -\text{Tr}(\rho \log \rho) . \quad (1)$$

The entanglement entropy S scales as the boundary surface that divides two subsystems, through which degrees of freedom of both sides are correlated. Srednicki proposed the area law $S \sim L^{d-1}$ in [1], where L is a length and d is the number of spatial dimensions. This entropy can be seen as the uncertainty on physical states on the other side of the boundary [3,4]. The larger the entropy, the larger the uncertainty. However, in one spatial dimension, out of criticality and when interactions are local the area law holds and the entanglement entropy saturates to a finite value. But, at a conformally invariant

critical point, when correlation length diverges, the entanglement entropy diverges logarithmically as the length of the subsystem [4,5]

$$S \sim \log \frac{L}{\epsilon}, \tag{2}$$

where ϵ is a short-length scale, namely, it is a UV cut-off. It is needed to put this length scale by hand in quantum field theories, while ϵ is the lattice spacing in discrete systems. For higher spatial dimensions, local interactions and gapped systems are not necessary conditions to the area law. In fact, critical bosonic systems still obey the area law.

Thus, in the $d = 1$ case and under critical conditions, quantum states are highly entangled and the entropy area law does not hold. It is worth studying such cases, answering the question: can we get more information about a critical state? The purpose of this work is to develop a procedure for doing so.

2. Materials and Methods

Quantum systems can be discretized and simulated on a computer. A very useful approach is the tensor network one [6], which works only with those physical states that lie on a little corner of the whole Hilbert space, reproducing them with a nested network of tensors. The effect of this approximation is to gain in complexity. On the contrary, the attempt done in this work is exact. The analysis focuses on critical phases of two spin chain models with periodic boundary conditions, the Heisenberg XXZ and the simple Ising model, whose Hamiltonians are

$$H_{XXZ} = \sum_{i=1}^n \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y \right) + \Delta \sum_{i=1}^n \sigma_i^z \sigma_{i+1}^z, \tag{3}$$

$$H_{Is} = - \sum_{i=1}^n \sigma_i^z \sigma_{i+1}^z - \lambda \sum_{i=1}^n \sigma_i^x, \tag{4}$$

where σ^α are the Pauli matrices, λ and Δ are the couplings. These models are usually analyzed in the study of the entanglement [7,8]. The Heisenberg XXZ model has two critical points, but only the one with $\Delta = 1$ is conformally invariant. In such a phase the model belongs to the free bosons universality class. The Ising model has a unique critical point at $\lambda = 1$ and the model belongs to the free fermions universality class.

The analysis consists of using the eigenstates of the Ising model near criticality to gain information about the XXZ ground state at the conformally invariant critical point. This choice relies on the fact that the Ising model is exactly solvable and at the critical point the eigenstates are already entangled. This analysis is performed in two ways. First, a change of basis is performed on the XXZ critical ground state from the computational basis to the one made of Ising eigenstates. Then, the Shannon entropies of the two basis are computed. Second, the Ising ground state is slowly adapted to the XXZ one via adiabatic evolution. In both cases, computations are executed on a Python code, increasing the number n of spins.

2.1. Ising Basis

In order to describe the Heisenberg XXZ ground state near criticality in a new basis, a basis made of the Ising eigenstates near criticality is built up. The new set of nonzero coefficients reproduces the ground state. However, not all of them are relevant when there are degenerations. A selection process is needed in order to extract the best combination of independent coefficients, which is the one with the closest norm to 1. The expectation is that a basis made of entangled states makes the uncertainty about the XXZ ground state lower. This is quantified by the Shannon entropy of the selected coefficients, defined as

$$H = - \sum_i |c_i|^2 \log |c_i|^2. \tag{5}$$

H is computed and its scaling against n in the Ising basis is compared to H in the computational basis.

2.2. Adiabatic Evolution

Quantum states evolve according to the Schroedinger equation, but if the evolution is slow enough the state remains close to the ground state of the system [9]. During this adiabatic evolution no other states are excited. The second part of the analysis consists of computing the adiabatic evolution from the Ising ground state near criticality to the critical Heisenberg XXZ ground state. The time T needed for the adiabatic evolution is inversely proportional to g_{min}^2 , where g_{min} is the minimum energy gap. Considering the Hamiltonian

$$H(s) = (1 - s)H_{Is} + s H_{XXZ} \tag{6}$$

with $0 \leq s \leq 1$, the Ising ground state near criticality evolves to the critical XXZ ground state. If g_{min} approaches to zero while n increases, a phase transition is encountered and T blows up.

3. Results

3.1. Ising Basis

The number of selected coefficients grows exponentially as $N \sim 2^{bn}$, where $b \sim 0.5$, suggesting that at large n the number of Ising eigenstates needed to reproduce the Heisenberg ground state tends to $2^{n/2}$. From information theory point of view, an initial sequence of data can be transmitted with less nonzero components if a basis is wisely chosen.

In the new basis, the Shannon entropy H is lower than it was originally, but it scales linearly with n as it was in the computational basis. The outcome is shown in Figure 1. At the very end, there is no net gain in information.

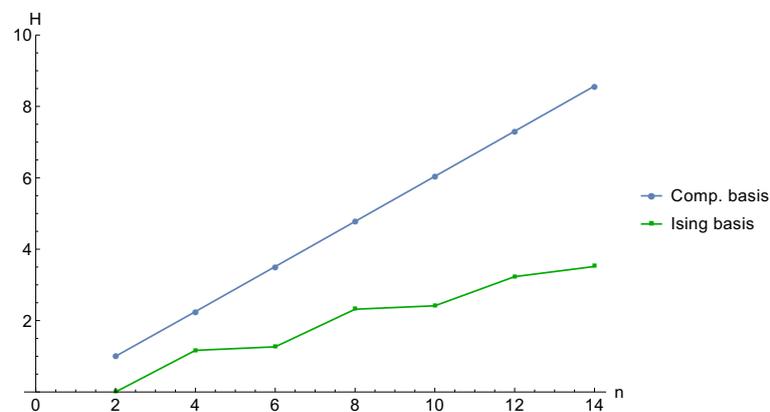


Figure 1. Shannon entropy of the critical ground state of the Heisenberg XXZ model. Computational basis in blue, Ising basis in green. Both entropies scale linearly with n .

3.2. Adiabatic Evolution

The time T needed for the adiabatic evolution comes out to scale exponentially with the dimension of the chain, as Figure 2 says. The Ising ground state encounters a quantum phase transition to adapt itself into the Heisenberg ground state.

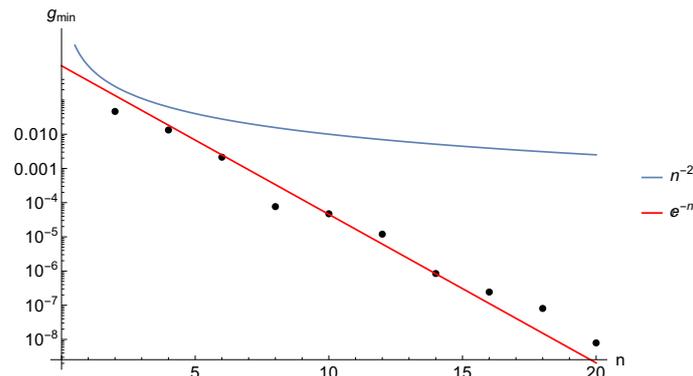


Figure 2. Minimum gap against n in the adiabatic evolution, from the critical ground state of the Ising model to the critical ground state of the Heisenberg XXZ model. The red line represents e^{-n} , making the time of adiabatic evolution scale exponentially. Blue line shows n^{-2} for comparison purpose. Log scale on y axis.

4. Discussion

Under critical conditions the entropy area law does not hold, since the entanglement entropy diverges logarithmically and the uncertainty on a quantum state is large. A basis made of Ising eigenstates near criticality carries information about the entanglement and can be used to reproduce the critical Heisenberg XXZ ground state, slightly reducing the uncertainty. However, the two models stand in two different critical phases and there is no net gain in information with such a procedure.

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