Neutron Stars in $f(R)$-Gravity and Its Extension with a Scalar Axion Field

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Abstract: We present a brief review of general results about non-rotating neutron stars in simple R^2 gravity and its extension with a scalar axion field. Modified Einstein equations are presented for metrics in isotropical coordinates. The mass–radius relation, mass profile and dependence of mass from central density on various equations of state are given in comparison to general relativity.

1. Introduction

Neutron stars are very interesting objects for the possible verification of not only physical models of dense matter but various theories of modified gravity. The most simple R^2 gravity and its possible extensions were investigated as a possible alternative to general relativity in many papers (see, for example, [1,2]). The main motivation comes from cosmology with the discovery of the accelerated expansion of universe [3–5].

According to the standard approach, this acceleration occurs due to nonzero vacuum energy consisting of nearly 70% of the global energy budget of the universe. The remaining 28%, clustered in galaxies and clusters of galaxies, consists of baryons (only 4%) and cold dark matter (CDM), the nature of which is unclear. Another paradigm is the description of cosmological acceleration in frames of modified gravity [6–8]. It is interesting to note that a unified description of cosmological evolution, including epochs of matter and radiation dominance, is possible in the $f(R)$ theory [9–14].

However, in context of modification of general relativity, one need consider not only the cosmological level, but stellar structures too, especially compact relativistic objects (neutron stars and black holes). The possible deviations from GR can be detected due to the extremely strong gravitational field in the centers of relativistic stars.

This paper presents a brief review of general results about non-rotating neutron stars in simple R^2 gravity and its extension with a scalar field. Our review was based mainly on results obtained in papers [15–17]. The mass–radius relation, mass profile and dependence of mass on central density are given in comparison with general relativity. From a methodological point of view, we used metrics in isotropical coordinates for the deriving of modified Einstein equations. In these coordinates, equations take relatively simple forms. Inclusion of R^2-term leads to additional equation for scalar curvature.

For illustration we consider two equations of state (EoS) for nuclear matter—GM1 [18] and APR [19].

2. Basic Equations for Non-Rotating Stars in Isotropic Coordinates

Einstein equations from general relativity have the following form (in the natural system of units with $G = c = 1$):

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}, \tag{1}$$

where $R_{\mu\nu}$ are components of the Ricci tensor, $R = g^{\mu\nu}R_{\mu\nu}$ is the scalar curvature and $T_{\mu\nu}$ is the energy-momentum tensor of matter.

In a case of $f(R)$ gravity (first, second and third derivatives of $f(R)$ on R should exist) with the action

$$S = \frac{1}{16\pi} \int f(R)\sqrt{-g}d^4x \tag{2}$$

equations became more complex

$$f_R R_{\mu\nu} - \frac{f}{2}g_{\mu\nu} - \left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\right)f_R = 8\pi T_{\nu\mu}. \tag{3}$$

Here and further on, we omit argument R for $f(R)$. The covariant D'Alambertian $\Box = \nabla^\mu\nabla_\mu$ is introduced and f_R simply means df/dR.

One can rewrite (3) in equivalent form

$$f_R R_{\mu\nu} - \frac{1}{2}(f_R R - f)g_{\mu\nu} - \left(\frac{1}{2}\Box + \nabla_\mu\nabla_\nu\right)f_R = 8\pi\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right), \tag{4}$$

where T is the trace of energy-momentum tensor. It is useful to make a foliation of spacetime by spacelike hypersurfaces Σ_t with unit vector n_α orthogonal to them. Therefore, components of metric $\gamma_{\alpha\beta}$ induced on Σ_t are

$$\gamma_{\alpha\beta} = g_{\alpha\beta} + n_\alpha n_\beta. \tag{5}$$

We consider the case of the non-rotating stars for which the metric components do not depend on time:

$$ds^2 = -N^2dt^2 + \gamma_{ij}dx^idx_j, \tag{6}$$

Here N is lapse function. Then one needs to project Equations (4) twice onto hypersurface Σ_t, twice along to normal vector $\vec{n}$ and once along $\vec{n}$ and Σ_t. As a result, one obtains three equations:

$$f_R\left(D_iD_jN - N\left\{{}^3R_{ij} + KK_{ij} - 2K_{ik}K^k{}_j\right\}\right) = \tag{7}$$

$$= 4\pi N\left[(\sigma - \epsilon)\gamma_{ij} - 2\sigma_{ij}\right] - \frac{1}{2}(f_R R - f)N\gamma_{ij} - N\left(\frac{1}{2}\gamma_{ij}\Box + D_iD_j\right)f_R,$$

$$f_R({}^3R + K^2 - K_{ij}K^{ij}) = 16\pi\epsilon + f_R R - f + 2D^iD_if_R, \tag{8}$$

$$f_R(D_jK^j{}_i - D_iK) = 8\pi p_i - n^\mu\nabla_\mu(D_if_R), \tag{9}$$

In these equations K_{ij} are components of the tensor of extrinsic curvature and $K = K^i_i$. The 3-dimensional covariant derivatives D_i are defined via 3-dimensional Christoffel symbols ${}^3\Gamma^i{}_{jk}$:

$$D_iD_jN = \frac{\partial^2 N}{\partial x^i\partial x^j} - {}^3\Gamma^k{}_{ij}\frac{\partial N}{\partial x^k}, \tag{10}$$

$$D_jK^j{}_i = \frac{\partial K^j{}_i}{\partial x^j} + {}^3\Gamma^j{}_{jk}K^k{}_i - {}^3\Gamma^k{}_{ji}K^j{}_k, \tag{11}$$

$$D_iK = \frac{\partial K}{\partial x^i}. \tag{12}$$

The components of the 3-dimensional Ricci tensor $^3R_{ij}$ and scalar curvature can be calculated via $^3\Gamma^i_{\ jk}$ and its partial derivatives from standard relations.

On the right-hand sides of Equations (7)–(9), quantities ϵ, σ_{ij} and p_i are defined by relations from energy-momentum tensor:

$$\epsilon = n^\mu n^\nu T_{\mu\nu},$$

$$\sigma_{ij} = \gamma^\mu_i \gamma^\nu_j T_{\mu\nu}, \quad \sigma = \sigma^i_i. \tag{13}$$

$$p_i = -n^\mu \gamma^\nu_i T_{\mu\nu}.$$

and have a sense of energy density, components of stress tensor and a vector of energy flux density correspondingly.

Then we take the trace of Equation (7):

$$f_R D_i D^i N = N f_R (^3R + K^2 - 2K_{ik}K^{ik}) + 4\pi N(\sigma - 3\epsilon) + \tag{14}$$

$$+\frac{3}{2}N(f - f_R R) - \frac{3}{2}N\Box f_R - N D_i D^i f_R.$$

From Equation (8) it follows that

$$f_R(^3R + K^2) = f_R K_{ij}K^{ij} + 16\pi\epsilon + f_R R - f + 2D_i D^i f_R$$

and therefore one can rewrite the previous equation as

$$f_R D_i D^i N = N f_R K_{ij}K^{ij} + 4\pi N(\epsilon + \sigma) - \frac{1}{2}N(f_R R - f) - \tag{15}$$

$$-\frac{3}{2}N\Box f_R + N D_i D^i f_R.$$

In the case of a non-rotating star, all metric functions depend only on the radial coordinate. We use isotropic spatial coordinates with a metric in the form

$$ds^2 = -N^2(r)dt^2 + A^2(r)(dr^2 + r^2 d\Omega^2). \tag{16}$$

For this metric, one can find that

$$K = 0, \quad K_{ij}K^{ij} = 0.$$

Three-dimensional scalar curvature is

$$^3R = -\frac{4}{A^2}\left(\triangle^r_{(3)}\ln A + \frac{1}{2}\left(\frac{d\ln A}{dr}\right)^2\right).$$

Hereinafter, $\triangle^r_{(n)}$ means the radial part of Laplace operator in n-dimensional euclidean space; i.e.,

$$\triangle^r_{(n)} = \frac{d^2}{dr^2} + \frac{n-1}{r}\frac{d}{dr}.$$

The energy-momentum tensor in the case of spherical symmetry is simply $T^{nu}_\mu = \text{diag}(-\epsilon, p, p, p)$ where p is pressure of matter, and therefore

$$\sigma^r_r = \sigma^\theta_\theta = \sigma^\phi_\phi = p, \quad \sigma = 3p.$$

After algebraic calculations, the equations for metric functions can be presented as (it is useful to introduce functions $\eta = \ln(AN)$ and $\nu = \ln N$):

$$f_R \triangle_{(3)}^r \nu + \frac{1}{2} \triangle_{(3)}^r f_R = 4\pi A^2 (\epsilon + 3p) - \frac{A^2}{2}(f_R R - f) - f_R \frac{d\eta}{dr} \frac{d\nu}{dr} - \tag{17}$$

$$- \frac{1}{2} \frac{d\eta}{dr} \frac{df_R}{dr} - \frac{d\nu}{dr} \frac{df_R}{dr}.$$

$$f_R \triangle_{(4)}^r \eta + \triangle_{(4)}^r f_R = 16\pi A^2 p - A^2 (f_R R - f) - f_R \left(\frac{d\eta}{dr}\right)^2 - 2\frac{d\eta}{dr} \frac{df_R}{dr} \tag{18}$$

For 4-dimensional scalar curvature, one can obtain an equation from the trace of Einstein's equation:

$$\triangle_{(3)}^r f_R = \frac{8\pi}{3} A^2 (3p - \epsilon) - \frac{A^2}{3}(f_R R - 2f) - \frac{d\eta}{dr} \frac{df_R}{dr}. \tag{19}$$

Outside the star, the following conditions on η, ν and R should be imposed:

$$\nu \to 0, \quad \eta \to 0, \quad R \to 0 \quad \text{for} \quad r \to \infty$$

from the condition of asymptotical flatness on spatial infinity. In general relativity, the solution of Einstein's equations outside the star has the form:

$$A = \left(1 + \frac{M}{2r}\right)^2, \quad N = \left(1 - \frac{M}{2r}\right)\left(1 + \frac{M}{2r}\right)^{-1}. \tag{20}$$

where parameter M has sense of gravitational mass. Therefore, the gravitational mass of a star can be defined from the asymptotic behavior of A at $r \to \infty$:

$$M = 2 \lim_{r \to \infty} r(\sqrt{A} - 1).$$

One should also take into account that the circumferential radius $\tilde{r}$ is

$$\tilde{r} = Ar.$$

Note that in the following the symbol "r" in the figures means circumferential radial coordinate. Tilde is omitted for simplicity.

As illustrative example, we consider R^2 gravity for which

$$f = R + \alpha R^2.$$

Interesting results were obtained for $f(R) = R^{1+\epsilon}$ gravity in [20]. The mass–radius relations in metric and torsional R^2 gravity were investigated in [21]. For a recent review of compact star models in modified theories of gravity, see [22–24] and references therein.

3. R-Square Gravity with a Scalar Axion Field

From results of simple R^2-gravity, it follows that manifestations of modified gravity are observable only in cases of large contribution from the R^2-term. The addition of a scalar field in the simple model with non-minimal interaction with gravity allows one to construct solutions for which R^2-term plays a significant role only inside the star.

How can we motivate this extension of the simple model? Astrophysical data about bullet cluster and cluster MACSJ0025 (see ([25–28]) speak in favor of the particle nature of dark matter. For a long time it was considered that dark matter was nothing other than so-called WIMPs from models of supersymmetry in particle physics. Unsuccessful experiments in WIMP searches (see for example [29–33]) gave rise to other hypotheses. In particular, a realistic explanation is that dark matter consists of axions [34–44]. In contrast to failed experiments for WIMPs, there are some indications in favor of the existence of axions (see [45–52]). Axion emission can appears in the process of the cooling of

neutron stars [53]. Masses of axions can be very low (theoretical estimations give value in the wide range $\sim 10^{-12} - 10^{-3}$ eV). The possibility of axions detection is based on axion-photon interaction in the presence of magnetic fields [54–56].

The contribution to action from a free axion scalar field ϕ with mass m_a is assumed in the following form

$$S_\phi = \int d^4x \sqrt{-g} \left(-\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m_a^2\phi^2 \right). \tag{21}$$

The solution for axion field ϕ with spherical symmetry can be considered as a core with typical size $\sim m^{-1}$ (or h/mc in SI units). The radius of a neutron star is 10–15 km; therefore, the size of the axion core and the radius of the star are comparable for $m_a \sim 10^{-11}$ eV. The interaction term in the Lagrangian equation is in simple form

$$\mathcal{L}_{int} = \frac{\beta R^2 \phi}{16\pi}$$

In this case the equation for scalar field $\phi = \phi(r)$ is written as

$$\triangle_{(3)}^r \phi = A^2 m_a^2 \phi - \frac{A^2}{16\pi}\beta R^2 - \frac{d\phi}{dr}\frac{d\eta}{dr}. \tag{22}$$

Therefore, one can consider the possibility of the existence of an axion "core" containing dark matter in the center of the star. The contribution to energy density from such a core is negligible itself, and therefore could not influence the parameters of a star. However, the assumption of coupling $\sim R^2\phi$ can lead to non-trivial deviations from general relativity.

For the case of function $f_R = f_R(R, \phi)$ depending also on scalar field ϕ, Equations (17)–(19) are valid, but one need only take into account that radial derivatives of $f(R, \phi)$ are

$$\frac{df_R}{dr} = f_{RR}\frac{dR}{dr} + f_{R\phi}\frac{d\phi}{dr},$$

$$\frac{d^2 f_R}{dr^2} = f_{RR}\frac{d^2 R}{dr^2} + f_{RRR}\left(\frac{dR}{dr}\right)^2 + f_{R\phi}\frac{d^2\phi}{dr^2} + f_{R\phi\phi}\left(\frac{d\phi}{dr}\right)^2 + 2f_{RR\phi}\frac{dR}{dr}\frac{d\phi}{dr}.$$

The Equations (17)–(19) (and (22) for model with axion field) can be integrated with boundary conditions at spatial infinity and given central density ϵ_c. The surface of star corresponds to $\epsilon = p = 0$. We use also the consequence of Bernoulli theorem according to which for non-rotational star

$$H + \nu = \text{const},$$

where H is so called log-enthalpy $H = \ln\left(\frac{\epsilon+p}{n_b m_b}\right)$. Here n_b is particle density and m_b is mean baryon mass. Therefore, from function $\nu(r)$ we can also define dependence of energy density and pressure from radial coordinate in process of integration. We use the self-consistent-field method for resolution of equations (this method for rotating stars in general relativity is described in detail, for example, in [57]).

4. Discussion of Results

An illustration of the dependencies of the gravitational mass of a star on radius and central density are given on Figure 1 for model (21) in comparison with general relativity and simple R^2-gravity. We use two well-known equations of state from nuclear physics; namely, GM1 (without hyperons) and APR. From Figure 1 one can see that for some value of central density (for given EoS), masses and radii of star configurations are very close to values in general relativity. Below this density, the radii and masses decrease in comparison with general relativity. The opposite situation takes place for larger densities. Deviation from general relativity is maximal for stars with maximal masses for a given EoS.
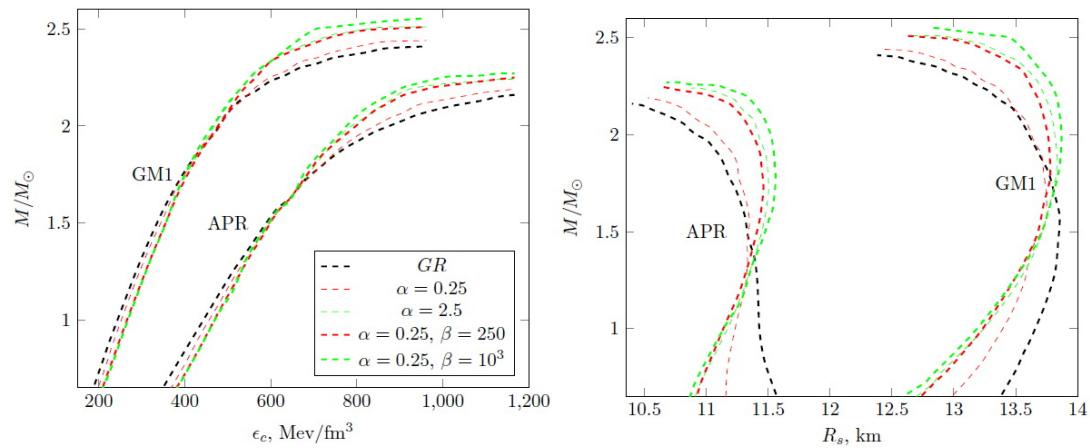
Figure 1. Gravitational mass versus central density (left panel) and radius (right panel) using equations of state (EoS) GM1 and APR for some values of α and β (in units of r_g^2 where r_g is gravitational radius of Sun) in comparison with general relativity ($\alpha = 0$). For mass of axion field, hereinafter, we take value $m_a = 0.1$ in units of r_g^{-1}.

Additionally, it is interesting to consider the dependence of mass confined by sphere with radius r from r (see Figure 2). In general relativity, $m(r) = M_s = \text{const}$ is constant outside the star surface. But in R^2 gravity and its extension with a scalar field, there is a contribution to gravitational mass outside the surface of a star. The gravitational mass confined by the star's surface for two models is always smaller than M_s. For relatively large central densities, the additional contribution to gravitational mass overcomes this smallness, and the gravitational mass for a distant observer increases.
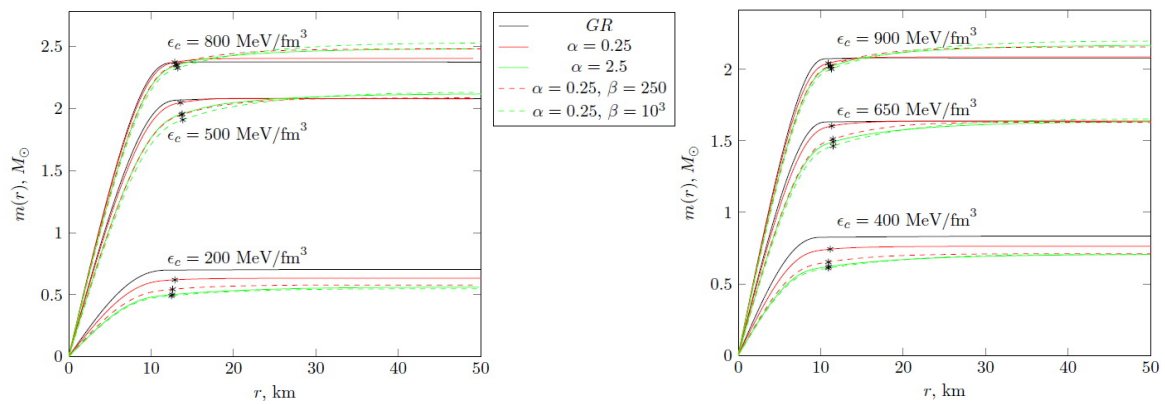
Figure 2. Mass profiles for some central densities using EoS GM1 (left panel) and APR (right panel) for some values of α and β in comparison with general relativity. Asterisks hereinafter designate points for which $\epsilon = p = 0$ (surface of star).

These features can be understood from the behavior of scalar curvature (Figure 3). Outside the star surface, there is an area in which scalar curvature is nonzero, in contrast with general relativity [16]. For large α and for a model with axion field scalar curvature inside, the star slowly decreases the curvature. Some results about masses and radii are given in Table 1. There is some equivalence between pure R^2-gravity and model (21): for example, for $\alpha = 0.25$ and $\beta = 250$ we have similar results for mass as in the simple model for $\alpha = 2.5$. The explanation is simple. From the dependence of the scalar field from radial coordinate (Figure 4) one can estimate the contribution of the $\sim \beta\phi$ term into an effective value of $\tilde{\alpha}$:

$$\tilde{\alpha} = \alpha + \beta\phi.$$

The model with an axion field can be considered as an R^2-model with non-constant parameter $\tilde{\alpha}$. If mean value of $\tilde{\alpha}$ is $\approx \alpha$, results for the mass and radius of the star will be similar. The dependence of scalar field $\phi(r)$ also demonstrates that the contribution of ϕR^2 grows with central density.

Table 1. Parameters of compact stars (masses and radii) in general relativity ($\alpha = \beta = 0$), simple R^2-gravity ($\beta = 0$) and for R^2-gravity with an axion field for some values of central energy density using two EoS. In the last column, the corresponding values of curvature in the center of star are given.

ϵ_c, MeV/fm^3	α, r_g^2	β, r_g^2	$M/M_\odot$	R_s, km	R_c, r_g^{-2}
			GM1 EoS		
	0	0	0.70	13.42	0.048
	0.25	0	0.63	12.95	0.034
200	2.5	0	0.56	12.52	0.0092
	0.25	250	0.58	12.61	0.014
	0.25	1000	0.55	12.46	0.0059
	0	0	2.07	13.55	0.049
	0.25	0	2.07	13.58	0.050
500	2.5	0	2.11	13.79	0.019
	0.25	250	2.08	13.73	0.016
	0.25	1000	2.12	13.85	0.0068
	0	0	2.37	12.76	−0.012
	0.25	0	2.39	12.84	0.026
800	2.5	0	2.48	13.11	0.015
	0.25	250	2.48	13.07	0.0092
	0.25	1000	2.52	13.22	0.0057
			APR EoS		
	0	0	0.83	11.47	0.087
	0.25	0	0.76	11.17	0.058
400	2.5	0	0.72	10.95	0.014
	0.25	250	0.71	10.95	0.018
	0.25	1000	0.71	10.90	0.0073
	0	0	1.63	11.30	0.067
	0.25	0	1.63	11.33	0.066
650	2.5	0	1.63	11.50	0.021
	0.25	250	1.64	11.45	0.018
	0.25	1000	1.65	11.53	0.0076
	0	0	2.07	10.93	−0.031
	0.25	0	2.08	10.94	0.031
900	2.5	0	2.17	11.23	0.016
	0.25	250	2.16	11.18	0.014
	0.25	1000	2.20	11.32	0.0062

One notes also, the weak dependence of mass increasing δm from the value of parameter β. If β increases, the mean value of curvature inside of star decreases, and then the contribution of $\beta\phi^2 R$ grows not so rapidly as β. In principle there is some upper limit on δm in this model (as for in simple R^2-gravity) close to considered. One should note that δm is the same for GM1 and APR EoS for high masses. For example, $\delta m \approx 0.15 M_\odot$ in comparison to general relativity for $\alpha = 0.25$ and $\beta = 10^3$. The main question of course is the possibility of discrimination between these models and general relativity. Unfortunately, we have no well established mass–radius dependence for neutron stars from observations (only masses can be measured with high accuracy). One should mention recent papers [58–60] in which the authors considered limits on mass and radius for pulsar PSR J0030+0451. Some EoS can be excluded due to these data, and for example, APR EoS of course is under question in general relativity. For our model this statement is also valid, because for intermediate mass, the possible

value of the radius differs from the GR value negligibly in comparison with the error of measurements from NICER (∼1 km). For GM1 EoS satisfying these data, the picture is the same: because GR fits these data well, our theory is also valid. The second difficulty is the uncertainty in the details of the equation of state for dense matter. We hope that further progress in astronomical observations and high energy physics can give an answer to this question. Additionally, it is useful to consider EoS-independent relations for neutron star properties [61,62]. We plan to address this issue in future papers.
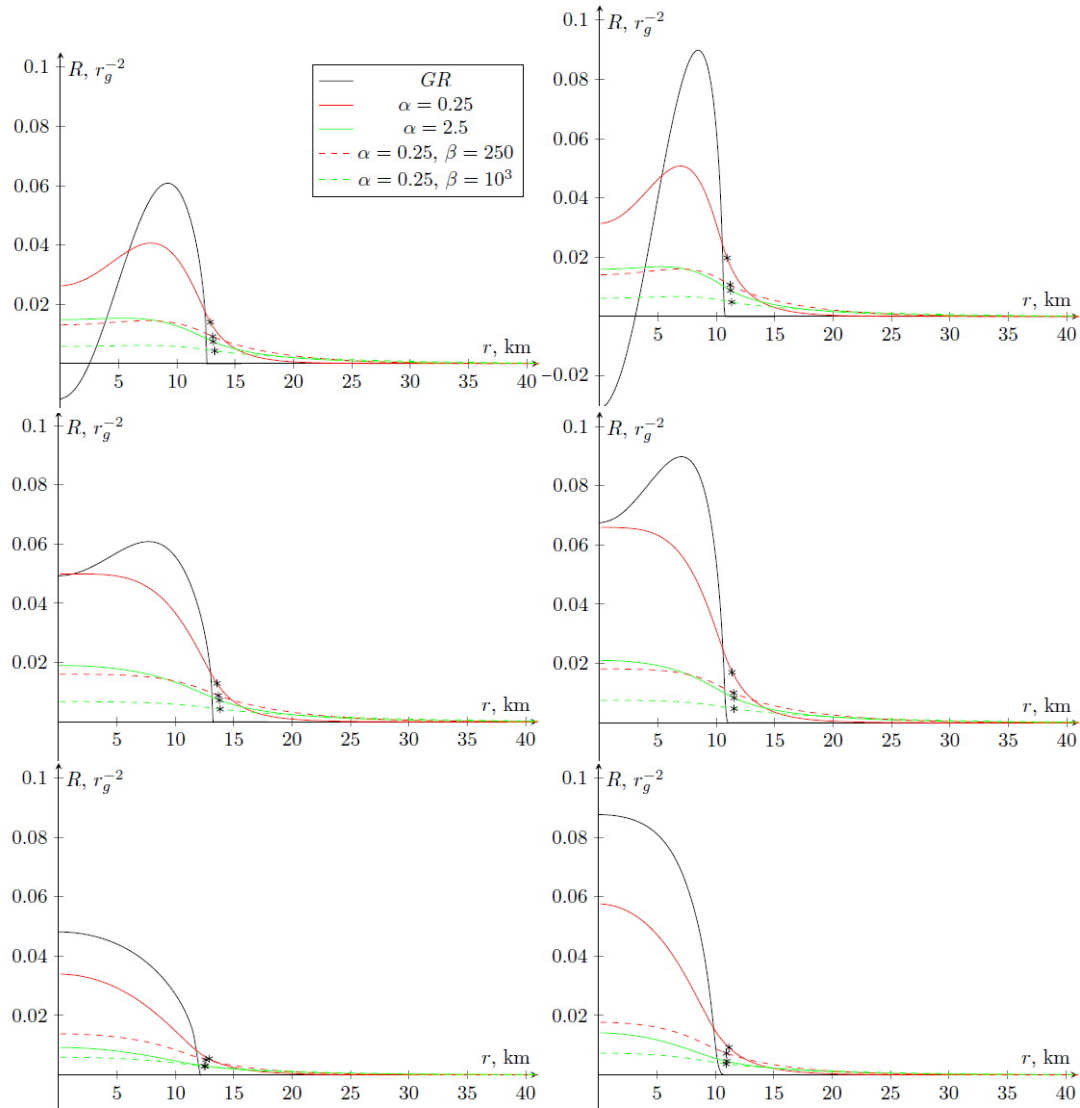
Figure 3. Dependence of scalar curvature (in units of r_g^{-2}) on radial coordinates using EoS GM1 (left panel) and APR (right panel) for some values of central density and parameters α and β. For GM1 EoS, ϵ_c is 800, 500 and 200 MeV/fm^3 (up to down), and for APR, EoS—900, 650 and 400 MeV/fm^3 (up to down).
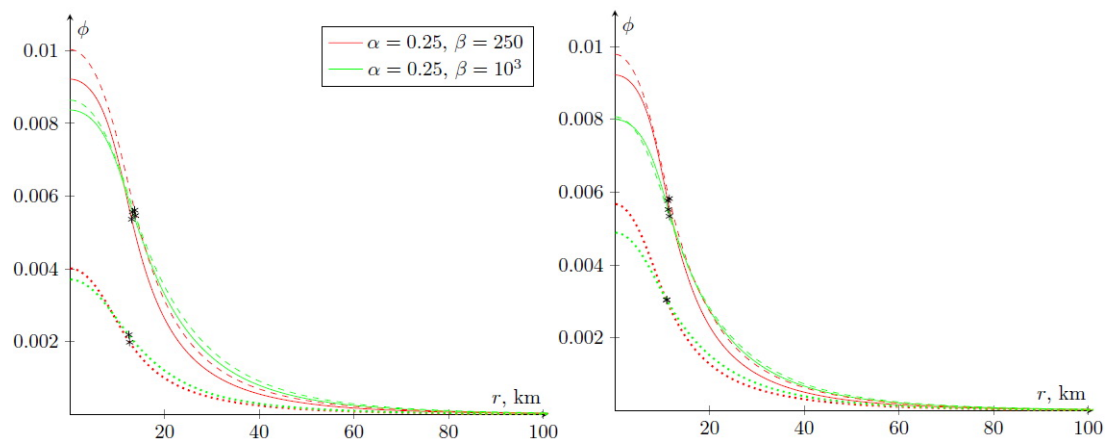
Figure 4. Dependence of scalar field on the radial coordinates using GM1 (left panel) and APR EoS (right panel) for some values of central density and parameter β in comparison with general relativity for $\alpha = 0.25$. Solid, dashed and dotted lines correspond to $\epsilon_c = 800, 500, 200$ MeV/fm^3 (GM1) and $\epsilon_c = 900, 650, 400$ MeV/fm^3 (APR) correspondingly.

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