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Parameters and Pulsation Constant of Cepheid

Sergei V. Sinitsyn

MEPhI, National Research Nuclear University, 115409 Moscow, Russia; s.sinitsyn@mail.ru

Abstract: The analysis of fifty empirical period-radius relations and forty-three empirical period-luminosity relations is performed for the Cepheids. It is found that most of these relations have significant systematic errors. A new metrological method is suggested to exclude these systematic errors using the new empirical metrological relations and the empirical temperature scale of the various samples of the Cepheids. In this regard, the reliable relations between the mass, radius, effective surface temperature, luminosity, absolute magnitude on the one hand, and the pulsation period on the other hand, as well as the reliable dependence of the radius on the mass are determined for the Cepheids of types δ *Cephei* and δ *Scuti* from the Galaxy. These reliable relations permit us to accurately determine the empirical value of the pulsation constant for the Cepheids of both types for the first time. It is found that the pulsation constant very weakly depends on the pulsation period of the Cepheid, contrary to the known theoretical calculation. Hence, the Cepheids pulsate almost as a unified whole and homogeneous spherical body in wide ranges of a star's mass and evolutionary state with an extremely inhomogeneous distribution of stellar substance over its volume. Therefore, it is first suggested that the pulsation of the Cepheid is, first of all, the pulsation of the almost unified whole and homogenous shell of its gravitational mass. This pulsation is triggered by well-known effects; for example, the local optical opacity of the stellar substance and overshooting, using the usual pulsation of the stellar substance.

Keywords: Cepheid variable star; star fundamental parameters; star gravitational mass

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1. Introduction

In [1], discrete and stepwise gravitational effects were found in the evolutionary expansion and nucleosynthesis of the components of a detached double-lined eclipsing system. In particular, it was found that, in this binary star, the absolute and relative evolutionary expansions of the first and second components are their transitions, respectively, between the areas of the temporal deceleration of the absolutely evolutionary expansion and between the areas of temporarily coordinated evolutionary expansion with temporal localization in them. That is, discrete and stepwise gravitational effects were found in the outer part of a star. In addition, a discrete gravitational effect was found in the nucleosynthesis of the first and second components, namely, along the axis of the relation of the reduced luminosities of these components. That is, a discrete gravitational effect was found in the inner part of a star. In this regard, in this binary star, there are some discrete systems that create these stepwise and discrete effects. It was suggested that these systems are the gravitational masses of the first and second components and the general gravitational mass of the binary star.

In this regard, it is of interest to study further the expansion and compression of the gravitational mass of a star using the example of such variable stars as the Cepheids [2–4]. The Cepheids are a type of variable stars that pulsate radially, varying in both diameter and temperature. They change in brightness with a well-defined stable period and amplitude. A strong direct relationship exists between a Cepheid's luminosity and its pulsation period. The Cepheids are important cosmic benchmarks for scaling galactic and extragalactic distances.

Typical representatives of the Cepheids are the classical Cepheids and, first of all, the Cepheid δ *Cephei*. Other Cepheids are also known, that is, dwarf Cepheids. The Cepheid δ *Scuti* is their typical representative. Further, only Galaxy classical Cepheids and Galaxy dwarf Cepheids, that pulsate with fundamental frequency, are analyzed. Hereinafter, the Cepheids of types δ *Cephei* and δ *Scuti* are denoted as the Cepheids δCep and δSct, respectively. In [3] it is empirically found that the Cepheids δCep and δSct have common linear relations between the radius and absolute magnitude on the one hand and the pulsation period on the other hand. Moreover, it is namely those Cepheids that pulsate with fundamental frequency [4]. In the Hertzsprung–Russell diagram, these Cepheids form an instability band [5,6]. In the upper and lower parts of the instability band there are the Cepheids δCep and δSct, respectively [4,6]. Some of them are the high-amplitude Cepheids δSct, that is, HADS-Cepheids [7]. The lower and upper parts of the instability band are separated by an area of Cepheid deficiency [4]. According to Figure 1 from [6], in the Hertzsprung–Russell diagram, the instability band is extended from about 4400 K to 8500 K and from about 4 to 10^5 of the Sun's luminosities, respectively. Therefore, these variable stars are in a wide range of evolutionary states; that is, from a normal dwarf to almost a red giant and a red supergiant. Moreover, these variable stars are in a wide range of masses; that is, about (1–10) solar masses.

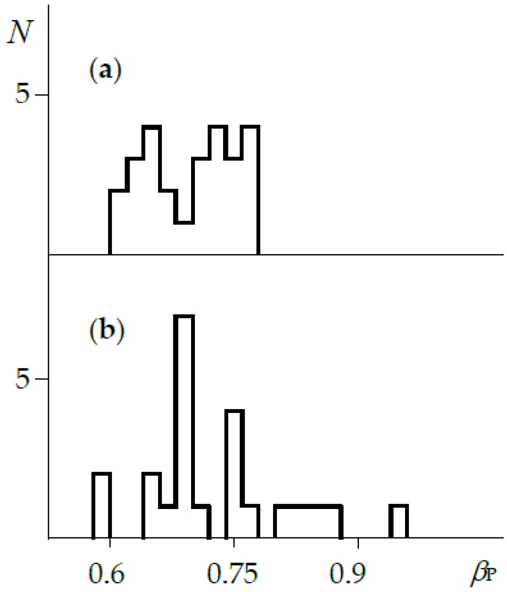

Figure 1. The distribution of *PR* relations for the Cepheids from 1966 to 2009 (**a**) and since 2009 (**b**).

The major parameter of a star's pulsation is the constant pulsation (Q). For a unified whole and homogeneous pulsating spherical body, it is true that $Q = P\rho^{1/2}$ [8], where P and ρ are the pulsation period and the volume mass density of this body, respectively. Further, P is the pulsation period of a star. For a pulsating star, Q and P are determined by day. Any pulsating star is not a unified whole and homogeneous pulsating body; therefore, for pulsating star, Q must be dependent on P. However, the variables of type β *Cephei* [9], the Cepheids of type δ *Scuti* [10], and the variables of type *RR Lyrae* [11] have an empirical $\langle Q \rangle$ = (0.033–0.036) day for the fundamental frequency of the radial star's pulsation. It is astonishing since they are very different pulsating stars with extremely inhomogeneous distributions of stellar substance over their volumes [12]. The first two variables are the main sequence stars. The last variables are the very evolved stars. In addition, the first variables are massive stars and the last two variables are small stars. For the fundamental frequency of the Cepheid's radial pulsation, let us determine the empirical dependence of Q on P; that is, in the range of about (1–10) solar masses and from a normal dwarf to almost a red giant and a red supergiant. Such determination has not been performed till now. However, the theoretical calculations of Q have been performed for the fundamental frequency of the Cepheid's radial pulsation [13,14].

For the determination of the empirical dependence of Q on P the empirical dependences of Cepheid's mass and radius on P must be determined. Therefore, in Section 2, the metrological foundation of the determination of Cepheid's parameters is introduced. The reliability of this foundation is confirmed. In Section 3, the analysis of all empirical period-radius and period-luminosity relations since 1966 are performed for the Cepheids. Significant systematic errors are found in most of these relations. The metrological method of elimination of these systematic errors is suggested. The reliable relations between the radius and the absolute magnitude on the one hand, and the pulsation period on the other hand, are determined for the Cepheids of types δ *Cephei* and δ *Scuti* from the Galaxy. In Section 4, the reliable relations between the mass, effective surface temperature, luminosity on the one hand, and the pulsation period on the other hand, as well as the reliable dependence of the radius on the mass, are determined for the Cepheids of types δ *Cephei* and δ *Scuti* from the Galaxy. In Section 5, the accurate empirical dependence of Q on P is determined. This dependence is compared with theoretical calculations [13,14] and other empirical data at the end.

2. Parameters of Cepheid

Hereinafter, the index of sol indicates that it belongs to the Sun. M, R, L, T_e are the mass, radius, luminosity and effective surface temperature of a star, respectively. Further, only a star's parameters, averaged over its pulsation period, are considered. The symbol $\langle\rangle$ indicates the averaging of such parameters over a sample of stars or the entire volume of a star.

In [1], as the result of the analysis of empirical data from catalogs [15–18], for the components of detached double-lined eclipsing systems on the main sequence it is found that

$$L/L_{sol} = \eta(M/M_{sol})^{\gamma}, \tag{1}$$

where η and γ are some positive constant parameters. In [19], the analysis of empirical data shows that (1) is valid also for the components of Algol-type binaries on the main sequence. Therefore, let us assume that (1) is valid for the Cepheids. Further, it shows that this assumption is true.

As it is known for a star, it is valid that [20]

$$L/L_{sol} = (R/R_{sol})^2(T_e/T_{sol})^4 \tag{2a}$$

$$M_V = M_{b(sol)} - (5/2)\log(L/L_{sol}) - BC_V, \tag{2b}$$

where M_V is the absolute magnitude, M_b is the bolometric magnitude, and BC_V is the bolometric correction.

In [1], as the result of the analysis of empirical data from catalogs [15–18], for the components of detached double-lined eclipsing systems at $0.445 \leq M/M_{sol} < 14.10$ it is found that

$$R/R_{sol} = \kappa(M/M_{sol})^{\nu}, \tag{3}$$

where κ and ν are some positive constant parameters. Therefore, let us assume that (3) is valid for the Cepheids. Further, it shows that this assumption is true.

According to [3,21], $\log(T_e) \geq 3.64$ and $\log(T_e) \leq 3.93$ are valid for the Cepheids δCep and δSct, respectively. Thus, the Cepheids are approximately in the range of $3.64 \leq \log(T_e) \leq 3.93$. Several of the dependences of BC_V on $\log(T_e)$ are known for this temperature range [22–27]. The dependence of BC_V on $\log(T_e)$ is weak when $3.64 \leq \log(T_e) \leq 3.93$. This temperature range ($T_e \approx$ (4400–8500)K) happens to be the critical temperature region at which helium is completely ionized. It is known that T_{sol} = 5772 K [28]; that is, $\log(T_{sol})$ = 3.7613. Therefore, let us assume that in the linear approximation at $3.64 \leq \log(T_e) \leq 3.93$

$$BC_V = BC_{V(\text{sol})} + b_T \log(T_e/T_{sol}), \tag{4}$$

where b_T is some constant coefficient. Further, it shows that this assumption is true.

Note that for the Cepheids (1–4) are the metrological foundation of the determination of their R, M and L. In addition, (1–4) are with respect to the logarithmic axes. Therefore, only $\log(R/R_{sol})$, $\log(M/M_{sol})$, $\log(L/L_{sol})$, $\log(T_e/T_{sol})$, $\log(P)$, $\log(T_e)$ are used in further next relations.

According to [3,29,30], for the Cepheids, an empirical linear relation between P and R (PR relation) and an empirical linear relation between P and M_V (PM_V relation) are valid. The empirical PR relation and PM_V relation are determined as, respectively,

$$\log(R/R_{sol}) = \alpha_P + \beta_P \log(P), \tag{5a}$$

$$M_V = A_P + B_P \log(P) \tag{5b}$$

where α_P, β_P and A_P, B_P are some constant coefficients. From (1), (2a), (3) and(5a) it follows that

$$\log(\eta/\kappa^{\gamma/\nu}) = 4\log(T_e/T_{sol}) + [2 - (\gamma/\nu)](\alpha_P + \beta_P \log(P)) \tag{6}$$

It is seen that, for the Cepheids, $\log(T_e)$ is also the function of $\log(P)$. From (2b), (4), (5b) and (6) it follows that

$$A_P = M_{b(sol)} - BC_{V(sol)} - (1/4)\{(b_T + 10)\,[\alpha_P(\gamma/\nu) + \log(\eta/\kappa^{\gamma/\nu})] - 2\alpha_P b_T\} \tag{7a}$$

$$B_P = -(1/4)\,[(b_T + 10)(\gamma/\nu) - 2b_T]\beta_P \tag{7b}$$

Taking into account (5b), the right parts of (7a) and (7b) are constant. Hence, η, γ, κ, ν, a_T, b_T,α_P, β_P are constant, too. Thus, (1)–(4) are valid and, thereby, the above assumptions are true. This is important for the metrological foundation of the Cepheids.

3. Radius, Absolute Magnitude and Pulsation Period of Cepheid

Let us analyze the known empirical PR relations and PM_V relations. They can be used to find and estimate the systematic errors (δ) of known empirical α_P, β_P, A_P and B_P from (5).

At least fifty empirical PR relations [3,29–61] are known, to date, for the Cepheids from the Galaxy. Figure 1a,b shows the distributions of the PR relations along the axis β_P from 1966 to 2009, and since 2009, respectively. Two wide peaks are visible in the range of (0.606–0.679) and (0.706–0.771) in Figure 1a. Two narrow peaks are visible in the range of (0.680–0.698) and (0.740–0.755) in Figure 1b. That is, the first peak shifts towards higher values over time.

At least forty-three empirical PM_V relations (PM_V) [3,5,29,50,51,55,56,58,60–86] are known, to date, for the Cepheids from the Galaxy. Figure 2 shows the distribution of the PM_V relations along the axis B_P since 1988. Three peaks are visible in the range of (−2.689–−2.671), (−2.789–−2.767), and (−2.950–−2.900). Let us determine the values of β_P and B_P, which are valid, in Figures 1 and 2, respectively.

Figure 3 shows the distribution of the fifty empirical PR relations of (5a). In the first approximation, the PR relations form a linear dependence of α_P on β_P. This indicates that in most of the PR relations, there is one systematic $\delta(\alpha_P)$ and one systematic $\delta(\beta_P)$, which are connected to each other by a linear law in the first approximation and are significantly larger than any random $\delta(\alpha_P)$ and $\delta(\beta_P)$ and other systematic $\delta(\alpha_P)$ and $\delta(\beta_P)$. This circumstance has not received attention, yet. Let us use it and define the relation between α_P and β_P in the linear approximation. It notes that the random $\delta(\alpha_P)$ and $\delta(\beta_P)$ are independent and can be comparable to each other. In addition, the other systematic $\delta(\alpha_P)$ and $\delta(\beta_P)$ can be independent and comparable to each other. Therefore, the least square method (LSM) must be used, simultaneously, along both axis α_P and axis β_P in the linear approximation. That is, the square deviations of empirical data are minimized along both the axis of α_P and the axis of β_P at the same time, using the linear relation between α_P and β_P. Moreover, it excludes eight *PR* relations that differ significantly on β_P and α_P from others. Then, it follows that

$$\alpha_P = 1.893 - 1.080\beta_P \quad (8)$$

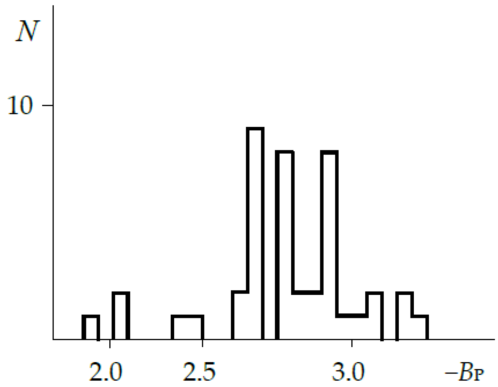

Figure 2. The distribution of the PM_V relations for the Cepheids since 1988.

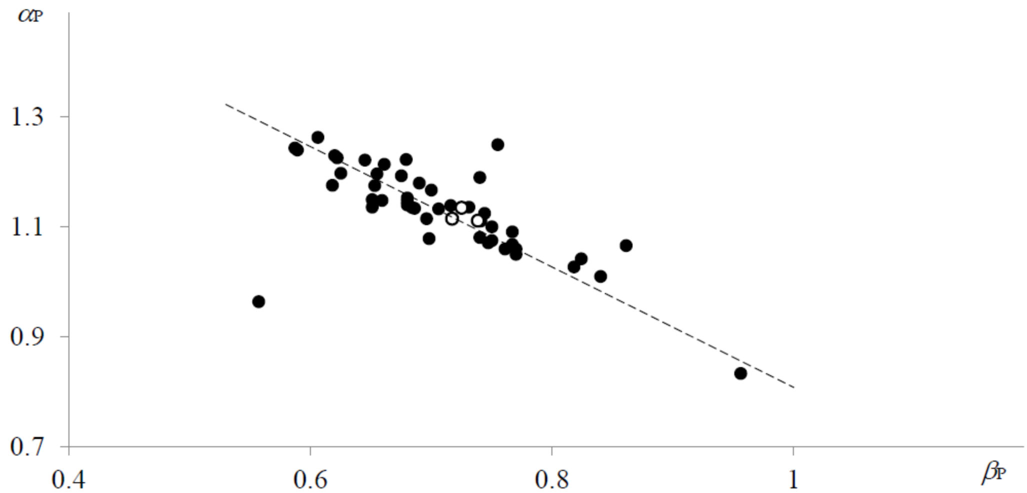

Figure 3. The distribution of 50 empirical *PR* relations according to the empirical data: ○—the Cepheids of type *δCephei*, •—the Cepheids of types *δCephei* and *δScuti*, _ _ _ _—(8).

Figure 3 shows (8). According to Figure 3 and (5a) and (8), most of the fifty *PR* relations intersect with each other near the point of $\log(P) = 1.080$ and $\log(R/R_{sol}) = 1.893$. That is, most of these relations differ from each other, first of all, by β_P. Therefore, there is a significant systematic $\delta(\beta_P)$. The use of (8) allows us to take into account this systematic $\delta(\beta_P)$ and, thereby, the significant systematic $\delta(\alpha_P)$, and also to minimize the random $\delta(\alpha_P)$ and $\delta(\beta_P)$ and the other systematic $\delta(\alpha_P)$ and $\delta(\beta_P)$. Thus, the analysis of the fifty *PR* relations, from 1966 to 2021, finds significant systematic $\delta(\beta_P)$ and $\delta(\alpha_P)$ in most of these relations.

Figure 4 shows the distribution of the forty-three empirical PM_V relations of (5b). In the first approximation, the PM_V relations form a linear dependence of A_P on B_P. This indicates that in most of the PM_V relations there is one systematic $\delta(A_P)$ and one systematic $\delta(B_P)$, which are connected to each other by a linear law in the first approximation and are significantly larger than any random $\delta(A_P)$ and $\delta(B_P)$ and other systematic $\delta(A_P)$ and $\delta(B_P)$. This circumstance has not received attention, yet. Let us use it and define the relation between A_P and B_P in the linear approximation. This notes that the random $\delta(A_P)$ and $\delta(B_P)$

are independent and can be comparable to each other. In addition, the other systematic $\delta(A_P)$ and $\delta(B_P)$ can be independent and comparable to each other. Therefore, LSM must be used, simultaneously, along both the axis A_P and the axis B_P in the linear approximation. That is, the square deviations of empirical data are minimized along both the axis of A_P and along the axis of B_P at the same time, using the linear relation between A_P and B_P. Moreover, it excludes twelve PM_V relations that differ significantly on B_P and A_P from others. Then, it follows that

$$A_P = -4.999 - 1.308B_P \tag{9}$$

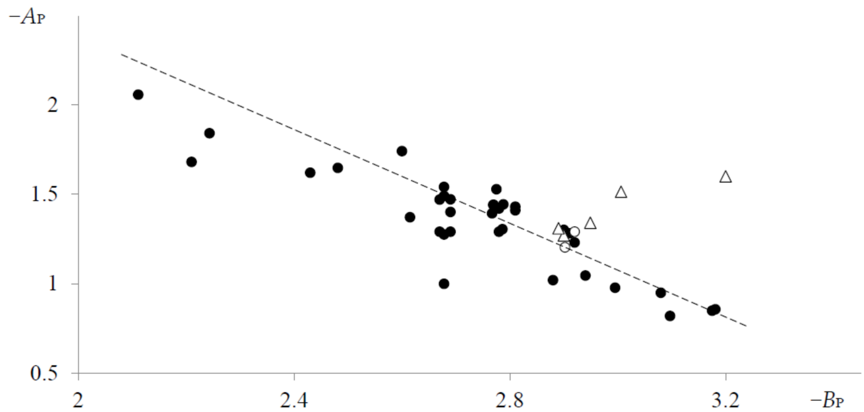

Figure 4. The distribution of 43 empirical PM_V relations according to the empirical data: △—the Cepheids of type *δCephei*, •—the Cepheids of types *δCephei* and *δScuti*, ○—the Cepheids of type *δScuti*, _ _ _ _—(9).

Figure 4 shows (9). According to Figure 4 and (5b) and (9), most of the forty-three PM_V relations intersect with each other near the point of $\log(P) = 1.308$ and $M_V = -4.999$. That is, most of these relations differ from each other, first of all, by B_P. Therefore, there is significant systematic $\delta(B_P)$. The use of (9) allows us to take into account this systematic $\delta(B_P)$ and, thereby, the significant systematic $\delta(A_P)$, and also to minimize the random $\delta(A_P)$ and $\delta(B_P)$ and the other systematic $\delta(A_P)$ and $\delta(B_P)$. Thus, the analysis of the forty-three empirical PM_V relations, from 1988 to 2021, finds significant systematic $\delta(A_P)$ and $\delta(B_P)$ in most of these relations.

The significant systematic $\delta(\alpha_P)$, $\delta(\beta_P)$ and $\delta(A_P)$, $\delta(B_P)$ must be excluded. Combining (2), (4), (5), (8) and (9), this obtains the metrological relation between β_P and B_P, taking into account systematic $\delta(\alpha_P)$,$\delta(\beta_P)$ and $\delta(A_P)$, $\delta(B_P)$

$$5(1.080 - \log(P))\beta_P = -(1.308 - \log(P))B_P + (10 + b_T)\log(T_e/T_{sol}) + 4.464 - M_{b(sol)} + BC_{V(sol)} \tag{10}$$

The use of (10) allows us to find reliable β_P and B_P and, thereby, to exclude significant systematic $\delta(\beta_P)$ and $\delta(B_P)$. Further, the use of (8) and (9) allows us to find reliable α_P and $A_P$and, thereby, to exclude significant systematic $\delta(\alpha_P)$ and $\delta(A_P)$. Therefore, only β_P and B_P are analyzed further.The metrological method, using (8)–(10), has not been used until now.

Note that $\log(T_{sol})$, $M_{b(sol)}$, $BC_{V(sol)}$ and $\log(P)$, $\log(T_e)$ must be known to determine β_P and B_P using (10). It is known that $\log(T_{sol}) = 3.7613$ (Section 2). According to [87,88], $M_{b(sol)} = (4.7554 \pm 0.0004)$ and $BC_{V(sol)} = -(0.107 \pm 0.002)$. The analysis of the temperature dependence of BC_V on $\log(T_e)$ [22–27] shows that $\langle b_T \rangle$ is equal to (2.2–2.8) in the range of $3.64 \leq \log(T_e) \leq 3.93$ (Section 2). For the determination of β_P and B_P, using a sample of the Cepheids is required; that is, $\langle \log(T_e) \rangle$ and $\langle \log(P) \rangle$ instead of $\log(T_e)$ and $\log(P)$. The use of $\langle \log(T_e) \rangle$ minimizes any random $\delta(\log(T_e))$. Therefore, the accuracy of the determination of β_P and B_P increases. Moreover, this sample of the Cepheids must have $\langle \log(P) \rangle$ as far from 1.080 and 1.308; that is, the values of β_P and B_P must be sensitive to each other in (10).

At least five relatively large samples [61,89–92] are known for the Cepheids from the Galaxy. Two and three samples of these are the sample of the Cepheids *δCep* [61,92] and the Cepheids δSct [89–91], respectively. The samples of the Cepheids δCep have a$\langle\log(P)\rangle$ close to 1.108. Therefore, each of these samples is divided into two subsamples. The first and second subsamples for $\log(P) < 0.93$ and $\log(P) > 0.97$ are valid, respectively. The samples and subsamples for $\langle\log(T_e)\rangle$ and $\langle\log(P)\rangle$ are shown in Table 1. These samples have the same temperature scale in the first approximation. That is, using LSM along the axis $\log(T_e)$; for these, it is true that

$$\log(T_e) = (3.812 \pm 0.002) - 0.064\log(P) \tag{11}$$

Table 1. The samples of the Cepheids and their parameters.

Data	N [a]	$\langle\log P\rangle$	$\langle\log T_e\rangle$	№ [b]
[89]	39	−1.0653	3.8790	*1*
[90]	24	−1.0382	3.8776	*2*
[91]	26	−0.8485	3.8697	*3*
	38	0.9468	3.7523	
[92]	21	0.7554	3.7656	*4a*
	17	1.1831	3.7358	*4b*
	53	0.9331	3.7515	
[61]	33	0.7613	3.7635	*5a*
	20	1.2165	3.7316	*5b*

[a] The number of the Cepheids in the sample; [b] The ordinal number of the dependence in Figure 5.

Here, the deviation of the first coefficient is equal to three standard deviations of the sample mean.

The samples and subsamples deviate from (11) by not more than 0.52% or 30K. Note that the samples were formed from different Cepheids for which their T_e were determined from 1972 to 2021 and by different scientists.

In contrast to the Cepheids δCep, for the Cepheids δSct, thePM_V relations are mainly in the narrow range along the axis B_P, namely, from to −3.00 to −2.89 [73,77,83,85]. Moreover, for the Cepheids δCep and δSct, all general PM_V relations [3,29] are also in this range. Therefore, let us determine the β_P and B_P for the Cepheids δSct first. According to (10), the result of the calculation of $BC_{V(\mathrm{sol})}$ depends on b_T.

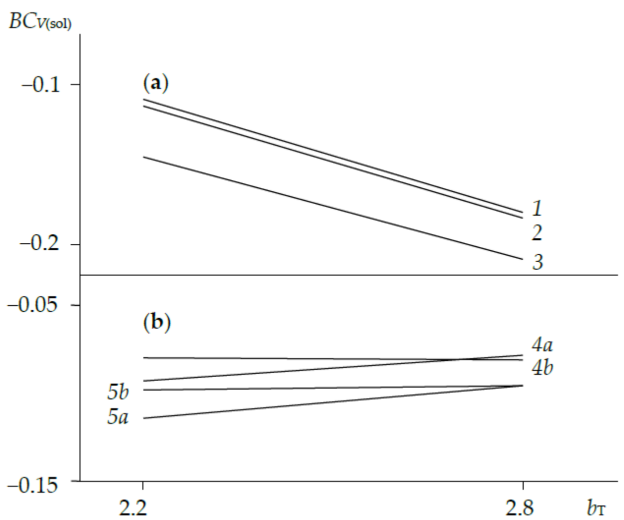

Figure 5. The dependencies of the calculation result of $BC_{V(\mathrm{sol})}$ on b_T, according to (10) and the empirical date from Table 1: for the Cepheids of type $\delta Scuti$ at $B_P = -2.89$, $\beta_P = 0.736$ (**a**); for the Cepheids of type $\delta Cephei$ at $B_P = -2.776$, $\beta_P = 0.7315$ (**b**). The ordinal numbers of these dependencies are according to Table 1.

In Figure 5a, as an example, at $B_P = -2.89$, the dependences of this result on b_T are shown for the samples [89–91] when $BC_{V(sol)} = -0.109$ at $b_T = 2.2$ for the sample [89]. This condition corresponds to $\beta_P = 0.736$. The other two dependences of the samples [90,91] are shifted to negative values. These shifts are due to the fact that the temperature scales of samples [90,91] are shifted from the temperature scale of sample [89] by 6K and 51K towards higher values, respectively. Using (10) and the data of Table 1, the set of β_P is calculated for each sample [89–91] at $-3.00 \le B_P \le -2.89$, $2.2 \le b_T \le 2.8$ and $-0.109 \le BC_{V(sol)} \le -0.105$. The analysis of these sets shows that $-3.00 \le B_P \le -2.89$ corresponds to $0.736 \le \beta_P \le 0.771$. Note that, for the Cepheids *δCep* and *δSct*, the general *PR* relations [3,30] are also in this range along the axis β_P in Figure 1b. Hence, using (5), (8) and (9), for the Cepheids *δSct*, it is true that

$$\log(R/R_{sol}) = (1.079 \pm 0.020) + (0.754 \pm 0.018)\log(P) \tag{12a}$$

$$M_V = -(1.15 \pm 0.08) - (2.945 \pm 0.055)\log(P) \tag{12b}$$

Hereinafter, in any linear relation or dependence the deviations of the first and second coefficients anticorrelate in sign. These coefficients and their deviations determine the upper and lower boundaries of the area in which the reliable relation or the reliable dependence exist. In turn, the coefficient deviations are determined using empirical-metrological (8), (9) and the areas in which the reliable values of β_P and B_P exist. The last areas are determined using empirical-metrological (10) and the indetermination of b_T, and the errors of the temperature scales of samples from Table 1. Thus, in any linear relation or dependence, the coefficient deviations are determined using empirical-metrological (8)–(10) and the indetermination of b_T, the errors of the temperature scales of samples from Table 1.

As an example, for the Cepheids *δCep,* the results of the calculation of $BC_{V(sol)}$ on b_T using (10) are shown for the samples [61,92] in Figure 5b when $BC_{V(sol)} = -0.105$ at $b_T = 2.8$ for the first and second subsamples of the sample [92]. This condition corresponds to $\beta_P = 0.7315$. For the sample [61], these results are shifted to positive values. This shift is due to the fact that the temperature scale of sample [61] is shifted from the temperature scale of sample [92] by 23K towards smaller values. Note that B_P is not fixed here, because two subsamples are used for each sample. Therefore, B_P and β_P are determined simultaneously using the intersection of two dependences of the calculation results of $BC_{V(sol)}$ on b_T for the first and second subsamples. Using (10) and the data of Table 1, the set of β_P is calculated for each sample [61,92] at $2.2 \le b_T \le 2.8$ and $-0.109 \le BC_{V(sol)} \le -0.105$. The analysis of these sets shows that for the sample [61], $-2.885 \le B_P \le -2.815$ and $0.734 \le \beta_P \le 0.757$ are valid. In addition, the sample [92] $-2.783 \le B_P \le -2.717$ and $0.713 \le \beta_P \le 0.735$ are valid. Hence, by using (5, 8, 9) and generalizing the calculation results, for the Cepheids *δCep*, it is true that

$$\log(R/R_{sol}) = (1.099 \pm 0.024) + (0.735 \pm 0.022)\log(P) \tag{13a}$$

$$M_V = -(1.34 \pm 0.12) - (2.80 \pm 0.08)\log(P) \tag{13b}$$

Note that in (12) and (13), for each coefficient, its δ is determined by using the temperature scale and the range of b_T. For example, for the Cepheids *δCep* at $b_T = 2.2$, B_P and β_P are equal to (−2.83–−2.72) and (0.713–0.738), respectively. Along with that, at $b_T = 2.8$, B_P and β_P are equal to (−2.88–−2.77) and (0.732–0.757), respectively. However, for the Cepheids *δSct,* the dependence of β_P on b_T is relatively weak. For example, β_P is equal to (0.736–0.765) and (0.743–0.771) at ab_T equal to 2.2 and 2.8, respectively, and $-3.00 \le B_P \le -2.89$.

According to the above, in Figure 2, for the Cepheids *δSct,* the peak in the range of (−2.950–−2.900) along the axis B_P corresponds to the peaks in the range of (0.706–0.770) and (0.740–0.755) along the axis β_P in Figure 1b and 1a, respectively. For the Cepheids *δCep,* the peak in the range of (−2.789–−2.767) along the axis B_P corresponds to the same peaks along the axis β_P in Figure 1. Thus, in Figure 1b and 1a along the axis β_P, the peaks in the range of (0.706–0.770) and (0.740–0.755) are valid for the Cepheids *δSct* and *δCep*. In

Figure 2, along the axis B_P, the peaks in the range of (-2.950–-2.900) and (-2.789–-2.767) are valid for the Cepheids δSct and δCep, respectively.

In this regard, the Cepheids δCep $b_T = 2.2$ and the temperature scale of sample [92] are more probable than $b_T = 2.8$ and the temperature scale of sample [61], respectively. Therefore, for the Cepheids δCep, it is more probable that $\log(R/R_{sol}) = (1.109 \pm 0.014) + (0.725 \pm 0.012)\log(P)$ and

$$M_V = -(1.370 \pm 0.075) - (2.775 \pm 0.058)\log(P). \tag{13c}$$

Note that the Cepheids δSct and δCep are in extremely different evolutionary states and have significantly different masses. The first variables are a normal dwarf and the second variables are almost a red giant and a red supergiant. However, from the comparison of (12) and (13), it follows that for the Cepheids δSct and δCep, their PR relations are close to each other. Their PM_V relations are close to each other, too. Hence, some general PR and PM_V relations can be suggested for the Cepheid δSct and δCep in the first approximation. Such relations can be PR and PM_V relations [3], namely,

$$\log(R/R_{sol}) = (1.1116 \pm 0.0060) + (0.7385 \pm 0.0060)\log(P) \tag{14a}$$

$$M_V = -(1.203 \pm 0.041) - (2.902 \pm 0.030)\log(P) \tag{14b}$$

Note that (14a) is very close to (8) and (14b) corresponds to (9) in the limits of its δ.

Thus, the analysis of all known empirical PR relations and PM_V relations allows us to find and estimate the significant systematic $\delta(\beta_P)$, $\delta(\alpha_P)$, $\delta(A_P)$, $\delta(B_P)$, and to eliminate them, and also decrease random $\delta(\alpha_P)$, $\delta(\beta_P)$, $\delta(A_P)$, $\delta(B_P)$. In its turn, this allows us to find reliable empirical PR relations and PM_V relations.

4. Radius, Mass, Luminance, Temperature and Pulsation Period of Cepheids

Taking into account the region of the existence of the Cepheids δSct and δCep along the axes B_P and β_P (Section 3) and using (7b), it follows that, for them, γ/ν is equal to (1.650 ± 0.010) and (1.619 ± 0.011), respectively. According to [1], the analysis of the empirical data [15–18] shows that $\gamma = 4$ at $0.445 \leq M/M_{sol} < 14.10$ for the components of detached double-lined eclipsing systems on the main sequence. In addition, in [19], the analysis of the empirical data shows that γ is equal to (3.92 ± 0.05) and (3.86 ± 0.05) in this mass range for the first and second components of the detached Algol binaries, respectively, on the main sequence. The Cepheids δSct are the main sequence stars at $M/M_{sol} \approx$ (1.5–2.0) [3]. Therefore, for these Cepheids, $\gamma = 4$ is valid. Then, taking into account (7), (12) and (13), for the Cepheids δCep, $\gamma = 4$ is valid, too. This condition can be confirmed in another way. For the Cepheids δCep and δSct, $-1.408 \leq \log(P) \leq 1.8378$ is valid.The lower and upper limits are from [91] and [68], respectively. Therefore, as follows from (12a), (13a) and(14a), for the Cepheids, $\log(R/R_{sol})$ increases by (2.3–2.5) when $\log(P)$ increases from -1.408 to 1.8378; that is, when M/M_{sol} increases by about an order of magnitude. Hence, taking into account (3), the Cepheids δCep and δSct have $\nu \approx$ (2.3–2.5). Further, taking into account γ/ν, the Cepheids have $\gamma \approx$ (3.72–4.16); that is, about 4. Thus, for the Cepheids δCep and δSct, ν is equal (2.470 ± 0.015) and (2.424 ± 0.015) at $\gamma = 4$, respectively. According to [1], the main sequence stars have $\nu = 3$ at $0.445 \leq M/M_{sol} < 14.10$.

For the Cepheids δSct, let us calculate κ using (6) and (12a) and the empirical data of the samples [89–91] (Table 1). In this regard, let us assume that the Cepheids δSct have $\eta = 5.31$. Further, it shows that this assumption is true. Then, for the Cepheids δSct at $\eta = 5.31$ and $\gamma = 4$, it follows that

$$\log(R/R_{sol}) = (0.091 \pm 0.012) + (2.424 \pm 0.015)\log(M/M_{sol}) \tag{15}$$

Further, taking into account the above calculations, from (1), (12a) and (15) at η = 5.31 and γ = 4 it follows that

$$\log(M/M_{sol}) = (0.407 \pm 0.013) + (0.311 \pm 0.008)\log(P) \tag{16a}$$

$$\log(L/L_{sol}) = (2.354 \pm 0.050) + (1.244 \pm 0.031)\log(P) \tag{16b}$$

On the other hand, without η and γ from (11) and (12a) it follows that

$$\log(L/L_{sol}) = (2.360 \pm 0.046) + (1.251 \pm 0.036)\log(P) \tag{17}$$

The coefficients of (16b) and (17) are the same in the limits of their δ. Therefore, η = 5.31 and γ = 4 are valid for the Cepheids *δSct*. Hence, the above assumptions are true.

Taking into account the above calculations, from (2a), (12a) and (16b) it follows that

$$\log(T_e) = (3.810 \pm 0.004) - (0.066 \pm 0.002)\log(P) \tag{18}$$

It is seen that the coefficients of (11) and (18) are the same in the limits of their δ.

For the Cepheids *δCep*, let us calculate κ using (6) and(13a) and the empirical data of the samples [61,92] (Table 1). In this regard, let us assume that the Cepheids *δCep* have η = 5.31. Further, it shows that this assumption is true. Then, for the Cepheids *δCep* at η = 5.31 and γ = 4, it follows that

$$\log(R/R_{sol}) = (0.050 \pm 0.014) + (2.470 \pm 0.016)\log(M/M_{sol}) \tag{19}$$

Further, taking into account the above calculations, from(1), (13a) and (19) at η = 5.31 and γ = 4 it follows that

$$\log(M/M_{sol}) = (0.424 \pm 0.013) + (0.298 \pm 0.011)\log(P) \tag{20a}$$

$$\log(L/L_{sol}) = (2.423 \pm 0.050) + (1.190 \pm 0.043)\log(P) \tag{20b}$$

On the other hand, without η and γ from (11) and (13a) it follows that

$$\log(L/L_{sol}) = (2.400 \pm 0.054) + (1.213 \pm 0.044)\log(P) \tag{21}$$

The coefficients of (20b) and (21) are the same in the limits of their δ. Therefore, η = 5.31 and γ = 4 are valid also for the Cepheids *δCep*. Hence, the above assumptions are true.

Taking into account the above calculations, from (2a), (13a) and (20b) it follows that

$$\log(T_e) = (3.817 \pm 0.002) - (0.070 \pm 0.001)\log(P) \tag{22}$$

It is seen that the coefficients of (11) and (22) are very close to each other. Moreover, (15)–(18) and (19)–(22) are close to each other, too. Hence, the Cepheids *δSct* and *δCep* are almost unified pulsators. Note, that the first and second variable stars are in extremely different evolutionary states and have significantly different masses. The first variables are a normal dwarf and the second variables are almost a red giant and a red supergiant. Therefore, at the same time, (15)–(18) and (19)–(22) are valid in the different ranges of log(P). According to [51,68,91], the Cepheids *δSct* and *δCep* are in the ranges of $-1.408 \leq \log(P) \leq -0.541$ and $0.2889 \leq \log(P) \leq 1.8378$, respectively. Then, from (16a) and (20a), it follows that $0.88 \leq M/M_{sol} \leq 1.80$ and $3.16 \leq M/M_{sol} \leq 9.53$ for the Cepheids *δSct* and *δCep*, respectively.

5. Pulsation Constant of Cepheid

Let us assume that $Q = P\rho^{1/2}$ (Section 1) is valid for the Cepheid's pulsation. Then, it follows that

$$Q = P(M/M_{sol})^{1/2}/(R/R_{sol})^{3/2} \tag{23}$$

Hence, taking into account (12a, 16a) and (13a, 20a) and also the above calculations (Section 4), it follows that for the Cepheids δSct and δCep, respectively,

$$\log(Q) = -(1.414 \pm 0.025) + (0.025 \pm 0.023)\log(P) \quad (24a)$$

$$\log(Q) = -(1.436 \pm 0.030) + (0.046 \pm 0.028)\log(P), \quad (24b)$$

where Q is determined by day. In (24), the deviations of the first and second coefficients correlate in sign. From (24a,b) it is seen that, for the Cepheids δSct and δCep, Q is very weakly dependent on P, especially for the first of them. Moreover, (24a,b) are very close to each other. However, the first and second variable stars are in extremely different evolutionary states and have significantly different masses. Hence, for these variables and the fundamental frequency, Q depends very weakly on M and the volume distribution of their substance. According to (3), (5a) and (23), it follows that

$$\log(Q) = (1/2\nu - 3/2)\alpha_P - (1/2\nu)\log(\kappa) + [1 - (3/2 - 1/2\nu)\beta_P]\log(P)$$

Hence, $d(\log(Q))/d(\log(P)) = 0$ if $\beta_P = 2/(3 - 1/\nu)$. Then, taking into account (15) and (19), for the Cepheids δSct and δCep, $d(\log(Q))/d(\log(P)) = 0$ ifβ_P is equal to (0.7730 $\pm$ 0.0008) and (0.7707 $\pm$ 0.0008), respectively. That is, according to (12a) and (13a), along theaxis β_P, the upper boundaries of the regions of the existence of the Cepheids δSct and δCep turn out to be very close to the condition of $d(\log(Q))/d(\log(P)) = 0$, especially for the first of them.

In [13,14,93–96], there are the physical foundations and theoretical models of a star pulsation as a stellar substance pulsation. As it follows from (24b), the Cepheids δCep have Q = (0.038 $\pm$ 0.002) day at P = 1.95 days. At the same time, according to the theoretical calculation [13] using the formula [93], Q = 0.0364 day at the same value of P. It is seen that the first value of Q coincides with the result of the theoretical calculation. However, in (24b), $d(\log(Q))/d(\log(P))$ is about (2–3) times less than according to the theoretical calculations [13], which is important. According to these calculations, $d(\log(Q))/d(\log(P))$ = (0.110–0.156) for the Cepheids δCep. In addition, according to another theoretical calculation [14] and taking into account (18) and (22), for the Cepheids δSct, $d(\log(Q))/d(\log(P)) \approx 0.001$, but for the Cepheids δCep, $d(\log(Q))/d(\log(P)) \approx 0.15$, already. Thus, taking into account the results of the theoretical calculations and (24b), at least the pulsation of the Cepheid δCep is not determined by the pulsation of its substance.

In addition, as it follows from (24a), the Cepheids δSct have $\langle Q \rangle$ = (0.0366 $\pm$ 0.0039) day at $\langle P \rangle$ = 0.110 day [97]. They are the main sequence stars at $M/M_{sol} \approx$ (1.5–2.0) [3]. According to [10], for the Cepheids δSct, Q = (0.033 $\pm$ 0.006) day as it follows from the empirical data. Hence, (24a) is true. Therefore, (15)–(18) and, thereby, (19)–(22) are true, too. Along with this, on the main sequence, the variables of type β *Cephei* have $\langle Q \rangle$ = 0.033 day for the fundamental frequency but at $8 \leq M/M_{sol} \leq 20$ and $\langle M/M_{sol} \rangle = 12$ [9]. Hence, for pulsating main sequence stars, the fundamental frequency Q depends also very weakly on M and the volume distribution of their substance.

In addition, according to (24), for the Cepheids δSct, Q increases by (0–10)% when $\log(P)$ increases from -1.408 to -0.541. For the Cepheids δCep, Q increases by (7–30)% when $\log(P)$ increases from 0.2889 to 1.8378. As it follows from (23) and (24), for the Cepheids, $\langle \rho \rangle \propto 1/P^2$ is valid. Therefore, for the Cepheids δSct and δCep, Q increases by no more than 10% and 30% when $\langle \rho \rangle$ changes by two and three orders of magnitude, respectively. Moreover, for the Cepheids, Q increases by no more than 50% when $\log(P)$ increases from -1.408 to 1.8378. Here, $\langle \rho \rangle$ changes even by six and a half orders of magnitude. This confirms that the Cepheid δSct and δCep really pulsate almost like a unified whole and a homogeneous spherical body, especially for the first of them. At the same time, the distribution of a substance in a star is extremely inhomogeneous [12].

The above indicates that the pulsation of the Cepheids δCep or δSct is determined by the pulsation of some their almost unified whole and homogeneous elements but not the pulsation of their substance. This element is common to the entire volume of the Cepheid and does not depend on the distribution of the substance in this star. The shell of the star's gravitational mass should be suggested as such an element.

Then, the pulsation of the Cepheid is determined by the pulsation of the shell of its gravitational mass. The pulsation of the almost unified whole and homogeneous shell of the star's gravitational mass is triggered by the usual pulsation of the star's substance. In turn, the usual pulsation of the star's substance is triggered by the well-known effect of its local optical opacity [94].This effect is created by metal atoms [95].That is, the metallicity of the stellar substance determines the position of the pulsation band with respect to the axes $\log(T_e)$ and $\log(L)$; for example, for *RR Lyrae* [6,96] or slowly pulsating B-type stars and the variables of type *β Cephei* [6,94]. In addition, there may be many other factors affecting the formation path and evolution state of variable stars; for example, overshooting [98]. Convective overshoot is not only related to the formation and evolution of pulsating variablestars but also associated with many important celestial bodies and extreme physical processes, such as massive pulsating variable stars [99], white dwarfs [100], X-ray binaries [101], and high-magnetic pulsars [102,103].

6. Conclusions

For the Cepheids *δCep* and *δSct* from the Galaxy, the dependence of the radius on the mass and the relations between the mass, radius, effective surface temperature, luminosity, absolute stellar magnitude on the one hand, and the pulsation period on the other hand, are determined. In this regard, it is found that each of these Cepheids pulsates almost like a unified whole and homogeneous spherical body. However, each of these Cepheids has an extremely inhomogeneous distribution of its substance over its volume. This contradiction is valid for wide ranges of a star's mass and a star's evolutionary state. Therefore, it is suggested that the pulsation of any Cepheid is, first of all, the pulsation of the almost unified whole and homogenous shell of its gravitational mass. This pulsation is triggered by well-known effects; for example, the local optical opacity of a star's substance and overshooting, using the usual pulsation of a star's substance. Thus, the pulsation of a star is, in general, a more complex physical process than was assumed until now.

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