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New Fuzzy Numerical Methods for Solving Cauchy Problems

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Abstract: In this paper, new fuzzy numerical methods based on the fuzzy transform (F-transform or FT) for solving the Cauchy problem are introduced and discussed. In accordance with existing methods such as trapezoidal rule, Adams Moulton methods are improved using FT. We propose three new fuzzy methods where the technique of FT is combined with one-step, two-step, and three-step numerical methods. Moreover, the FT with respect to generalized uniform fuzzy partition is able to reduce error. Thus, new representations formulas for generalized uniform fuzzy partition of FT are introduced. As an application, all these schemes are used to solve Cauchy problems. Further, the error analysis of the new fuzzy methods is discussed. Finally, numerical examples are presented to illustrate these methods and compared with the existing methods. It is observed that the new fuzzy numerical methods yield more accurate results than the existing methods.

Keywords: fuzzy partition; fuzzy transform; new iterative method; Cauchy problems

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1. Introduction

In fact, most mathematical models in engineering and science requires the solution of ordinary differential equations (ODEs). Generally, it is difficult to obtain the closed form solutions for ODEs, especially, for nonlinear and nonhomogeneous cases. Many models often lead to ordinary differential equations which consist of Cauchy problems are an important branch of modern mathematics that arises naturally in different areas of applied sciences, physics, and engineering. Thus, many researchers start developing methods for solving Cauchy problems are of particular importance [1–3].

FT was coined by Perfilieva as a new mathematical method was developed [4]. The core idea of FT is a fuzzy partition of a universe into fuzzy subsets. The technique of FT has been successfully applied into other mathematical problems as well including image processing, analysis of time series and elsewhere [5–7]. This idea has been applied to Cauchy problems was first published as well as other numerical classical methods [8], by proposing generalized Euler and Euler- Cauchy methods, so that the Mid-point FT method was demonstrated in [9]. The success of these applications is due in part to the fact that FT is capable to accurately approximate any continuous function. Thus, we will propose new fuzzy numerical methods for Cauchy problems with help of the FT and new iterative method.

The motivation of the proposed study comes from the papers [3,8,10]. Numeric Solution to the Cauchy problem was considered and the authors showed that the error can be reduced by using FT with uniform fuzzy partitions [8,9]. At the same time, [10,11], the concept of generalized fuzzy partition was proposed. Besides others, a necessary and sufficient condition making it possible to design easily the generalized fuzzy partition was provided [12]. This is important for various practical applications of FT. Further [3], the authors have proposed modifications trapezoidal rule and Adams-Moulton methods (2 and 3-step) to solve ODEs based on the new iterative method was introduced [2].

In this paper, we discuss the problem that considered in [8,9]. The triangular and raised cosine generating function was replaced by new representations formulas for generalized uniform fuzzy partition of FT such as power of the triangular and raised cosine generating function. We study approximation properties of the FT based on powers of triangular and raised cosine generalized uniform fuzzy partition can be constructed in such way that the FT can reduce error. Also, we propose modifications in the FT introduced by I. Perfilieva [4] with respect to new representations formulas for generalized uniform fuzzy partition of FT and then the technique of FT is combined with traditional methods based on the new iterative method [2,3] to solve Cauchy problems. It is observed that the new methods proposed are more accurate results than the fuzzy approximation method [8,9].

This paper is organized as follows. In Section 2, we introduce the basic concepts and results of the FT with respect to the generalized uniform fuzzy partition needed throughout this paper. The main part of this paper is Sections 3 and 4, new representations for basic functions of FT, followed by the modified one step, 2-step , and 3-step based on new representations formulas for generalized uniform fuzzy partition of FT. In Section 5, numeric examples are discussed. Concluding remarks are presented in Section 6.

Throughout the paper, we denote by $\mathbb{N}$, $\mathbb{N}^+$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{R}^+$ the sets of natural (including zero), positive natural, integer, real , and positive real numbers, respectively.

2. Basic Concepts

In this section, we give some definitions and introduce the necessary notation in [10], which will be used throughout the paper. Throughout this section, we deal with an interval $[a, b] \subset \mathbb{R}$ of real numbers.

Definition 1. *(generalized uniform fuzzy partition) Let $x_i \in [a, b]$, $i = 1, \ldots, n$, be fixed nodes such that $a = x_1 < \ldots < x_n = b$, $n \geq 2$. We say that the fuzzy sets $A_i : [a, b] \to [0, 1]$ constitute a generalized fuzzy partition of $[a, b]$ if for every $i = 1, \ldots, n$ there exists $h > 0$ such that $x_0 = x_1$, $x_n = x_{n+1}$, $[x_i - h, x_i + h] \subseteq [a, b]$ and the following conditions are fulfilled:*

1. *(positivity and locality) – $A_i(x) > 0$ if $x \in (x_{i-1}, x_{i+1})$ and $A_i(x) = 0$ if $x \in [a, b] \setminus (x_{i-1}, x_{i+1})$;*
2. *(continuity) – A_i is continuous on $[x_{i-1}, x_{i+1}]$;*
3. *(covering) – for $x \in [a, b]$, $\sum_{i=1}^{n} A_i(x) > 0$.*

Fuzzy sets $A_1, \ldots, A_n$ are called basic functions. It is important to remark that by conditions of locality and continuity, $\int_a^b A_i(x)dx > 0$. A generalized of uniform fuzzy partition of $[a, b]$ is defined for equidistant nodes, i.e., for all $i = 1, \ldots, n-1$, $x_i = x_{i+1} + h$, where $h = (b - a) / (n - 1)$ and two additional properties are satisfied,

4. *$A_i(x_i - x) = A_i(x_i + x)$ for all $x \in [0, h]$, $i = 2, \ldots, n-1$;*
5. *$A_i(x) = A_{i-1}(x - h)$ and $A_{i+1}(x) = A_i(x - h)$ for all $x \in [x_i, x_{i+1}]$, $i = 2, \ldots, n-1$;*

then the fuzzy partition is called h-uniform generalized fuzzy partition. Throughout this paper, we will write generalized uniform fuzzy partition instead of h-uniform generalized fuzzy partition.

Definition 2. *(generating function) A function $K : [-1, 1] \to [0, 1]$ is called a generating function if it is assumed to be even, continuous and $K(x) > 0$ if $x \in (-1, 1)$. The function $K : [-1, 1] \to \mathbb{R}$ is even if for all $x \in [0, 1]$, $K(-x) = K(x)$.*

The following definition recall the concept of generalized fuzzy partition which can be easily extended to the interval $[a, b]$. We assume that $[a, b]$ is partitioned by $A_1, \ldots, A_n$, according to Definition 1.

Definition 3. *A generalized uniform fuzzy partition of interval* $[a,b]$*, determined by the triplet* (K, h, a)*, can be defined using generating function K (Definition 2). Then, basic functions of a generalized uniform fuzzy partition are shifted copies of K defined by*

$$A_i(x) = K\left(\frac{x - x_i}{h}\right), \ \ x \in [x_i - h, x_i + h],$$

for all $i = 1, \ldots, n$*. The parameter h is called the bandwidth or the shift of the fuzzy partition and the nodes* $x_i = a + ih$ *are called the central point of the fuzzy sets* $A_1, \ldots, A_n$*.*

Remark 1. *A fuzzy partition is called Ruspini if the following condition*

$$A_i(x) + A_{i+1}(x) = 1, \ i = 1, \ldots, n-1, \tag{1}$$

holds for any $x \in [x_i, x_{i+1}]$*. This condition is often called Ruspini condition.*

3. New Representations of Basic Functions for Particular Cases

In this section, we propose two subsection, new representations of basic functions constitute a generalized uniform fuzzy partition of interval $[a,b]$ and then FT technique based on new representations of basic functions.

3.1. Power of the Triangular and Raised Cosine Generalized Uniform Fuzzy Partition

Two types of basic functions, triangular and sinusoidal shaped membership functions, were proposed by [4,8]. Later [13], the authors considered different shapes for the basic functions of fuzzy partition. Furthermore, a generalized fuzzy partition appeared in connection with the notion of a higher-degree F-transform [11]. Its even weaker version was implicitly introduced to satisfy the requirements of image compression [14]. Recently, the different conditions for generalized uniform fuzzy partitions was proposed by [10,12]. Table 1 provides the definition two types of generating function, triangular and raised cosine generating functions [7,10–12,15].

Table 1. Generating functions of strong uniform fuzzy partition.

Triangular Generating Function	Raised Cosine Generating Function
$\max\{1 - \lvert x \rvert, 0\}$	$\frac{1}{2}(1 + \cos(\pi x))_{\lvert[-1,1]}$

In the following, we present new representations for generating function. In particular, we present three new representations, based on the triangular and raised cosine generating functions: two generating function based on the triangular generating functions and one generating function based on the raised cosine generating function.

Definition 4. *(natural order triangular generating function) Let* $K_{T_i^m} : \mathbb{R} \to [0,1]$*,* $i = 1,2$*, be defined by*

$$1.\ K_{T_1^m}(x) = \begin{cases} (1 - |x|)^m, & |x| \le 1, \\ 0, & otherwise \end{cases} = \min\left((1 - |x|)^m, 1\right), \tag{2}$$

$$2.\ K_{T_2^m}(x) = \begin{cases} 1 - (|x|)^m, & |x| \le 1, \\ 0, & otherwise \end{cases} = \min\left(1 - (|x|)^m, 1\right), \tag{3}$$

are called power of the triangular (shaped) generating functions, when $m \in \mathbb{N}^+$*.*

Definition 5. *(odd natural order raised cosine generating function) Let* $K_{C^m} : \mathbb{R} \to [0,1]$ *be defined by*

$$K_{C^m}(x) = \begin{cases} \frac{1}{2}\left(1+\cos^m(\pi x)\right), & |x| \le 1; \\ 0, & otherwise. \end{cases} \quad (4)$$

is called power of the raised cosine generating function , when m *is an odd natural number (i.e.,* $m = 2k-1,\ k \in \mathbb{N}^+$*).*

Remark 2. *Particularly, we can check the validity of Equation (4) using the following relation*

$$\begin{aligned} K_{C^m}(x) &= \begin{cases} \frac{1}{2}\left(1+\cos^m(\pi x)\right), & |x| \le 1, \\ 0, & otherwise. \end{cases} \\ &= \begin{cases} \frac{1}{2}\left(1+\sin^m\left(\frac{\pi}{2}(2x+1)\right)\right), & |x| \le 1, \\ 0, & otherwise. \end{cases} \end{aligned}$$

Lemma 1. *If* $K_{T_i^n}(x)$, $i = 1,2$, $(K_{C^m}(x))$ *determines power of the triangular (raised cosine) generating functions, then*

$$1.\ \int_{-1}^{1} K_{T_1^n}(x)\,dx = \frac{2}{n+1}, \qquad 2.\ \int_{-1}^{1} K_{T_2^n}(x)\,dx = \frac{2n}{n+1}, \qquad 3.\ \int_{-1}^{1} K_{C^m}(x)\,dx = 1,$$

or equivalent

$$1.\ \int_{-h}^{h} K_{T_1^n}\left(\tfrac{t}{h}\right)dt = \frac{2h}{n+1}, \qquad 2.\ \int_{-h}^{h} K_{T_2^n}\left(\tfrac{t}{h}\right)dt = \frac{2nh}{n+1}, \qquad 3.\ \int_{-h}^{h} K_{C^m}\left(\tfrac{t}{h}\right)dt = h,$$

where $0 \le \left|\frac{2}{n+1}\right| \le 1$, $1 \le \left|\frac{2n}{n+1}\right| \le 2$ *, h be positive real numbers, m is an odd natural number and* $n \in \mathbb{N}^+$*.*

Proof. The proof can be easily obtained by using integration methods within the boundaries and then substitution $x = t/h$. □

On the basis of Definitions 4 and 5, Lemma 1, and according to Definition 3, we can also be defined using generating function αK for $\alpha > 0$ (in general, not necessarily satisfy Ruspini condition). Thus, basic functions of a generalized uniform fuzzy partition are shifted copies of αK defined by

$$A_k(x, x_0) = \alpha K\left(\frac{x - x_0}{h} - k\right),\ x \in [x_{i-1}, x_{i+1}]. \quad (5)$$

In particular, let $K_{T_1^m}$, $K_{T_2^m}$, (and K_{C^m}) be power of the triangular (and raised cosine) generating function defined above. We will say that a generalized uniform fuzzy partition is power of a triangular (or of raised cosine) generalized uniform fuzzy partition if its generating function K belongs to $\alpha K_{T_1^m}$, $\alpha K_{T_2^m}$, (or αK_{C^m}) whenever $\alpha = 1/\left(\int_{-1}^{1} K(t)dt\right)$. Indeed, the equality α immediately follows from $\int_{-1}^{1} \alpha K_{T_1^m}(t)\,dt = 1 \Rightarrow \alpha = 1/\left(\int_{-1}^{1} K_{T_1^m}(t)\,dt\right)$. In the following, we modified the definition a triangular and raised cosine generalized uniform fuzzy partition by propose that power of the triangular and raised cosine generalized uniform fuzzy partitions can be simply using the equality $\alpha = 1/\left(\int_{-1}^{1} K(t)dt\right)$.

Definition 6. *Let* $m \in \mathbb{N}^+$*. A system of fuzzy sets* $\{A_k \mid k \in \mathbb{Z}\}$ *defined by*

$$1.\ A_k(x, x_0) = \alpha K_{T_1^m}\left(\tfrac{x-x_0}{h} - k\right),\ \alpha = \frac{m+1}{2}, \quad (6)$$

$$2.\ A_k(x, x_0) = \alpha K_{T_2^m}\left(\tfrac{x-x_0}{h} - k\right),\ \alpha = \frac{m+1}{2m}, \quad (7)$$

is called power of the triangular generalized uniform fuzzy partition of the real line determined by the triplet $(K_{T_i^m},\ h,\ x_0)$, $i = 1, 2$. *Further, let m is an odd natural number. A system of fuzzy sets* $\{A_k \mid k \in \mathbb{Z}\}$ *defined by*

$$3.\ A_k(x, x_0) \quad = \quad \alpha K_{C^m}\left(\tfrac{x-x_0}{h} - k\right),\ \ \alpha = 1, \tag{8}$$

is called power of the raised cosine generalized uniform fuzzy partition of the real line determined by the triplet $(K_{C^m}, h,,\ x_0)$. *The parameter h is bandwidth of the fuzzy partition and* $x_0 + kh = x_k$.

Definition 7. *Let* $x_1 < \ldots < x_n$ *be fixed nodes within* $[a, b] \subset \mathbb{R}$, *such that* $x_1 = a$, $x_n = b$ *and* $n \geq 2$. *We consider nodes* $x_1, \ldots, x_n$ *are equidistant, with distance (shift)* $h = (b-a)\ /\ (n-1)$. *A system of fuzzy sets* $B_1, \ldots, B_n:\ [a, b] \to [0, 1]$ *be power of a triangular and raised cosine generalized uniform fuzzy partitions of* $[a, b]$ *if it is defined by*

$$B_k(x) \quad = \quad \begin{cases} A_k(x, a), & x \in [a, b], \\ 0, & \text{otherwise}. \end{cases} \quad \text{or equivalent} \quad B_k(x) \quad = \quad \begin{cases} \alpha K\left(\frac{x-x_k}{h}\right), & x \in [a, b], \\ 0, & \text{otherwise}. \end{cases} \tag{9}$$

where $x_k = a + kh$. *In the sequel, we denote K for a generating function determined by the Formulas (2)–(4). Further,* α, $A_k(x, a)$, $k = 1, \ldots, n$, *are determined by the Formulas (6)–(8).*

Lemma 2. *If* $B_k(x)$ *determines power of the raised cosine generalized uniform fuzzy partition of* $[a, b]$, *then* $B_k(x)$ *satisfied Ruspini condition (1) when m (see (4)) is an odd natural number.*

Proof. Indeed, if $x \in [a, b]$, there exists $k \in \{1, \ldots, n-1\}$ such that $x \in [x_k, x_{k+1}]$. By (4) and (8), and Remark 1, we get

$$\begin{aligned} B_k(x) + B_{k+1}(x) = A_k(x, a) + A_{k+1}(x, a) &= \alpha K_{C^m}\left(\frac{x - x_k}{h}\right) + \alpha K_{C^m}\left(\frac{x - x_{k+1}}{h}\right), \\ &= \frac{1}{2}\left(1 + \cos^m\left(\pi\left(\frac{x - x_k}{h}\right)\right)\right) + \frac{1}{2}\left(1 + \cos^m\left(\pi\left(\frac{x - x_{k+1}}{h}\right)\right)\right), \\ &= 1 + \frac{1}{2}\left(\cos^m\left(\frac{\pi}{h}(x - x_k)\right) + \cos^m\left(\frac{\pi}{h}(x - x_{k+1})\right)\right). \end{aligned}$$

By the properties of trigonometric functions, notice that$\cos(\theta + \pi) = -\cos(\theta)$, it is easy to see that

$$\cos^m\left(\pi\left(\frac{x - x_k}{h}\right)\right) + \cos^m\left(\pi\left(\frac{x - x_{k+1}}{h}\right)\right) = \cos^m\left(\pi\left(\frac{x - x_{k+1}}{h}\right) + \pi\right) + \cos^m\left(\pi\left(\frac{x - x_{k+1}}{h}\right)\right).$$

Thus, if m is an odd natural number, the result is 0. □

In the following, if K is a normal generating function (i.e., $K(0) = 1$, not necessarily satisfy Ruspini condition), we use generating function αK for $\alpha > 0$, where $(\alpha K)(x) = \alpha \cdot K(x)$.

Lemma 3. *If basic functions* B_k, $k = 1, \ldots, n$, *of a generalized uniform fuzzy partition are shifted copies of* αK, $\alpha > 0$, *defined by the Formula (5) and moreover, K is normal as an additional condition. Then, for each* $k = 1, \ldots, n$, $B_k(x_k) = \alpha$, $x_k \in [x_k - h, x_k + h]$.

Proof. A generating function K is said to be normal if $K(0) = 1$. By the Formula (5) and a generating function K is normal, we get $B_k(x_k) = \alpha K\left(\frac{x_k - x_k}{h}\right) = \alpha K(0) = \alpha > 0$. □

Corollary 1. *Let the assumptions of Lemma 3 be fulfilled, but fuzzy sets* B_k, $k = 1, \ldots, n$, $n \geq 2$, *determined by Definition 7. Then, for each* $k = 1, \ldots, n$, $B_k(x_k) = \alpha$, $x_k \in [x_k - h,\ x_k + h]$, *where* α *is defined by Definition 7.*

Proof. Indeed, the proof immediately follows from Definition 7 and Lemma 3. □

Corollary 2. *Let the assumptions of Lemma 3 be fulfilled, but fuzzy sets* $B_k,\ k = 1, \ldots, n,\ n \geq 2$, *determined by Definition 3. Then, for each* $k = 1, \ldots, n$, $B_k(x_k) = 1,\ \ x_k \in [x_k - h, x_k + h]$.

3.2. New FT Based Power of the Triangular and Raised Cosine Generalized Uniform Fuzzy Partition

In this subsection, we present the main principles of F-transform detailed in [8,10,11] that are modified with respect to power of the triangular and raised cosine generalized uniform fuzzy partition. Further, we will show that FT components with respect to power of the triangular and raised cosine generalized uniform fuzzy partition can be simplified and approximated of an original function, say f.

Definition 8. *Let* f *be a continuous function on* $[a, b]$ *and* $B_k(t)$, $k = 1, \ldots, n$, *be power of the triangular and raised cosine generalized uniform fuzzy partition of* $[a, b]$, $n \geq 2$. *A vector of real numbers* $F[f] = (F_1,\ F_2, \ldots,\ F_n)$ *given by*

$$F_k = \frac{\int_a^b f\,(t)\,B_k(t)\,dt}{\int_a^b B_k(t)\,dt}, \tag{10}$$

for $k = 1, \ldots, n$ *is called the direct FT of* f *with respect to power of the triangular and raised cosine generalized uniform fuzzy partition* B_k.

In the following, we assume a generating function K in the Formulas (2)–(4). We will simplify the representation (10).

Lemma 4. *Let* $f \in C\,([a, b])$ *and according to Definition 7, fuzzy sets* $B_k,\ k = 1, \ldots, n,\ n \geq 2$, *be power of a triangular and raised cosine generalized uniform fuzzy partition of* $[a, b]$ *with a generating function* K, *then representation (10) of direct FT can be simplified as follows for* $k = 1, \ldots, n$

$$F_k = \frac{\int_{-1}^{1} f\,(th + t_k)\,K(t)\,dt}{\int_{-1}^{1} K(t)\,dt} = \frac{\int_{-h}^{h} f\,(t + t_k)\,K(\frac{t}{h})\,dt}{\int_{-h}^{h} K(\frac{t}{h})\,dt}.$$

Proof. In this proof, we will write a generating function K instead of (2)–(4). By Definition 7, we get

$$B_k\,(t) = \alpha K\left(\frac{t - t_k}{h}\right),\ \ t \in [t_k - h, t_k + h]\,,$$

for $k = 1, \ldots, n$, $t_0 = t_1,\ t_{n+1} = t_n$, and substituting $u = \frac{t - t_k}{h}$ and then substituting $t = s/h$. Thus, we get

$$\int_{t_{k-1}}^{t_{k+1}} f\,(t)\,B_k(t)\,dt = \alpha h \int_{-1}^{1} f\,(th + t_k)\,K(t)\,dt = \alpha \int_{-h}^{h} f\,(t + t_k)\,K(\frac{t}{h})\,dt$$

$$\int_{t_{k-1}}^{t_{k+1}} B_k(t)\,dt = \alpha h \int_{-1}^{1} K(t)\,dt = \alpha \int_{-h}^{h} K(\frac{t}{h})\,dt$$

and its corresponding results with representation (10). □

Indeed, the previous lemma holds for every fuzzy partition generated by a kernel. Now, we will simplify the above given expressions for the coefficients $F[f] = (F_1,\ F_2, \ldots,\ F_n)$ in the representation (10) even more. This fact is very important for applications which are more flexible and consequently easier to use.

Lemma 5. *Let the assumptions of Lemma 4 be fulfilled. Then, the coefficients $F[f] = (F_1, F_2, \ldots, F_n)$ in the expression (10) of the FT component F_k of f as follows:*

$$F_k = \frac{1}{h}\int_a^b f(t)\, B_k(t)\, dt = \frac{\alpha}{h}\int_a^b f(t)\, K\left(\frac{t-t_k}{h}\right) dt, \tag{11}$$

for $k = 1, \ldots, n$, where interval $[a, b]$ is partitioned by power of the triangular and raised cosine generalized uniform fuzzy partition $B_1, \ldots, B_n$ and α is defined by Definition 7.

Proof. Let $k \in \{1, \ldots, n\}$ and consider set of fuzzy sets $B_k(x)$ from power of the triangular and raised cosine generalized uniform fuzzy partition of $[a, b]$ in (9). We will prove the equality $\int_{t_{k-1}}^{t_{k+1}} B_k(t)\, dt = h$. We get by virtue of Lemmas 1 and 4, and (6):

$$\int_{t_{k-1}}^{t_{k+1}} B_k(t)\, dt = \int_{t_{k-1}}^{t_{k+1}} A_k(t,a), dt = \int_{t_k-h}^{t_k+h} \left(\frac{m+1}{2}\right) K_{T_1^m}\left(\frac{t-t_k}{h}\right) dt = h\int_{-1}^{1}\left(\frac{m+1}{2}\right) K_{T_1^m}(t)\, dt = h,$$

where h is the bandwidth of the fuzzy partition and $t_k = a + kh$. Similarly, the other Formulas (7) and (8) will be proved and then its corresponding in the expression (10). □

Lemma 6. *Let $f \in C[a, b]$. Then for any $\varepsilon > 0$ there exist $n_\varepsilon \in \mathbb{N}$ and $B_1, \ldots, B_{n_\varepsilon}$ be basic functions form power of the triangular and raised cosine generalized uniform fuzzy partition of $[a, b]$. Let F_k, $k = 1 \ldots, n$, be the integral FT components of f with respect to $B_1, \ldots, B_{n_\varepsilon}$. Then for each $k = 1 \ldots, n_\varepsilon - 1$ the following estimations hold: $|f(t) - F_i| \leq \varepsilon$ for each $t \in [a, b] \cap [t_k, t_{k+1}]$ and $i = k, k+1$.*

Proof. see [4]. □

Corollary 3. *Let the conditions of Lemma 6 be fulfilled. Then for each $k = 1 \ldots, n_\varepsilon - 1$ the following estimations hold: $|F_k - F_{k+1}| < \varepsilon$.*

Proof. According to [4,16], let $t \in [a, b] \cap [t_k, t_{k+1}]$. Then by Lemma 6, for any $k = 1, \ldots, n-1$ we obtain $|f(t) - F_k| < \varepsilon/2$ and $|f(t) - F_{k+1}| < \varepsilon/2$. Thus, $|F_k - F_{k+1}| \leq |f(t) - F_k| + |f(t) - F_{k+1}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. □

The following theorem estimates the difference between the original function and its direct FT with respect to power of the triangular and raised cosine generalized uniform fuzzy partition.

Theorem 1. *Let $f(t) \in C^2[a, b]$ and the conditions of Lemma 5 be fulfilled. Then for $k = 1, \ldots, n$*

$$F_k = \alpha f(t_k) + \mathcal{O}\left(h^2\right), \tag{12}$$

where $\alpha > 0$ or α is defined by Definition 7.

Proof. By locality condition for Definition 1, Lemmas 3 and 5, and according to the proof of Lemma 9.3 [8], using the trapezoid formula with nodes t_{k-1}, t_k, t_{k+1} to the numerical computation of the integral, we get for $\alpha > 0$

$$\begin{aligned}
F_k &= \frac{1}{h}\int_{t_{k-1}}^{t_{k+1}} f(t)\, B_k(t)\, dt, \\
&= \frac{1}{h}\cdot\frac{h}{2}\left(f(t_{k-1})\, B_k(t_{k-1}) + 2f(t_k)\, B_k(t_k) + f(t_{k+1})\, B_k(t_{k+1})\right) + \mathcal{O}\left(h^2\right), \\
&= f(t_k)\, B_k(t_k) + \mathcal{O}\left(h^2\right) = f(t_k)\, A_k(t_k, a) + \mathcal{O}\left(h^2\right), \\
&= f(t_k)\, \alpha K(0) + \mathcal{O}\left(h^2\right), \\
&= \alpha f(t_k) + \mathcal{O}\left(h^2\right).
\end{aligned}$$

□

Definition 9. *Let $F[f] = (F_1, F_2, \ldots, F_n)$ be direct FT of a function $f \in C[a,b]$ with respect to the fuzzy partition $B_k(t)$, $k = 1, \ldots, n$ of $[a,b]$. Then, the function $\hat{f}$ defined on $[a,b]$*

$$\hat{f}(t) = \frac{\sum_{k=1}^{n} F_k B_k(t)}{\sum_{k=1}^{n} B_k(t)}, \tag{13}$$

is called the inverse FT of f.

Corollary 4. *Let the assumptions of Lemma 2 and moreover, Let $\hat{f}(t)$ be the inverse FT of f with respect to power of the raised cosine generating function . Then, for all $t \in [a,b]$ the following holds: $\hat{f}(t) = \sum_{k=1}^{n} F_k B_k(t)$.*

Proof. This proof immediately follows from Defintion 9, Lemma 2 and then using $\sum_{k=1}^{n} B_k(t) = 1$. □

The following lemma estimates the difference between the original function and its inverse FT.

Lemma 7. *Let the assumptions of Theorem 1 and let $\hat{f}(t)$ be the inverse FT of f with respect to the fuzzy partition of $[a,b]$ is given by Definition 7. Then, for all $t \in [a,b]$ the following estimation holds:*

$$\hat{f}(t) = \alpha f(t_k) + \mathcal{O}\left(h^2\right). \tag{14}$$

Proof. Let $t \in [a,b]$ so that $x \in [t_k, t_{k+1}]$ for some $k = 1, \ldots, n$. By Theorem 1,

$$\begin{aligned}\hat{f}(t) - \alpha f(t_k) &= \frac{\sum_{k=1}^{n} F_k B_k(t)}{\sum_{k=1}^{n} B_k(t)} - \alpha f(t) = \frac{\sum_{k=1}^{n} F_k B_k(t)}{\sum_{k=1}^{n} B_k(t)} - \frac{\sum_{k=1}^{n} \alpha f(t_k) B_k(t)}{\sum_{k=1}^{n} B_k(t)} \\ &= \frac{\sum_{k=1}^{n} (F_k - \alpha f(t_k)) B_k(t)}{\sum_{k=1}^{n} B_k(t)} = \mathcal{O}\left(h^2\right).\end{aligned}$$

□

Corollary 5. *Let the assumptions of Lemma 7, then $\left|\hat{f}(t) - f(t)\right| < \varepsilon$.*

Proof. The proof easily follows from the proof of Lemma 7 and then using Lemma 6 as follows:

$$\left|\hat{f}(t) - f(t)\right| = \frac{\sum_{k=1}^{n} |F_k - f(t)| B_k(t)}{\sum_{k=1}^{n} B_k(t)} < \varepsilon.$$

□

Remark 3. *According to the Definitions 1 and 2, if the normality is considered to be an additional condition for generating function (i.e., $K(0) = 1$) and generalized uniform fuzzy partition of $[a,b]$ satisfies $A_k(x_k) = \alpha$, $\alpha > 0$, then it is easy to see that the inverse FT $\hat{f}(t_k) = F_k$ for all $k = 1, \ldots, n$. This is true for Definition 7. Moreover, if orthogonality condition (Ruspini condition (1)) is replaced by covering condition in Definition 1 and generalized uniform fuzzy partition of $[a,b]$ satisfies $A_k(x_k) = \alpha = 1$, then it is easy to also see that the inverse FT $\hat{f}(t_k) = F_k$ for all $k = 1, \ldots, n$. This is true for Formula (8) only.*

Important property of the direct FT as well as inverse FT is their linearity, namely, given $f, g \in C[a,b]$ and $\alpha, \beta \in R$, if $h = \alpha f + \beta g$, then $F[h] = \alpha F[f] + \beta F[g]$ and $\hat{h} = \alpha \hat{f} + \beta \hat{g}$. In the next section, we present new fuzzy numerical methods based on the FT and a new iterative method to numeric solution of the Cauchy problem.

4. New Fuzzy Numerical Methods for Cauchy Problem

Consider the initial value problem (IVP) for the Cauchy problem:

$$y' = f(t, y), \quad y(t_1) = y_1, \quad a = t_1 \leq t \leq t_n = b. \tag{15}$$

where $y_1 \in \mathbb{R}$ and f is continuous function on $[a, b] \times \mathbb{R}$ and satisfies Lipschitz condition. In fact, the analytical solution of problem (15) is often difficult and sometimes impossible to obtain. Instead, numerical analysis is interested with obtaining approximate solutions with errors within reasonable bounds. Thus, a usage of fuzzy numerical methods seems to be suitable.

In [8,9], the authors have presented Euler method and Mid-point rule, based on FT to numeric solution of Cauchy problem (15). A new iterative method (NIM) has been proposed for solving linear (nonlinear) functional equations, ordinary differential equations and delay differential equations [2,3].

In this section, we present three new schemes to solve Cauchy problem (15), that use the FT and NIM. Our motivation stems from the classical approach, trapezoidal rule (1-step) and Adams Moulton methods (2 and 3-step). For the rest of this paper, suppose that we are given the Cauchy problem (15), where the function f on $[a, b]$ are sufficiently smooth and we assume that all necessary requirements for constructing the FT of the solution of Cauchy problem (15) are fulfilled. Now, we present numerical Scheme I, II, and III. The first scheme uses 1-step method, while the second one uses 2-step method, and the third uses 3-step method.

4.1. Numeric Scheme I: Modified Trapezoidal Rule Based on FT and NIM for Cauchy Problem

In the present subsection, we will construct a numeric scheme of the more advanced method known as the Trapezoidal Rule. Recall that it is a one-step method with second-order accuracy, which can be considered as a Runge–Kutta method. We propose modification of trapezoidal rule based on FT and NIM for solving Cauchy problem. Modification of the trapezoidal rule can be improved by the FT to solve Cauchy problem (15). We contributed to numeric methods of Cauchy problem (15) by scheme provides formulas for the FT components, Y_k, $k = 2, \ldots, n-1$, of the unknown function $y(t)$ with respect to choose some power of the triangular (or raised cosine) generalized uniform fuzzy partition, $B_1, \ldots, B_n$, of interval $[a, b]$ with parameter h to approximate solution of Cauchy problem (15). The first, choose the number $n \geq 2$ and compute $h = (b-a)/(n-1)$, then construct the generalized uniform fuzzy partition of $[a, b]$ using Definition 7. Note that each function B_k spans over three nodes t_{k-1}, t_k, t_{k+1}, $k = 2, \ldots, n-1$. Nevertheless, $B_k(t_{k-1}) = B_k(t_{k+1}) = 0$ and $B_k(t_k) = 1$. Now, we apply the FT and NIM to Cauchy problem (15) and obtain the numeric Scheme I for $k = 1, \ldots, n-1$ as follows (see [3,8] for technical details):

$$\begin{aligned} Y_1 &= y_1, \\ Y^*_{k+1} &= Y_k + hF_k/2, \\ Y^{**}_{k+1} &= Y^*_{k+1} + hF^*_{k+1}/2, \\ Y_{k+1} &= Y_k + h\left(F_k + F^{**}_{k+1}\right)/2, \end{aligned} \tag{16}$$

where

$$F_k = \frac{\int_a^b f(t, Y_k)B_k(t)dt}{\int_a^b B_k(t)dt}, \quad F^*_{k+1} = \frac{\int_a^b f(t, Y^*_{k+1})B_{k+1}(t)dt}{\int_a^b B_{k+1}(t)dt}, \quad F^{**}_{k+1} = \frac{\int_a^b f(t, Y^{**}_{k+1})B_{k+1}(t)dt}{\int_a^b B_{k+1}(t)dt}. \tag{17}$$

In the sequel, the approximate solution of Cauchy problem (15) can be obtained using the inverse FT as follows:

$$y_n(t) = \sum_{k=1}^{n} Y_k B_k(t). \tag{18}$$

4.2. Numeric Scheme II: Modified 2-Step Adams Moulton Method Based on FT and NIM for Cauchy Problem

The Scheme I uses 1-step method for solving Cauchy problem (15). In this subsection, we improve 2-step Adams Moulton method using FT and NIM for solving Cauchy problem (15). The 2-step Adams Moulton method can be improved to effectively approximate the solution of (15) by the FT components, Y_k, $k = 2, \ldots, n-1$, of the unknown function $y(t)$ with respect to choose some power of the triangular (or raised cosine) generalized uniform fuzzy partition (9). Let $Y_1 = y_1$ and $Y_2 = y_2$ if possible; otherwise, we can compute FT component Y_2 from numeric Scheme I. Analogously to [3,8], we apply the FT and NIM to Cauchy problem (15) and obtain the numeric Scheme II in the following form for $k = 2, \ldots, n-1$:

$$\begin{aligned} Y^*_{k+1} &= Y_k + h\left(8F_k - F_{k-1}\right)/12, \\ Y^{**}_{k+1} &= Y^*_{k+1} + 5hF^*_{k+1}/12, \\ Y_{k+1} &= Y_k + h\left(8F_k - F_{k-1} + 5F^{**}_{k+1}\right)/12, \end{aligned} \tag{19}$$

where

$$F_{k-1} = \frac{\int_a^b f(t, Y_{k-1})B_{k-1}(t)dt}{\int_a^b B_{k-1}(t)dt}, \qquad F_k = \frac{\int_a^b f(t, Y_k)B_k(t)dt}{\int_a^b B_k(t)dt},$$

$$F^*_{k+1} = \frac{\int_a^b f(t, Y^*_{k+1})B_{k+1}(t)dt}{\int_a^b B_{k+1}(t)dt}, \quad \text{and } F^{**}_{k+1} = \frac{\int_a^b f(t, Y^{**}_{k+1})B_{k+1}(t)dt}{\int_a^b B_{k+1}(t)dt}.$$

Then, obtain the desired approximation for y by the inverse FT (18) applied to $[Y_1, \ldots, Y_n]$.

4.3. Numeric Scheme III: Modified 3-Step Adams Moulton Method Based on FT and NIM for Cauchy Problem

In this subsection, we improve 3-step Adams Moulton method using FT and NIM for solving Cauchy problem (15). The 3-step Adams Moulton method can be improved to effectively approximate the solution of (15) by the FT components, Y_k, $k = 2, \ldots, n-1$, of the unknown function $y(t)$ with respect to choose some power of the triangular (or raised cosine) generalized uniform fuzzy partition (see Definition 7), $B_1, \ldots, B_n$, of interval $[a, b]$ with parameter $h = (b-a)/(n-1)$, $n \geq 2$. Let $Y_1 = y_1$, $Y_2 = y_2$ and $Y_3 = y_3$ if possible; otherwise, we can compute FT components Y_2 and Y_3 from numeric Scheme I. Now, we apply the FT and NIM to Cauchy problem (15) and obtain the following numeric Scheme III for $k = 3, \ldots, n-1$ (see [3,8] for technical details):

$$\begin{aligned} Y^*_{k+1} &= Y_k + h\left(19F_k - 5F_{k-1} + F_{k-2}\right)/24, \\ Y^{**}_{k+1} &= Y^*_{k+1} + 9hF^*_{k+1}/24, \\ Y_{k+1} &= Y_k + h\left(19F_k - 5F_{k-1} + F_{k-2} + 9F^{**}_{k+1}\right)/24, \end{aligned} \tag{20}$$

where

$$F_{k-2} = \frac{\int_a^b f(t, Y_{k-2})A_{k-2}(t)dt}{\int_a^b A_{k-2}(t)dt}, \qquad F_{k-1} = \frac{\int_a^b f(t, Y_{k-1})A_{k-1}(t)dt}{\int_a^b A_{k-1}(t)dt}, \quad F_k = \frac{\int_a^b f(t, Y_k)A_k(t)dt}{\int_a^b A_k(t)dt},$$

$$F^*_{k+1} = \frac{\int_a^b f(t, Y^*_{k+1})A_{k+1}(t)dt}{\int_a^b A_{k+1}(t)dt}, \text{ and } \quad F^{**}_{k+1} = \frac{\int_a^b f(t, Y^{**}_{k+1})A_{k+1}(t)dt}{\int_a^b A_{k+1}(t)dt}.$$

In the sequel, the inverse FT (18) approximates the solution $y(t)$ of the Cauchy problem (15).

4.4. Error Analysis of Fuzzy Numeric Method for Cauchy Problem

In this subsection, we present error analysis for numeric scheme I only, because the technique of error analysis for rest numeric schemes (Schemes II and III) can be obtained analogously. Consider the Formula (16). If $y(t_k) = y_k$ and Y_k denote the exact solution and the numerical solution and substituting the exact solution in the Formula (16), we get

$$\begin{aligned} y^*_{k+1} &= y_k + hF^e_k/2, \\ y^{**}_{k+1} &= y^*_{k+1} + hF^{e*}_{k+1}/2, \\ y_{k+1} &= y_k + h\left(F^e_k + F^{e**}_{k+1}\right)/2, \end{aligned} \tag{21}$$

where

$$F_k^e = \frac{\int_a^b f(t,y_k)B_k(t)dt}{\int_a^b B_k(t)dt}, \quad F_{k+1}^{e*} = \frac{\int_a^b f(t,y_{k+1}^*)B_{k+1}(t)dt}{\int_a^b B_{k+1}(t)dt}, \quad F_{k+1}^{e**} = \frac{\int_a^b f(t,y_{k+1}^{**})B_{k+1}(t)dt}{\int_a^b B_{k+1}(t)dt}, \tag{22}$$

and the truncation error T_k of the Scheme I is given by

$$T_k = \frac{y_{k+1} - y_k}{h} - \frac{1}{2}\left(F_k^e + F_{k+1}^{e**}\right). \tag{23}$$

Rearranging (16), we get

$$0 = \frac{Y_{k+1} - Y_k}{h} - \frac{1}{2}\left(F_k + F_{k+1}^{**}\right). \tag{24}$$

If we denote the error $e_{k+1} = Y_{k+1} - y_{k+1}$ and subtracting (24) from (23), so:

$$T_k h = e_{k+1} - e_k - \frac{h}{2}\left(F_k - F_k^e\right) - \frac{h}{2}\left(F_{k+1}^{**} - F_{k+1}^{e**}\right). \tag{25}$$

Lemma 8. *Let f is assumed to be sufficiently smooth function of its arguments on $[a,b]$ and satisfies the Lipschitz condition with the constant L with respect to y, then we get for $k = 1, \ldots, n$,*

$$|e_{k+1}| \le |e_k|(1+c) + Th \quad \text{and} \quad \left|F_k^e - F_{k+1}^{e**}\right| \le LhM_2$$

where $c = hL + \frac{h^2L^2}{2} + \frac{h^3L^3}{8}$, $T = \max\limits_{1\le k\le n} |T_k|$, M_2 is upper bound for f, and F_k^e, F_{k+1}^{e**} are determined by Formula (22).

Proof. By hypothesis, f satisfies the Lipschitz condition and using Lemma 5, Formulas (16), (17), (21) and (22), we get

$$\begin{aligned}
|F_k - F_k^e| &\le \frac{1}{h}\left|\int_a^b f(t,Y_k)B_k(t)dt - \int_a^b f(t,y_k)B_k(t)dt\right| \le L|e_k| \\
|F_{k+1}^* - F_{k+1}^{e*}| &\le \frac{1}{h}\left|\int_a^b f(t,Y_{k+1}^*)B_{k+1}(t)dt - \int_a^b f(t,y_{k+1}^*)B_{k+1}(t)dt\right| \le L\left|Y_{k+1}^* - y_{k+1}^*\right| \\
|F_{k+1}^{**} - F_{k+1}^{e**}| &\le \frac{1}{h}\left|\int_a^b f(t,Y_{k+1}^{**})B_{k+1}(t)dt - \int_a^b f(t,y_{k+1}^{**})B_{k+1}(t)dt\right| \le L\left|Y_{k+1}^{**} - y_{k+1}^{**}\right| \\
\left|Y_{k+1}^* - y_{k+1}^*\right| &\le |(Y_k + hF_k/2) - (y_k + hF_k^e/2)| \le |e_k|\left(1 + \frac{hL}{2}\right) \\
\left|Y_{k+1}^{**} - y_{k+1}^{**}\right| &\le \left|(Y_{k+1}^* + hF_{k+1}^*/2) - (y_{k+1}^* + hF_{k+1}^{e*}/2)\right| \le |e_k|\left(1 + \frac{hL}{2}\right)^2 \\
|e_{k+1}| &\le |e_k| + \frac{hL}{2}|e_k| + \frac{hL}{2}\left|y_{k+1}^{**} - Y_{k+1}^{**}\right| + Th \\
&\le |e_k| + \frac{hL}{2}|e_k| + \frac{hL}{2}|e_k|\left(1 + \frac{hL}{2}\right)^2 + Th \\
&= |e_k|\left(1 + hL + \frac{h^2L^2}{2} + \frac{h^3L^3}{8}\right) + Th
\end{aligned}$$

Furthermore, by using $|f(t, y(t))| \leq M_2$, we get

$$\begin{aligned}
\left|F_k^e - F_{k+1}^{e**}\right| &= \left|\frac{1}{h}\left(\int_a^b \left(f(t, y_k) - f(t+h, y_{k+1}^{**})\right) B_k(t) dt\right)\right| \\
&= \left|\frac{1}{h}\left(\int_a^b \left(f(t, y_k) - f(t, y_{k+1}^{**}) + f(t, y_{k+1}^{**}) - f(t+h, y_{k+1}^{**})\right) B_k(t) dt\right)\right| \\
&\leq L\left|y_k - y_{k+1}^{**}\right| \\
&= \frac{Lh}{2}\left|-F_k^e - F_{k+1}^{e*}\right| \\
&= \frac{Lh}{2}\left|\frac{1}{h}\int_a^b \left(f(t, y_k) + f(t, y_{k+1}^{*})\right) B_k(t) dt\right| \\
&\leq LhM_2
\end{aligned}$$

This completes the proof. □

Theorem 2. *Consider the the numeric Scheme I (16), where $f \in C^2[a, b]$ and satisfies the Lipschitz condition with the constant L with respect to y. Then, the solution Y_k, $k = 1, \ldots, n$, obtained by the numeric scheme I (16) for solving Cauchy problem (15) satisfies*

$$|e_k| = |Y_k - y_k| \leq \frac{hM}{2L} e^{kc}, \tag{26}$$

where $c = hL + \frac{h^2L^2}{2} + \frac{h^3L^3}{8}$, M_1, M_2 are upper bound for f', f, respectively, on $[a, b]$, and $M_1 + M_2L = M$.

Proof. By hypothesis, y'' exists and bounded on $[a, b]$ with $\max_{a \leq t \leq b}|y''(t)| = M_1$ by assuming that $f \in C^2[a, b]$. Then, using Lemma 8, (23) and Taylor's theorem for $k = 1, \ldots, n-1$, we get

$$\begin{aligned}
T_k &= \frac{y_{k+1} - y_k}{h} - \frac{1}{2}\left(F_k^e + F_{k+1}^{e**}\right) \\
&= \frac{1}{2}hy''(\xi_k) + f(t_k, y_k) - \frac{1}{2}\left(F_k^e + F_{k+1}^{e**}\right) \\
&= \frac{1}{2}hy''(\xi_k) + f(t_k, y_k) - F_k^e + \frac{1}{2}F_k^e - \frac{1}{2}F_{k+1}^{e**} \\
&= \frac{1}{2}hy''(\xi_k) + \frac{1}{2}\left(F_k^e - F_{k+1}^{e**}\right)
\end{aligned}$$

where $\xi_k \in [t_k, t_{k+}]$. Now, using Lemma 8

$$\begin{aligned}
T = \max_{1 \leq k \leq n}|T_k| &\leq \frac{1}{2}h\left|y''(\xi_k)\right| + \frac{LhM_2}{2} \\
&\leq \frac{h}{2}(M_1 + LM_2) = \frac{hM}{2}
\end{aligned}$$

Now, by virtue of Lemma 8 and we have used $e_1 = 0$, $(1+c)^k \leq e^{kc}$, we get for $k = 1, \ldots, n$

$$\begin{aligned}
|e_k| &\leq \frac{(1+c)^k - 1}{c} Th \leq \frac{(1+c)^k}{L + \frac{hL^2}{2} + \frac{h^2L^3}{8}} T \\
&\leq \frac{T}{L} e^{kc} \leq \frac{hM}{2L} e^{kc}
\end{aligned}$$

where $c = hL + \frac{h^2L^2}{2} + \frac{h^3L^3}{8}$. Thus, if the step length $h \to 0$, then for all k, the error,$|e_k|$ converges to zero. So the method is convergent. This completes the proof. □

5. Numerical Examples

In this section, we present examples of the Cauchy problem (15).

Example 1. *Consider the following initial value problem with initial conditions $y(0) = 1$ and with a smooth right-hand function*

$$y'(t) = t^2 - y, \quad t \in [0, 2]. \tag{27}$$

Example 2. *Consider the Cauchy problem (15) with oscillating right-hand function. We take $f(t, y) = 1 + 2y\cos(t^2) + \sin(2t^2)$, $t(\frac{\pi}{2}) = 2.1951$, $a = \frac{\pi}{2}$ and $b = \frac{3\pi}{2}$.*

The results are listed in Tables 2–4 by fuzzy numerical methods proposed in this paper with respect to case $K_{T_1^{201}}$ and Table 5 by fuzzy numerical methods proposed in this paper with respect to case $K_{T_1^1}$, $K_{T_1^3}$, $K_{T_1^{201}}$, K_{C^1}. The Euclidean distance is given by Norm ℓ_2 defined as $\|Y - y(t)\|_2 = \sqrt{\sum_k (Y_k - y(t_k))^2}$ and mean square error (MSE) defined as MSE $= \frac{1}{n}(\|Y_k - y(t_k)\|_2)^2$. This is an easily computable quantity for a particular sample. Concluding remarks are summarized as follows:

- In view of Table 2, a comparison between the Euler method (Euler-FT) [8], the Mid-point rule (Mid-FT), Scheme I and II [9] and three new schemes (16), (19) and (20) in this paper for Example 1. We can easily observe from Table 2, the better results (in comparison with the Euler-FT method [8]) are obtained by the three new schemes in this paper and the best result (in comparison with the Scheme I, II and II) is obtained by the Scheme III. Also, the better results (in comparison with the Mid-point rule (Mid-FT), Scheme I and II [9]) are obtained by the Scheme II (19) and Scheme III (20) in this paper where all fuzzy numerical methods used the FT components and the best approximation is shown by the Scheme III (20) with FT components.

Table 2. Comparison of numeric results for Example 1. The columns contain the exact and seven approximate solutions of the Cauchy problem (27) with a smooth right-hand function: the first three approximate solution is obtained by the three new schems ((16), (19) and (20)), the fourth approximate solution by the Euler-FT [8] with FT components and the last three by the schemes are proposed in [9]. The best approximation is shown by the Scheme III proposed above (20) with FT components.

t_i	Solution $y(t)$	Proposed Scheme I	Proposed Scheme II	Proposed Scheme III	Euler-FT in [8]	Mid-FT in [9]	Scheme I in [9]	Scheme II in [9]
0	1	1	1	1	1	1	1	1
0.1	0.905163	0.905350	0.905163	0.905163	0.900166	0.905162	0.904392	0.904297
0.2	0.821269	0.821605	0.821322	0.821269	0.811316	0.8213	0.819722	0.819741
0.3	0.749182	0.749630	0.749274	0.749221	0.734351	0.749235	0.746860	0.747182
0.4	0.689680	0.690208	0.689798	0.689742	0.670083	0.689786	0.686592	0.687391
0.5	0.643469	0.644047	0.643602	0.643546	0.619241	0.643611	0.639629	0.641061
0.6	0.611188	0.611788	0.611324	0.611271	0.582484	0.611397	0.606615	0.608821
0.7	0.593415	0.594012	0.593543	0.593495	0.560402	0.593665	0.588129	0.591239
0.8	0.590671	0.591243	0.590781	0.590741	0.553528	0.590998	0.584697	0.588828
0.9	0.603430	0.603956	0.603513	0.603483	0.562342	0.603799	0.596795	0.602053
1	0.632121	0.632581	0.632168	0.632149	0.587274	0.632571	0.624851	0.631332
1.1	0.677129	0.677507	0.677132	0.677127	0.628714	0.677618	0.669253	0.677045
1.2	0.738806	0.739085	0.738757	0.738768	0.687009	0.739381	0.730353	0.739535
1.3	0.817468	0.817635	0.817360	0.817388	0.762475	0.818075	0.808466	0.819111
1.4	0.913403	0.913443	0.913229	0.913276	0.855394	0.914099	0.903881	0.916053
1.5	1.026870	1.026772	1.026624	1.026692	0.966021	1.027588	1.016856	1.030615
1.6	1.158103	1.157857	1.157779	1.157869	1.094586	1.158915	1.147625	1.163024
1.7	1.307316	1.306911	1.306909	1.307022	1.241294	1.308138	1.296400	1.313489
1.8	1.474701	1.474127	1.474205	1.474343	1.406331	1.47562	1.463372	1.482195
1.9	1.660431	1.659681	1.659842	1.660006	1.589864	1.661347	1.648715	1.669312
2	1.864665	1.863636	1.863899	1.864097	1.779378	1.865684	1.852585	1.874993

- In Tabel 3, a comparison of MSE and a comparison of Norm ℓ_2 for Examples 1 and 2. We can easily observe, the best results are obtained by the three new schemes in this paper and the better results (in comparison with the other numerical classical methods) are obtained by all fuzzy numerical methods used the FT components except Euler-FT [8] for these examples.

Table 3. The values of MSE and Norm ℓ_2 for Example 1– 2.

Method	Ex.1		Ex.2	
	Norm ℓ_2	MSE	Norm ℓ_2	MSE
Proposed Scheme I	2.21945×10^{-03}	2.34569×10^{-07}	3.42892×10^{-01}	5.59882×10^{-03}
Proposed Scheme II	1.28684×10^{-03}	7.88551×10^{-08}	3.76033×10^{-01}	6.73336×10^{-03}
Proposed Scheme III	9.28253×10^{-04}	4.10311×10^{-08}	2.15401×10^{-01}	2.20942×10^{-03}
Euler-FT [8]	2.20790×10^{-01}	2.32134×10^{-03}	$3.74484 \times 10^{+00}$	6.67801×10^{-01}
Mid-FT [9]	2.56525×10^{-03}	3.13357×10^{-07}	6.73731×10^{-01}	2.16149×10^{-02}
Scheme I [9]	3.54973×10^{-02}	6.00026×10^{-05}	8.42893×10^{-01}	3.38319×10^{-02}
Scheme II [9]	1.90439×10^{-02}	1.72701×10^{-05}	5.90233×10^{-01}	1.65893×10^{-02}
Trapezoidal Rule	4.30423×10^{-02}	8.82208×10^{-05}	$1.93095 \times 10^{+00}$	1.77551×10^{-01}
2-Step Adams Moulton	3.49968×10^{-02}	5.83228×10^{-05}	$1.85289 \times 10^{+00}$	1.63485×10^{-01}
3-Step Adams Moulton	3.14968×10^{-02}	4.72405×10^{-05}	$1.57237 \times 10^{+00}$	1.17732×10^{-01}

- In view of Table 4, a comparison between the three new schemes (16), (19) and (20) in this paper and the Trapezoidal Rule, 2-Step Adams Moulton Method and 3-Step Adams Moulton Method based on Euler method for Example 2. We can easily observe from Table 4, the better results are obtained by the three new schemes in this paper and the best result (in comparison with the Scheme I, II and II) is obtained by the Scheme III.

Table 4. Comparison of numeric results for Example 2. The columns contain the exact and six approximate solutions of the Cauchy problem (27) with oscillating right-hand function: the first three approximate solution is obtained by the three new schems ((16), (19), and (20)), the last three approximate solution by the Trapezoidal Rule, 2-Step Adams Moulton Method and 3-Step Adams Moulton Method. The best approximation is shown by the Scheme III proposed above (20) with FT components.

t_i	Solution $y(t)$	Proposed Scheme I	Proposed Scheme II	Proposed Scheme III	Trap [1]	2-Step Adams [2]	3-Step Adams [3]
1.570796327	2.195062	2.195062	2.195062	2.195062	2.195062	2.195062	2.195062
1.727875959	1.883281	1.894259	1.883281	1.883281	1.860613	1.883281	1.883281
1.884955592	1.485003	1.511046	1.490853	1.485003	1.454428	1.463813	1.485003
2.042035225	1.185605	1.224868	1.191621	1.184830	1.172418	1.163839	1.177378
2.199114858	1.206758	1.256721	1.208147	1.202292	1.205264	1.180648	1.194336
2.35619449	1.688183	1.733796	1.675538	1.676788	1.638071	1.613025	1.638504
2.513274123	2.546629	2.558069	2.508798	2.525411	2.371288	2.370415	2.421052
2.670353756	3.420051	3.381690	3.362740	3.396118	3.110292	3.151241	3.228492
2.827433388	3.817594	3.751660	3.766365	3.802239	3.459038	3.534435	3.617396
2.984513021	3.479187	3.451288	3.463039	3.476930	3.153857	3.226956	3.285059
3.141592654	2.711291	2.760842	2.722280	2.707811	2.480046	2.498676	2.521585
3.298672286	2.305201	2.404556	2.301686	2.280860	2.168197	2.117053	2.127181
3.455751919	2.871345	2.942818	2.810863	2.818639	2.622265	2.558398	2.599754
3.612831552	4.080085	4.035446	3.952587	4.015230	3.555034	3.556356	3.660448
3.769911184	4.767095	4.645081	4.647076	4.733825	4.104830	4.188576	4.317767
3.926990817	4.209785	4.184589	4.183879	4.213375	3.643127	3.728492	3.801548
4.08407045	3.258243	3.383482	3.263962	3.224157	2.935940	2.895967	2.895039
4.241150082	3.481873	3.609332	3.386338	3.370921	3.111499	2.989980	3.008814
4.398229715	4.873146	4.799588	4.642440	4.733200	4.094280	4.055938	4.179701
4.555309348	5.501192	5.331327	5.311775	5.444657	4.561691	4.652699	4.813484
4.71238898	4.498591	4.551357	4.485128	4.493916	3.817903	3.867167	3.912209

[1] Trapezoidal Rule; [2] 2-Step Adams Moulton Method; [3] 3-Step Adams Moulton Method.

- In Tabel 5, a comparison between computation errors for three schemes based on the FT with respect to the power of the triangular and raised cosine generalized uniform fuzzy partition determined by Formula (9), where the advantage of the $K_{T_1^m}$ for Examples 1 and 2 is evident.

Table 5. The values of MSE and Norm ℓ_2 for Examples 1 and 2 by the three schemes with respect to the power of the triangular and raised cosine generalized uniform fuzzy partition are proposed in this paper. The best approximation is shown by using $K_{T_1^{201}}$.

Proposed Scheme	Case	Ex.1		Ex.2	
		Norm ℓ_2	**MSE**	**Norm ℓ_2**	**MSE**
I	$K_{T_1^1}$	7.81857×10^{-03}	2.91095×10^{-06}	6.38151×10^{-01}	1.93922×10^{-02}
	$K_{T_1^3}$	5.06528×10^{-03}	1.22176×10^{-06}	4.65538×10^{-01}	1.03203×10^{-02}
	$K_{T_1^{201}}$	2.21945×10^{-03}	2.34569×10^{-07}	3.42892×10^{-01}	5.59882×10^{-03}
	K_{C^1}	6.96371×10^{-03}	2.30920×10^{-06}	5.79002×10^{-01}	1.59640×10^{-02}
II	$K_{T_1^1}$	5.92425×10^{-03}	1.67127×10^{-06}	5.45616×10^{-01}	1.41761×10^{-02}
	$K_{T_1^3}$	3.58895×10^{-03}	6.13360×10^{-07}	4.29959×10^{-01}	8.80307×10^{-03}
	$K_{T_1^{201}}$	1.28684×10^{-03}	7.88551×10^{-08}	3.76033×10^{-01}	6.73336×10^{-03}
	K_{C^1}	5.18129×10^{-03}	1.27837×10^{-06}	5.01710×10^{-01}	1.19864×10^{-02}
III	$K_{T_1^1}$	5.31047×10^{-03}	1.34291×10^{-06}	4.42442×10^{-01}	9.32167×10^{-03}
	$K_{T_1^3}$	3.09350×10^{-03}	4.55702×10^{-07}	2.88083×10^{-01}	3.95199×10^{-03}
	$K_{T_1^{201}}$	9.28253×10^{-04}	4.10311×10^{-08}	2.15401×10^{-01}	2.20942×10^{-03}
	K_{C^1}	4.59684×10^{-03}	1.00624×10^{-06}	3.84860×10^{-01}	7.05320×10^{-03}

This constitutes an important improvement to previous methods which do not provide such information except in the methods such as Euler-FT proposed in [8] and Mid-FT , Scheme I, and Scheme II [9] for Cauchy problems by the more efficient way of computation approximate solutions. Thus, this study will be of particular importance.

6. Conclusions

We extended applicability of fuzzy numeric methods to the initial value problem (the Cauchy problem). We proposed three new numeric methods based on the FT and NIM and then analyzed their suitability. We considered in the case of the generalized uniform fuzzy partition is power of the triangular (raised cosine) generalized uniform fuzzy partition and showed that the newly proposed schemes outperform the Euler-FT [8] and Mid-FT , Scheme I, and Scheme II [9] especially on examples where the generalized uniform fuzzy partition is power of the triangular generalized uniform fuzzy partition by using generating function (2). Alos, the newly proposed schemes in this paper outperform the Trapezoidal Rule, 2-Step Adams Moulton Method and 3-Step Adams Moulton Method. Moreover, we proved that the Scheme I determines an approximate solution which converges to the exact solution. This constitutes an important improvement to previous results were coined by I. Perfilieva [8]. To conclude previous sections, the proposed schemes are more accurate and stable. In particular, these schemes can be used for solving initial value problem.

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