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Mass and Magnetic Moment of the Electron and the Stability of QED—A Critical Review

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Abstract: The anomalous magnetic moment of the electron, first calculated by Schwinger, lowers the ground state energy of the electron in a weak magnetic field. It is a function of the field and changes signs for large fields, ensuring the stability of the ground state. This has been shown in the past 50 years in numerous papers. The corresponding corrections to the mass of the electron have also been investigated in strong fields using semiclassical methods. We critically review these developments and point out that the calculation for low-lying excited states raises questions. Also, we calculate the contribution from the tadpole diagram, the relevance of which was observed only quite recently.

Keywords: QED; electron mass; magnetic moment

1. Introduction

Quantum electrodynamics (QED) is the fundamental theory for describing the interaction of matter and light. On the classical level, these are the Maxwell theory and the relativistic mechanics, joined by a covariant coupling. QED is the quantum version of these. It was formulated quite early in the history of quantum mechanics, beginning with the paper of Paul Dirac. One of the first calculations of QED effects was the effective Lagrangian of Werner Heisenberg and Hans Euler [1], and the first new effect predicted using QED was the scattering of light on light. About ten years later, in 1948, Hendrick Casimir found [2] a vacuum interaction between neutral conducting plates caused by the quantized electromagnetic field confined between the plates. These two effects can be interpreted as loop correction, or radiative correction, to external influences. In the first case, such influence is provided by a classical electromagnetic field, and in the second case, it is provided by a conductor boundary condition on the plates (or by the freely movable charges within the plates), which also constitute a classical object.

It must be mentioned that there is a different interpretation for these effects saying that the vacuum of QED is filled with a fluctuating electromagnetic field, the interaction of which with the mentioned influences causes the effects. However, in a more formal approach, there is no need to speak about fluctuating fields. Namely, the mentioned effects can be equivalently described as vacuum-to-vacuum transition amplitude in an external field or as the vacuum expectation value of the energy–momentum tensor in the presence of external influences. For details, we refer to one of many books on this topic [3].

Beyond the two mentioned effects go the radiative, or loop, corrections like the anomalous magnetic moment of the electron and the Lamb shift. The former one can be viewed as a correction to the mass of the electron and to its magnetic moment, μ . The magnetic moment can be expressed as $\mu = g\mu_B$ in terms of the Bohr magneton, μ_B , and the gyromagnetic ratio, g . From the Dirac equation, $g = 2$ follows, whereas the radiative corrections cause a deviation, $a_e = (g - 2)/2$, called the anomaly of the magnetic moment. In 1948, Julian Schwinger found [4] $a_e = \alpha/(2\pi)$ in the lowest order in the fine structure constant, α . This anomaly lowers the energy of the ground state of an electron in a magnetic



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field. Also, it results in a split of the excited energy levels. This split was measured in the so-called $g - 2$ experiments with high precision and showed an excellent agreement with the theory (including higher powers of α).

In a weak magnetic field, the lowering of the ground state energy is proportional to the magnetic field. However, for an increasing field (as well as for higher excited states), it becomes a function of the field and it changes sign before the total energy can reach zero, ensuring the stability of the ground state. It must be mentioned that in theories with higher spin ($s > 1/2$), there is no stability. An example is quantum chromodynamics (QCD), where the so-called Savvidy vacuum is unstable, as shown in Ref. [5]; for a recent review, see [6].

In line with these examples, the Casimir effect is not only a phenomenon arising from vacuum fluctuations, but it may also bear instabilities under certain boundary conditions. In Appendix A, we refer to the Robin boundary conditions as an example. Like in a strong magnetic field, for certain values of the parameters, the energy of the lowest state may be below zero and constitute the instability of the system. It must be mentioned that in the literature on the Casimir effect, a situation with instabilities is rarely considered, although it well deserves more attention.

The present paper is a critical review on the question of the stability of QED in a magnetic field. The point is the following. While for the ground state the stability can be shown using a quite simple calculation and for high excited states and/or strong magnetic field using asymptotic methods, for lower excited states and medium fields, one is left with numerical investigation. Such analysis was performed in Ref. [7].

However, as we point out here, there are questions about the methods used in Ref. [7]. In addition, recently, in Ref. [8], it was observed that a class of diagrams (one-particle irreducible ones) do not vanish in distinction from earlier belief and may give an additional contribution.

For this reason, in what follows, we are interested only in the one-loop correction to the mass of the electron in a homogeneous magnetic field. Therefore, we do not consider the motion of the electron in the direction of the field and use the simplified notation.

It worth noting that the motion of an electron in a magnetic field is a field of vital interest, not so much in connection with the stability of QED as in connection with the synchrotron radiation which appears on the tree level as well as from the imaginary part of the radiative correction to the electron mass, see [9] and the book [10], for example. An essential tool is the semiclassical approximation, i.e., mass correction for high excited levels, see [11–13].

The paper is organized as follows. In Section 2, we reproduce and discuss the formulas known in the literature with a focus on the proper time representation. In Sections 2.1 and 2.2, we discuss the problems which appear. In Section 3, we consider the one-particle irreducible (1PI) contribution. Section 4 gives conclusions of the study.

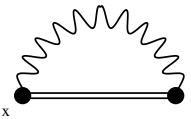
Throughout the paper, we use notations with $\hbar = c = 1$ for the reduced Planck's constant, \hbar , and the speed of light, c .

2. The Mass and the Magnetic Moment of the Electron in a Magnetic Field

We consider the effective Dirac equation, i.e., the Dirac equation with loop corrections,

$$(i\mathcal{D} - m)\psi(x) + \int dx' \Sigma(x, x')\psi(x') = 0, \quad (1)$$

in one loop approximation. Here, x represents a four-dimensional coordinate, ψ is the wave function, m denotes the mass of a particle. The covariant derivative is $\mathcal{D} = \gamma^\mu D_\mu$, with γ_μ the Dirac matrices, $D_\mu = \partial_\mu - ieA_\mu$, where e is the elementary charge, A_μ denotes the electromagnetic potential, the Greek indices take the values 0 (for the time coordinate), 1, 2, and 3 (space), and the self-energy operator, $\Sigma(x, x')$, reads



$$\Sigma(x, x') = \text{diagram} = -ie^2 D_{\mu\nu}(x, x') \gamma^\mu S(x, x') \gamma^\nu, \quad (2)$$

where the wavy line represents the photon propagator, $D_{\mu\nu}(x, x')$, the doubled line represents dressed electron propagator and the dots represent interaction vertices. The spinor propagator $S(x, x')$, which is 'exact' in the field A_μ , obeys

$$(i\mathcal{D} - m)S(x, x') = \delta(x - x'), \quad (3)$$

where $\delta(x - a)$ is the Dirac delta function.

From Equation (1), for the electron, a mass,

$$M = m + \Delta M, \quad (4)$$

follows with a mass correction ΔM . In the given order of approximation, ΔM is the expectation value of the self-energy operator in the unperturbed states.

From here on, we consider a homogeneous magnetic field, H . Then, the mass correction is the average

$$\begin{aligned} \Delta M(N, \zeta) &= \langle N, \zeta | \Sigma(x, x') | N, \zeta \rangle, \\ &= \int d^3x \int d^3x' \bar{\psi}_{M, \zeta}(\vec{x}) \Sigma(x, x') \psi_{M, \zeta}(\vec{x}) \end{aligned} \quad (5)$$

and it depends on the state of the electron. The eigenfunctions, taken in coordinate representation, read $\langle x | N, \sigma \rangle = e^{-iE_N t} \psi_{N, \zeta}(\vec{x})$, and they obey the Dirac equation

$$(i\mathcal{D} - m) | N, \zeta \rangle = 0. \quad (6)$$

The $\psi_{N, \zeta}(\vec{x})$ are the known eigenspinors, and the one-particle energy is

$$E_N = \sqrt{m^2 + p_3^2 + 2eHN}, \quad N = 0, 1, \dots, \quad (7)$$

where p_3 is the momentum third space-component.

The states are numbered by N , and the spin projection is $\zeta = \pm 1$. The number N consists of two parts,

$$N = n + \frac{1}{2}(1 - \zeta) \quad (8)$$

where $n = 0, 1, \dots$ numbers the Landau levels in the magnetic field. The ground state has $n = 0$ and $\zeta = 1$ (spin projection parallel to the field). The excited states are doublets, which are degenerated on the tree level and split by the radiative correction.

As soon as $\zeta^2 = 1$, the mass correction (5) can be written in the form

$$\Delta M(N, \zeta) = \Delta M_0(N) + \zeta \Delta M_\zeta(N), \quad N = 1, 2, \dots, \quad (9)$$

for the excited states. For the ground state, one has only

$$\Delta M(N, \zeta) = \Delta M_0(0). \quad (10)$$

A split like in the excited states is formally possible, but physically meaningless. Making a non-relativistic approximation in (7), one comes to

$$E_N = m + \frac{p_3^2}{2m} + \mu_B H \left(2n + 1 - g \frac{\zeta}{2} \right) + \dots \quad (11)$$

For the free Dirac equation, $g = 2$ follows, and from the loop correction, one has

$$g = 2(1 + a_e), \quad (12)$$

where a_e is the anomaly factor and $\mu_B a_e$ is the so-called anomalous magnetic moment. Comparing Equation (11) with Equation (7), one identifies

$$\Delta M(N, \zeta) = \mu_B H a_e. \quad (13)$$

This way, the anomaly factor becomes a function of both the field and the state.

For the calculation of the mass correction, there are two known methods. The first starts from the representation of the electron propagator in terms of the eigenspinors $\psi_{N,\zeta}(x)$ and the one-particle energies E_N , which is an eigenfunction representation. This way was used in Refs. [14,15]. The representation of the mass correction is in terms of an integral over the photon loop momentum (in polar coordinates $|k|$ and $\cos \theta$) and a sum over the intermediate states of the electron. It is convenient for strong fields since in that case only the lowest intermediate state contributes.

The second method uses the proper time representation of the propagators. In Ref. [16], the spinor propagator in the magnetic field was represented this way by performing the sum over the intermediate states. In Ref. [17], this representation was used to obtain an integral representation of the self-energy operator, $\Sigma(x, x')$.

In Ref. [18], a similar result was obtained using the algebraic procedure of Schwinger [19]. This method allows us to obtain the result through bypassing the summation of the eigenfunctions. The method was refined in a subsequent paper [20], and we follow the representation given there.

We use the following notations instead of the ones used in Ref. [20]. We change the integration variable $y \rightarrow x$. For the energy E , we insert E_N (7), and set $p_3 = 0$. In addition, we set $m = 1$ (except in the factor in front) and also $e = 1$ so that one has: $E/m \rightarrow E_N$, $(E^2 - m^2)/m^2 \rightarrow 2NH$ and $(E^2 - m^2)/(eH) \rightarrow 2N$. Finally, the prime is dropped: $\zeta' \rightarrow \zeta$. With this notation, Equation (32) from Ref. [20] reads

$$\Delta M(N, \zeta) = \frac{\alpha m}{2\pi} \int_0^\infty \frac{dx}{x} \int_0^1 du e^{-iux/H} (F - F_{\text{sub}}), \quad (14)$$

where

$$F_{\text{sub}} = 1 + u \quad (15)$$

is the ultraviolet subtraction and

$$\begin{aligned} F &= e^{-i(\beta - (1-u)x)2N} W, \\ W &= \frac{1}{\sqrt{\Delta}} \{ \cos(\beta - x) - i\zeta E_N \sin(\beta - x) + u[\cos(\beta + x) - i\zeta E_N \sin(\beta + x)] \\ &\quad + (1-u)2NH W_0 \}, \\ W_0 &= \frac{1-u}{\Delta} \cos(\beta - x) + \frac{u}{\Delta} \frac{\sin(x)}{x} \cos(\beta) - \cos(\beta + x). \end{aligned} \quad (16)$$

Equation (25) from Ref. [20] defines β as

$$\tan(\beta) = \frac{(1-u) \sin(x)}{(1-u) \cos(x) + u \sin(x)/x} \quad (17)$$

and Equation (29) of Ref. [20] defines

$$\Delta = (1-u)^2 + 2u(1-u) \frac{\sin(x) \cos(x)}{x} + u^2 \left(\frac{\sin(x)}{x} \right)^2, \quad (18)$$

which appears in the denominator.

Actually, these notations are not convenient enough. For that reason, we rewrite the notations. We start with the definition

$$h = 1 - u + u \frac{\sin(x)}{x} e^{-ix}. \quad (19)$$

Multiplying h by its complex conjugate, one can see that Δ factorizes,

$$\Delta = h h^*. \quad (20)$$

Further, let us consider

$$1 + i \tan(\beta) = \frac{h e^{ix}}{(1 - u) \cos(x) + u \sin(x)/x}, \quad (21)$$

which allows one to write

$$e^{2i(\beta-x)} = \frac{h}{h^*}. \quad (22)$$

Next, let us consider W_0 in Equation (16). We write the trigonometric functions as sum/difference of the corresponding exponentials, and with Equation (22) one arrives at

$$W_0 = \frac{1 - h e^{2ix}}{2\sqrt{h h^*}} + \text{c.c.}, \quad (23)$$

where ‘c.c.’ denotes the complex conjugate. Using these formulas, we rewrite F in Equation (16) in the form

$$\begin{aligned} F &= \left(\frac{h^*}{h}\right)^N e^{-2iuxN} W, \\ W &= \frac{1}{2h} (1 + \zeta E_N) (1 + u e^{-2ix}) + \frac{1}{2h^*} (1 - \zeta E_N) (1 + u e^{2ix}) \\ &\quad + (1 - u) \left[\frac{1 - h e^{2ix}}{2h h^*} + \text{c.c.} \right] 2NH. \end{aligned} \quad (24)$$

In this representation, the first two terms are interrelated by the complex conjugation and spin reversal: $\zeta \rightarrow -\zeta$. The third term is of a real value.

The expression (24) can be simplified by keeping in the third term only the contributions from the ‘1’ in the numerator,

$$\begin{aligned} W &= \frac{1}{2h} \left[(1 + u e^{-2ix}) (1 + \zeta E_N) - (1 - u) e^{-2ix} 2NH \right] \\ &\quad + \frac{1}{2h^*} \left[(1 + u e^{2ix}) (1 - \zeta E_N) - (1 - u) e^{2ix} 2NH \right] \\ &\quad + \frac{1 - u}{h h^*} 2NH. \end{aligned} \quad (25)$$

Finally, one arrives at

$$\begin{aligned} W &= \frac{e^{-ix}}{h} \left[\cos(x) (1 + \zeta E_N) - \frac{1 - u}{2} e^{-ix} E_N (E_N + \zeta) \right] \\ &\quad + \frac{e^{ix}}{h^*} \left[\cos(x) (1 - \zeta E_N) - \frac{1 - u}{2} e^{ix} E_N (E_N - \zeta) \right] \\ &\quad + \frac{1 - u}{h h^*} 2NH. \end{aligned} \quad (26)$$

The expression (26) coincides with that in Refs. [17] with $\sigma = -\zeta$, up to an overall factor E_N , of which Ref. [17] has more in the denominator (probably a typo).

In Ref. [7], the expression for ΔM was taken over from Ref. [17] (using different notations). Also using the operator method, in Ref. [21], a representation—Equation (3.18) in Ref. [21]—was derived which coincides with the above one. This representation was also taken over in Ref. [22], Equation (1), where, however, new notations were introduced, $s(x) = 1 - \sin(x)/x$, for instance. Regrettably, the last term in the second line in Equation (1) [22] has a misprint and coincides with Equation (1) if one substitutes $-us(x) \rightarrow 1 + u\sin(x)/x$.

In Ref. [7], a further set of notations was introduced. In Equation (8), the energy correction is split into real and imaginary parts. This is the expression which in Ref. [7] was used for the numerical evaluation. It can be obtained from Equation (26) with the application of Equation (7), which here takes the form $E_N^2 = 1 + 2NH$. Equation (26) then reads:

$$W = \frac{1}{h h^*} (A + 2NH B + i\zeta E_N C) \quad (27)$$

with

$$\begin{aligned} A &= (1-u)(1+u\cos(2x)) + u(1+u)\frac{\sin(2x)}{2x}, \\ B &= (1-u)\left(1 - (1-u)\cos(2x) - u\frac{\sin(2x)}{2x}\right), \\ C &= 2u(1-u)\sin(x)\left(-\cos(x) + \frac{\sin(x)}{2x}\right). \end{aligned} \quad (28)$$

Actually, Equation (28) is Equations (25) or (26), taken on a common denominator.

In order to investigate the structure of the expression F in Equation (14), it is necessary to look for the singularities in the complex plane. Denoting $x = \zeta + i\eta$, let us first look for the zeros of the expression h (19),

$$h = \frac{1}{2ix} \left(2i(\zeta + i\eta)(1-u) + u - ue^{2\eta}(\cos(2\zeta) - i\sin(2\zeta)) \right). \quad (29)$$

For $h = 0$, one obtains two equations:

$$e^{2\eta} \cos(2\zeta) = 1 - 2\frac{1-u}{u}\eta, \quad (30)$$

$$e^{2\eta} \sin(2\zeta) = -2\frac{1-u}{u}\zeta. \quad (31)$$

Equation (31) can be resolved for η , and from the quotient of Equations (30) and (31), one obtains:

$$\eta = \frac{1}{2} \ln \left(\frac{1-u}{u} \frac{-2\zeta}{\sin(2\zeta)} \right), \quad (32)$$

$$\tan(2\zeta) = \frac{-2(1-u)\zeta}{u - 2(1-u)\eta}. \quad (33)$$

Inserting η (32) into Equation (33), it can be seen that there are solutions for

$$\pi(k + 1/2) < \zeta < \pi(k + 1), \quad k = 0, 1, \dots \quad (34)$$

For solutions (34), from Equation (32), follows that $\eta > 0$.

This way, the zeros of h are in the upper half-plane. Straightforwardly, h^* has its zeros in the lower half-plane. For $u = 1$, these zeros reach the real axis in $\xi = \pi k$. For $u \rightarrow 0$, the zeros go to infinity, $x = \pi\left(k - \frac{1}{2}\right)\left(1 + \frac{1}{\ln(1/u)}\right)$, $\eta \sim \frac{1}{2} \ln \frac{1}{u}$. The zeros are shown in Figure 1.

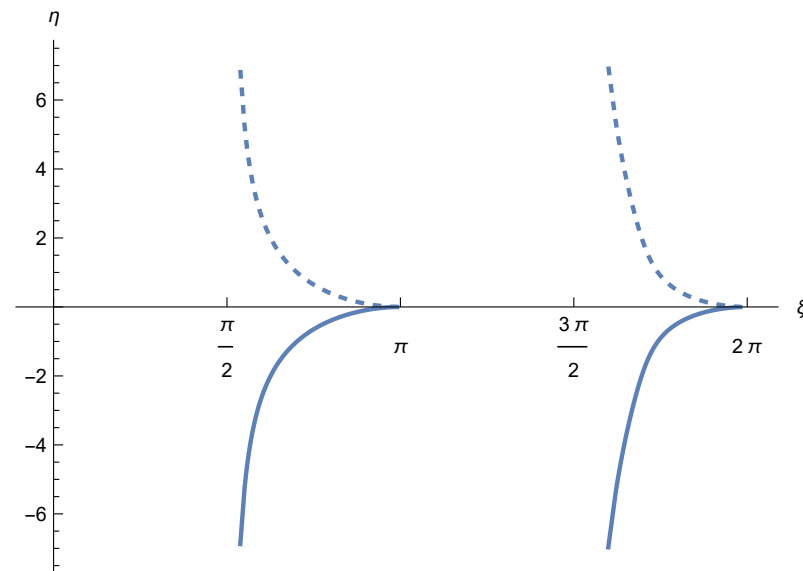


Figure 1. In the complex plane, $x = \xi + i\eta$, the locations of the zeros of h (19) (upper half-plane, dashed) and of h^* (lower half-plane). In these curves, the parameter u takes values from $u = 1$ at $\eta = 0$ to $u \rightarrow 0$ at $\eta \rightarrow \pm\infty$.

For the ground state, $N = 0$, $\zeta = 1$, the expressions simplify, and using Equations (24) and (25), one comes to

$$F_{\text{ground state}} = \frac{1}{h} \left(1 + ue^{-2ix}\right). \quad (35)$$

Since h has zeros only in the upper half-plane, according to the exponential factor in Equation (14), one may rotate the integration path towards the negative imaginary axis, $x \rightarrow -iy$. The result is a well-converging expression which allows for an immediate numerical evaluation. We repeat here in Figure 2 the representation given in Ref. [23].

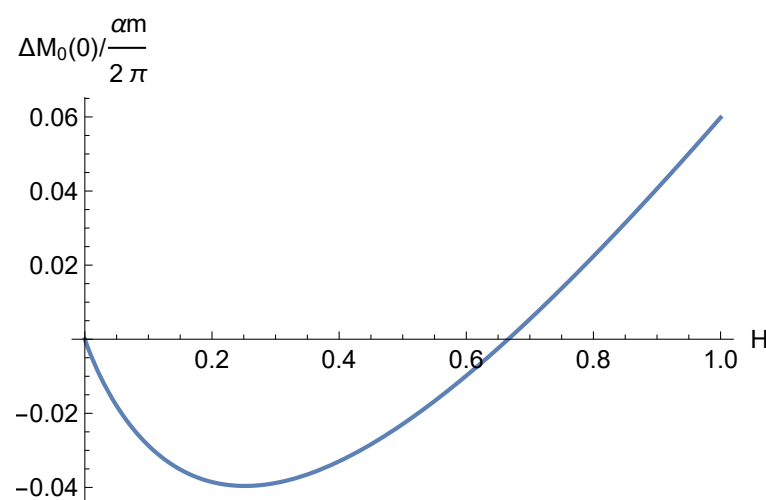


Figure 2. The mass correction $\Delta M_0(0)$, see Equation (10), for the ground state, divided by $\alpha m / (2\pi)$, as a function of the magnetic field. In the minimum, this function takes small values and must be additionally multiplied by α to compare with the rest mass.

For the excited states, the numerical evaluation is considerably more complicated. So far, only one attempt has been undertaken [7], which we consider in Section 2.1 just below.

2.1. On the Numerical Evaluation in Ref. [7] for the Excited States

In Section III of Ref. [7], it is mentioned that a numerical integration over the real x -axis is complicated because of the singularities in $x = \pi n$. A rotation of the integration path downwards in the complex plane, $x \rightarrow -iy$, would cross the singularities (see Figure 1). In Ref. [7], the authors use the following method. The u -integration is divided into two parts at some u_0 . For $0 \leq u \leq u_0$, there are no problems on the real axis and a numerical integration is possible. For $u_0 \leq u \leq 1$, they turn the integration path towards the imaginary axis, see Equation (11) in Ref. [7]. However, this is not possible. As one can see in Figure 1, there are zeros in the denominator for all values of u . From this, we conclude that the numerical results in Ref. [7] are questionable.

2.2. On the Strong Field Limit

The limit of the strong magnetic field, $H \rightarrow \infty$, was calculated for the ground state in Refs. [18,24]. Recently, using an expression similar to Equation (35) from Ref. [23], the limit was recalculated, including the constant term,

$$\Delta M_0(0) = \frac{\alpha m}{2\pi} \frac{5}{2} \left(\ln(2H) - \gamma_E - \frac{3}{2} \right)^2 + a + \mathcal{O}\left(\frac{1}{H}\right), \quad (36)$$

$$a \simeq 4.028717,$$

where γ_E is the Euler's constant.

For the low excited states ($N = 1, 2, \dots$), there is only one calculation of the strong field limit [24]. It is in terms of the eigenfunction method and it is only for the spin-dependent part. It delivered

$$\Delta M_\zeta(N) \simeq \frac{\alpha m}{2\pi} \frac{\ln(2H)}{N}. \quad (37)$$

This way, for a strong field, the mass correction is also positive.

2.3. The Mass Correction for Low-Lying Excited States

An attempt to calculate this mass correction from the formulas which were reproduced in Section 2.2 above hits the following problem. When turning the integration path in (14) into the complex plane, $x \rightarrow -iy$, besides the contributions from the poles which one crosses, the following integral shows up:

$$\Delta M(N, \zeta) = \frac{\alpha m}{2\pi} \int_0^\infty \frac{dy}{y} \int_0^1 du e^{-uy/H} (F - F_{\text{sub}}) + \text{pole terms}. \quad (38)$$

However, for $y \rightarrow \infty$, one observes an asymptotic behavior, using Equations (19) and (30) with $x \rightarrow -iy$,

$$F = \left(\frac{u e^{2(1-u)y}}{u + 2(1-u)y} + \mathcal{O}(1) \right)^N \left[\frac{y}{u} (1 - E_N \zeta) + \frac{y}{u + 2(1-u)y} (1 + E_N \zeta) + \mathcal{O}(e^{-2y}) \right], \quad (39)$$

where

$$h = 1 - u + \frac{u}{2y} + \mathcal{O}(e^{-2y}), \quad h^* = \frac{u}{2y} e^{2y} + \mathcal{O}(1) \quad (40)$$

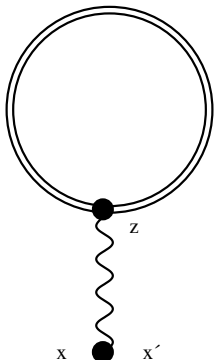
was used. In addition, there is the factor $e^{-uy/H}$ in the integrand. This way, one observes an exponential growth in the integrand for $y \rightarrow \infty$ for $H < \frac{uy}{2(1-u)N}$, i.e., for all finite H . Let us mention that this divergence is absent in the ground state, i.e., for $N = 0$. Also, this

divergence cannot be related to the infrared divergence of the mass operator $\Sigma(x, x')$ in Equation (2), since ΔM in Equation (5) is taken in states. Now, one could speculate on a cancellation from the pole contributions. But this is quite unlikely.

One may wonder how under such circumstances the weak-field expansion derived in Refs. [18,25,26] may be possible. The point is that for real x an expansion in the powers of u and x brings the power N down to factors N , and the resulting, oscillating integrals well may give finite answers. And they do, as follows from the mentioned papers. However, it is clear that the low-field expansion must be an asymptotic one.

3. The Contribution from the Tadpole Diagram

As observed quite recently in Ref. [8], 1PI diagrams can contribute to the effective Lagrangian in a homogeneous background field. The feature is that a tadpole diagram does not vanish. It contributes to the electron self-energy,



$$\Sigma^{1PI}(x, x) = \text{diagram} = ie\gamma_\nu D^{\mu\nu}(x, z) j_\mu(z) \delta(x - x'). \quad (41)$$

The tadpole diagram can be derived from the one-loop Heisenberg–Euler (HE) Lagrangian (see, e.g., [27]) that

$$L_{\text{HE}}^{1\text{-loop}} = \frac{1}{8\pi^2} \int \frac{dT}{T^3} e^{-mT^2} \left[\frac{\sqrt{2\mathcal{F}} T}{\tanh(\sqrt{2\mathcal{F}} T)} - \frac{2}{3} T^2 \mathcal{F} - 1 \right], \quad (42)$$

with $\mathcal{F} = \frac{1}{4}(F_{\mu\nu}F^{\mu\nu})$. To mention is the possibility of deriving it from the worldline methods as performed, for example, in Ref. [28] for the spinor propagator. Such a diagram contributes a current (in Ref. [8] called a *photon current*),

$$j_\nu(k) = ieB\epsilon_{\mu\nu 3} k^\mu \delta^{(4)}(k) \frac{\partial L_{\text{HE}}^{1\text{-loop}}}{\partial \mathcal{F}}, \quad (43)$$

where, for a constant purely magnetic field, B , directed along the z -axis, $F_{\mu\nu} = -\epsilon_{\mu\nu 3} B$, $\mathcal{F} = B^2/2$, k^μ is the 4-momentum, $\epsilon_{\mu\nu 3}$ is the antisymmetric Levi-Civita symbol, $\delta^{(4)}(k)$ is the four-dimensional Dirac delta, with [8]

$$\frac{\partial L_{\text{HE}}^{1\text{-loop}}}{\partial \mathcal{F}} = \frac{e^2}{8\pi} \left[4\zeta'(-1, \chi) - \chi (\zeta'(0, \chi) - \ln \chi + \chi) - \frac{1}{3} \ln \chi - \frac{1}{2} \right], \quad \chi = \frac{m^2}{2eB}, \quad (44)$$

following. In Equation (43), the indices μ and ν take only values 1 and 2. In Equation (44), $\zeta(s, \chi)$ stands for the Hurwitz zeta function.

As it stands, the expression (44) is zero due to the factor k in front of the delta function and the general theory of distributions, and from symmetry reasons as well. However, a more detailed investigation [8] made a regularization of the delta function by considering a non-constant background field. The reason for the vanishing of Equation (43) is that a constant background field cannot support momentum transfer and therefore the momentum k must be zero. However, in a non-constant field, this argument cannot be applied and, in the limit of removing the regularization, one comes to an undefined expression of the

type zero times infinity, to be considered in more details. The calculation realization rests on the ‘local constant field approximation’ [29], which allows us to use the expression of the one-loop effective Lagrangian for constant fields. This way, Ref. [8] calculated the 1PI contribution to the two-loop HE Lagrangian.

A similar effect appears also for the mass correction of the electron. From the Feynman rules and from Equation (41), a mass correction,

$$\Delta M^{1\text{PI}}(N, \zeta) = \langle N, \zeta | \Sigma^{1\text{PI}}(x, x') | N, \zeta \rangle, \quad (45)$$

follows, which adds to $\Delta M(N, \zeta)$ in Equation (5).

In the sense of some regularization, we take for the delta function in Equation (43) the expression

$$\delta_\epsilon^{(4)}(k) = \frac{e^{-k^2/4\epsilon}}{(4\pi\epsilon)^2}, \quad (46)$$

which is most convenient in the given case. Then, the mass correction $\Delta M^{1\text{PI}}(n, \zeta)$ in Equation (45) can be calculated using the states $|n, \zeta\rangle$; these states are known, see, e.g., [30]. In order to be close to the notations of Ref. [30], we change the notation from N from Section 2 to n , and note

$$|n, s, \zeta\rangle = \frac{e^{ip_3 z}}{\sqrt{L}} \frac{e^{il\varphi}}{\sqrt{2\pi}} \frac{\sqrt{\gamma}}{\sqrt{K_n(K_n + m)}} (\xi_+ \delta_{\zeta,1} + \xi_- \delta_{\zeta,-1}). \quad (47)$$

where the following notations, also close to those in Ref. [30], are used: $\gamma = B/2$, $K_n = \sqrt{m^2 + p_3^2 + 4\gamma n}$ and

$$\xi_+ = \begin{pmatrix} (K_n + m) I_{n-1,s}(\rho) e^{-i\varphi/2} \\ 0 \\ p_3 I_{n-1,s}(\rho) e^{-i\varphi/2} \\ i\sqrt{4\gamma n} I_{n,s}(\rho) e^{i\varphi/2} \end{pmatrix}, \quad \xi_- = \begin{pmatrix} 0 \\ i(K_n + m) I_{n,s}(\rho) e^{i\varphi/2} \\ \sqrt{4\gamma n} I_{n-1,s}(\rho) e^{-i\varphi/2} \\ -p_3 I_{n,s}(\rho) e^{i\varphi/2} \end{pmatrix}, \quad (48)$$

where $\rho = Br^2/2$ is the radial variable, ξ_\pm are the spinor factors for spin projection $\zeta = \pm 1$ and $n = 0, 1, \dots$ enumerates the energy levels (7). These levels are degenerated with respect to the orbital quantum number l , which is related by

$$l = n + s \quad (49)$$

to the principal quantum number $s = 0, 1, \dots$ and takes values $-\infty < l < n$. The radial wave functions $I_{n,s}(\rho)$ are given by

$$I_{n,s}(\rho) = (-1)^l \sqrt{\frac{n!}{s!}} \rho^{-l/2} e^{-\rho/2} L_n^{-l}(\rho) \quad (50)$$

in terms of Laguerre polynomials. Below, we set the electron mass $m = 1$.

Using Equations (47)–(50), the calculation of the matrix elements becomes just a computing task. Since the dependence on the momentum p_3 in the direction of the magnetic field can be restored by a Lorentz transform, we restrict ourselves to $p_3 = 0$. The mass correction reads

$$\langle n, s, \zeta | \Sigma^{1\text{PI}}(x, x') | n', s', \zeta' \rangle = \int d^4x \bar{\psi}_\zeta^{n,s}(x) i e \gamma_\nu \psi_{\zeta'}^{n',s'}(x) \int d^4z j_\mu(z) D^{\mu\nu}(z, x). \quad (51)$$

Using momentum representation for the current j_μ and the Euclidean propagator, one obtains:

$$\int d^4z j_\mu(z) D_{\mu\nu}(z, x) = \frac{1}{(2\pi)^4} \int d^4k \frac{e^{-ikx}}{k^2} j_\nu(k) \quad (52)$$

$$= \frac{2eB\varepsilon_{\nu\mu 3}}{(4\pi\epsilon)^2} \int_0^\infty dt \frac{\partial}{\partial x_\mu} \int d^4k e^{-\frac{k^2}{4t} - ikx - tk^2} = \frac{ieB}{32} \frac{\partial L_{HE}^{1-loop}}{\partial \mathcal{F}} \varepsilon_{\nu\mu 3} x_\mu. \quad (53)$$

We represent the vector x in Equation (53) as $x = r \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix}$. It is to be multiplied by the gamma matrices. Then,

$$x_2 \gamma_1 - x_1 \gamma_2 = \frac{r}{2i} \left((\gamma_1 - i\gamma_2) e^{i\varphi} - (\gamma_1 + i\gamma_2) e^{-i\varphi} \right). \quad (54)$$

With Equation (48), the spinor diagonal matrix elements turn into

$$\xi_\pm^\dagger (x_2 \gamma_1 - x_1 \gamma_2) \xi_\pm = -2i\sqrt{n}(K_n + m) I_{n,s}(\rho) I_{n-1,s}(\rho). \quad (55)$$

These matrix elements (55) are independent of the spin projection and, thanks to this, cannot contribute to the anomalous magnetic moment. Further, there is no contribution to the ground state ($n = 0$) and no contribution to a spin flip. As well, one can observe that the dependence on the azimuthal angle, φ , dropped out so that, from Equation (47), the conservation of angular momentum for non-diagonal matrix elements follows. Finally, one has to insert Equations (47) and Equation (55) into Equation (45) and are left with a radial integration,

$$\int_0^\infty d\rho \sqrt{\rho} I_{n,n-l}(\rho) I_{n-1,n-l}(\rho) = \sqrt{n}. \quad (56)$$

so that one arrives at

$$\begin{aligned} \Delta M^{1PI}(n, s) &= \frac{eB}{16} \frac{\partial L_{HE}^{1-loop}}{\partial \mathcal{F}} \frac{\sqrt{n}}{K_n} \int_0^\infty d\rho \sqrt{\rho} I_{n,n-l}(\rho) I_{n-1,n-l}(\rho). \\ &= \frac{eB}{16} \frac{\partial L_{HE}^{1-loop}}{\partial \mathcal{F}} \frac{n}{\sqrt{1 + 2eBn}}. \end{aligned} \quad (57)$$

Here, $\partial L_{HE}^{1-loop} / \partial \mathcal{F}$ is defined in Equation (44). As it turns out, this contribution to the mass corrections does not depend on the orbital momentum, l , similar to Equation (5). Examples for the dependence on the magnetic field are shown in Figure 3.

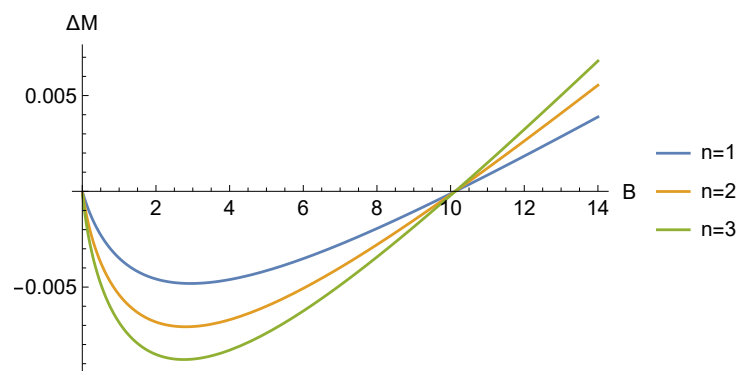


Figure 3. The mass correction ΔM^{1PI} in Equation (57), as a function of the magnetic field, B , for a few n values as indicated.

For the mass correction in a strong field, using the asymptotics of $\partial L_{\text{HE}}^{1-\text{loop}}/\partial \mathcal{F}$ derived in Equation B.6 of Ref. [8] one obtains

$$\Delta M^{1\text{PI}}(n, s) = \frac{\alpha e}{48\sqrt{2}\pi} \sqrt{eBn} \ln\left(\frac{eB}{\sqrt{2}}\right) + \mathcal{O}(\sqrt{B}). \quad (58)$$

This way, the contribution from the tadpole graph follows the general expectation, a negative contribution for small fields and a positive contribution for strong fields.

4. Conclusions

We reconsidered the calculation of the one-loop mass correction of the electron in a homogeneous magnetic field. There are two kind of representations known in the literature. One is in terms of eigenfunctions, and the other one uses the proper time representation and the operator method. We focused on the second case and compared the corresponding results obtained by different authors. In convenient notations, the result can be best written in the form $\Delta M(N, \zeta)$ in Equation (14), with W (26) (or (27)). Quite obvious misprints in the various papers are identified.

For studying the structure of the integrand F , Equation (26) is the most convenient since it has the simplest form of the denominators. On the real x -axis, one observes for $u = 1$, simple poles in $x = \pi\left(n + \frac{1}{2}\right)$, $n = 0, 1, \dots$. The ratio h/h^* in front is regular, and the last term in Equation (26) also has a simple pole if accounting for the integration over u and the factor $(1 - u)$ in the nominator. In the complex plane, one observes simple poles, the locations of which are shown in Figure 1.

As for the ground state, there are coinciding results from all the studies referred to here. The mass correction is shown in Figure 2 and its asymptotics for $H \rightarrow \infty$ are given in Equation (36). The finding demonstrates the stability of the ground state for arbitrary strength of the magnetic field.

For the low excited states, a similar result was obtained numerically in Ref. [7]. However, the method used for the calculation, as laid out in Section III in Ref. [7], raises questions as discussed in Section 2.1 here. More questions arise from the attempt to use the formulas shown in Section 2 for $N > 0$. As shown in Section 2.3, the rotation into the complex plane, $x \rightarrow -iy$, results in an exponential growth for $y \rightarrow \infty$, which should not have been there.

In addition, we calculated the so-far-overlooked contribution to the mass correction from the tadpole. The findings show no addition to the anomalous magnetic moment as well as no addition to the ground state energy. The tadpole contribution to the low excited states is shown in Figure 3. However, in order to compare this contribution with the contribution from Equation (2), more studies are necessary.

It worth mentioning that for high excited states, $N \gg 1$, which is of interest primarily for synchrotron radiation, a semiclassical approach delivers quite simple formulas which all demonstrate a growth in the mass correction in a magnetic field. We did not consider those calculations in the present paper.

In summary, the existing calculation for the mass correction of the electron in a magnetic field suggests no instability in QED. However, there are doubts in the correctness of the mentioned calculations, and a recalculation is advised.

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Appendix A. Robin Boundary Condition

In this Appendix, we refer to the basic formulas with the Robin boundary conditions, which are of interest in context with the Casimir effect and instabilities. The simplest case is a scalar field in four dimensions, obeying the wave equation and the boundary condition

$$(\partial_t^2 - \partial_{\parallel}^2 - \partial_z^2)\phi(\underline{x}) = 0, \quad (1 + \beta\partial_z)\phi(\underline{x})|_{z=0} = 0, \quad (\text{A1})$$

where $\partial_t = \partial/\partial t$, $\partial_{\parallel} = (\partial/\partial x, \partial/\partial y)$ the underlining in \underline{x} denotes a four-vector, β is some parameter and we consider $z \geq 0$. After a Fourier transform in time and with the directions parallel to the plate,

$$\phi(\underline{x}) = e^{-i\omega t + ik_{\parallel} x_{\parallel}} \varphi(z), \quad (\text{A2})$$

the equation $(-\omega^2 + k_{\parallel}^2 - \partial_z^2)\varphi(z) = 0$ follows. With the ansatz $\varphi(z) = e^{-\kappa z}$, from the boundary condition (A1), $\kappa = 1/\beta$ follows. One must assume $\beta > 0$ to obtain a normalizable solution and to arrive at

$$\omega^2 = k_{\parallel}^2 - \kappa^2 \quad (\text{A3})$$

for the frequency. For $k_{\parallel}^2 > \kappa^2$, the frequency is real. This solution describes a surface mode (known, for example, on the surface of metals). However, for $k_{\parallel}^2 < \kappa^2$, the frequency becomes imaginary and there will be a solution exponentially increasing in time. This is the mentioned instability, which is typically excluded from investigating the Casimir effect. It is interesting to note the similarities between Equations (A3) and (11).

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