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Fractional Laplacian Spinning Particle in External Electromagnetic Field

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Abstract: We construct a fractional Laplacian spinning particle model in an external electromagnetic field that generalizes a standard relativistic spinning particle model without anti-commuting spin variables. The one-dimensional fractional Laplacian in world-line variable λ governs the kinetic energy that is non-local in λ. The interaction between the particle's charge and the electromagnetic four-potential is non-local in λ, while the interaction between the particle's spin tensor and the electromagnetic field is standard. By applying the variational principle, we obtain the equations of motion for particle coordinates. We solve analytically the equations of motion in two particular cases: the constant electric and magnetic field. For more complex field configurations, the equations are, in general, non-local and non-linear. By making the assumption of a much weaker interaction term between the charge and four-potential compared with the interaction between spinning degrees of freedom and the electromagnetic field, we obtain approximate analytical solutions in the case of a quadratic electromagnetic potential.

Keywords: fractional Laplacian; fractional field theory; fractional particle; fractional calculus

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1. Introduction

The fractional Laplacian is a non-local pseudo-differential operator that has recently been used to model and investigate a wide range of physical and mathematical physical systems [1]. Models based on the fractional Laplacian have been successfully used to analyze the absorption and dispersion of compressional and shear waves in viscoelastic solids [2], fractional diffusion theory [3], non-linear fractional MOND-like gravity [4] and the Kuzmin-like gravitational model [5], the Scott correction for few electrons in atoms [6,7], the non-locality of the Pippard kernel in superconductivity and anomalous dimension for holographic conserved currents [8], long-distance processes that characterize tracer transport in both subsurface and surface environments [9], Polyakov's confinement mechanism for Maxwell's equations [10], soliton stabilization in $d = 1$ with possible applications to Bose–Einstein condensates [11], solitonic solutions to the non-linear Schroedinger equation [12], the quantization of non-local scalar fields with an extension problem [13], the conformal invariance in the long-range Ising model [14] and non-local scalar fields with an extension problem [15], just to mention a few applications across the spectrum of physics.

Among these applications, the fractional classical particles with a fractional Laplacian are of special interest, from both the conceptual point of view as well as from their possible applications. The fractional classical particle represents an embedding of the lowest dimensional non-local field theory into $\mathbb{R}^n$, which allows one to address questions about the classical observables and the geometry of the tangent and cotangent spaces, similar to the standard particle [16]. From a conceptual perspective, fractional classical particles can be considered as the classical approximation of quantum particles, which can aid in comprehending and elucidating the characteristics of the latter. In terms of practicality, fractional classical particles serve as the fundamental constituents of statistical systems, and comprehending their attributes can contribute to a better understanding of phenomena such as turbulent flows [17], anomalous transport [18], and fractional diffusion [19] at a more fine-grained scale. By constructing fractional particle models, we can obtain insights into the microscopic properties of statistical systems that obey Lévy processes [20] or long-range interactions [21].

In this paper, we introduce a particle model governed by the one-dimensional fractional Laplacian with a time variable. This model interacts with an external electromagnetic field via both electric charge and spin. For lack of a better name and to prevent confusion with particles possessing fractional spin or other fractional particle models, we label this model as the 'Fractional Laplacian Spinning Particle' (FLSP). It is important to note the existence of various semi-classical models describing spinning particles, both with and without anti-commutative variables associated with spin degrees of freedom [22]. In this work, we extend the classical relativistic spinning particle model proposed by Corben [23]—later generalized by Ellis [24]—which does not incorporate anti-commutative degrees of freedom. The merit of this model lies in its capacity to offer an intuitive description of the interaction between spin and the electromagnetic field. For alternative spin particle models, refer to [25]. In order to introduce the fractional Laplacian, we start with the covariant spinning particle formulation in which the world-line reparametrization invariance of the action is manifest [26]. The natural variable in this formulation is the world-line parameter, λ, and the particle's coordinates, x^a, are smooth functions on λ. In this formulation, the Lagrangian is polynomial in derivatives with respect to λ and the action can be generalized to massless particles. However, our construction, which relies on substituting the one-dimensional fractional Laplacian with a time variable into the standard covariant action, can be applied verbatim to all spinning particle models without anti-commutative variables.

The fractional Laplacian governs the particle's kinetic energy, introducing non-locality in world-line parameter λ. Another source of temporal non-locality arises from the interaction energy between the particle's charge and the four-potential. This represents the most general scenario achievable without assuming a delocalized spin or electromagnetic field, while considering the non-locality along the particle world-line. Additional assumptions about the non-local properties of the particle and field can give rise to distinct models. Conversely, adhering to the standard gauge invariance yields a less general model, where the interaction between charge and field follows the standard form.

Similar to the classical particle model, the description of the interaction between spin and the electromagnetic field poses two challenges. The first challenge involves determining the particle's trajectory in the presence of the spin tensor. Addressing this issue, as commonly performed in the literature, entails finding the equations of motion in the presence of a constant spin tensor field. The second challenge is to describe the precession of the electric and magnetic moments, which constitute components of the spin tensor. Given the intricate nature of the FLSP model and its fractional differential equations, this paper will focus solely on the first problem—specifically, deriving the equations of motion for the FLSP in the presence of a constant spin tensor field. Additionally, three examples will be discussed in this context. Describing the dynamics of the electric and magnetic moments in the fractional Laplacian setting requires a deeper understanding of fractional linear momenta and fractional torque, which is linked to the definition and geometric interpretation of the first-order fractional derivative and fractional Laplacian, aspects that are currently lacking.

The non-locality in the evolution along the world-line stems from the necessity to construct kinetic energy using fractional time derivatives. The fractional Laplacian is particularly advantageous due to its remarkable properties. Notably, when the particle's coordinates satisfy the extension problem, the FLSP model can be mapped into a local two-dimensional field theory [16].

In the traditional sense, derivatives with respect to world-line, such as standard first and second derivatives, are local operators — depending solely on the values of the function at a specific point on the world-line. However, the fractional Laplacian introduces a non-local operation, considering the function's history over a range of values of λ, resulting in a more intricate behavior.

The physical interpretation of this non-locality in λ is associated with memory effects generated by contributions from the position of the particle on the trajectory at past times through the fractional Laplacian. This leads to long-range correlations in the system, where events separated by significant time-like intervals exhibit correlation. Additionally, it gives rise to anomalous diffusion, where the mean square displacement of particles does not increase linearly along the world-line due to non-local interactions. This behavior is akin to fractional Brownian motion, characterized by stochastic long-range temporal correlation in the system's evolution. It is important to note that exploring these phenomena is beyond the scope of this work.

The paper is structured as follows. Section 2 provides a concise review of the fundamental properties of the fractional Lagrangian, essential for our work, following [27]. In Section 3, we develop the FLSP model, primarily by substituting the standard Laplacian with the fractional Laplacian—a common practice in the literature. Notably, the interaction term between charge and the electromagnetic four-potential is novel, incorporating a distinct fractional derivative related to the fractional Laplacian through a standard derivation. Within this section, employing the variational principle, we derive the equations of motion for both the particle coordinates and their Laplace potentials. These describe the particle's dynamics in the presence of electric and magnetic moments.

The derivation of the equations of motion diverges from a direct application of the Euler–Lagrange equations, as these lack a general form in the fractional Laplacian context. Instead, we rely on specific technical aspects of fractional calculus. The first crucial element is a well-known inversion property associated with the fractional Laplacian. The second is an original mathematical result, introduced here for the first time, mirroring an inversion property pertinent to the first-order fractional derivative.

In Section 4, we present two straightforward examples featuring analytic solutions for the equations of motion in constant electric and magnetic fields. However, in these simplified configurations, the interaction term between spin and the electromagnetic field is absent, rendering these solutions devoid of spin information. To explore the manifestation of this interaction, we investigate more complex field configurations later in Section 4. Nevertheless, for more general electromagnetic fields, the resulting equations of motion become highly non-local and non-linear, posing challenges for analytic analysis. Despite these complexities, by considering the limit of spin-field interaction stronger than charge-field interaction, we derive approximate analytic solutions that exhibit spin dependence. A specific case exemplifying this behavior is discussed, with the corresponding analytic solution presented. The paper concludes with a discussion of the obtained results and outlines potential avenues for future research. Throughout this work, we adopt the mostly-minus convention for the spacetime metric $\eta_{ab} = \mathrm{diag}(+,-,-,-)$, and use the natural units in which $c = 1$.

2. Fractional Laplacian

In this section, we review the basic properties of the fractional Laplacian that will be used in this work. Proofs and other properties can be found in [27]. Our presentation is for the one-dimensional case, and in a notation adapted to the construction of the fractional spinning particle model to be presented in the next section.

The fractional Laplacian is an pseudo-differential operator in mathematics that generalizes the notion of Laplacian spatial derivatives to fractional powers. It is often used to generalize certain types of partial differential equations with extensive applications in mathematics and physics, some of which have been reviewed in the introduction. The definitions of the fractional Laplacian often vary in the literature, but most of the time these definitions are equivalent. In this paper, we are going to use the equivalent definitions of the one-dimensional fractional Laplacian in terms of a singular operator and the Fourier transform, with an excursion in the fractional Laplacian viewed as a distribution.

The fractional Laplacian in a single variable is a non-local generalization of the classical Laplace operator. As a singular operator, the one-dimensional fractional Laplacian is defined by the Cauchy principal value of the following integral

$$(-\Delta_\lambda)^{\frac{\alpha}{2}} f(\lambda) \equiv c_\alpha \mathrm{P.V.} \int_{-\infty}^{\infty} \mathrm{d}\zeta \frac{f(\zeta+\lambda)-f(\lambda)}{|\zeta|^{1+\alpha}}, \tag{1}$$

where the constant is given by

$$c_\alpha = -\frac{2^\alpha}{\sqrt{\pi}} \frac{\Gamma\left(\frac{1+\alpha}{2}\right)}{\Gamma\left(-\frac{\alpha}{2}\right)}. \tag{2}$$

The fractional operator $(-\Delta_\lambda)^{\frac{\alpha}{2}}$ is well defined for functions $f \in C(\mathbb{R})$ that are bounded in the following sense

$$\int_{-\infty}^{+\infty} \mathrm{d}\lambda \frac{|f(\lambda)|}{(1+|\lambda|)^{1+\alpha}} < \infty. \tag{3}$$

By using the Fourier transform of $f(\lambda)$ defined as

$$\mathcal{F}[f](\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}\lambda f(\lambda) \mathrm{e}^{-\mathrm{i}\omega\lambda}, \tag{4}$$

we obtain the Fourier transform form of the fractional Laplacian

$$(-\Delta_\lambda)^{\frac{\alpha}{2}} f(\lambda) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}\omega\, |\omega|^\alpha \mathcal{F}[f](\omega) \mathrm{e}^{-\mathrm{i}\omega\lambda}. \tag{5}$$

The pseudo-differentiability and non-locality of the fractional Laplacian make its interpretation more difficult and complex. Among its more common interpretations are the fractional gradient, which is related to the uniform isotropic measure [28], generalized Dirichlet-to-Neumann operator, associated with Graham, Jenne, Mason, and Sparling operators (GJMS operators) on smooth metric spaces [29], or the number of particles that can move from one point to another in diffusion theory [27]. However, it is important to note that these are high-level interpretations and the actual behavior of functions under the fractional Laplacian can be quite complex and depends on the specific problem or application. In general, giving a geometric interpretation to the fractional operators in fractional calculus, which is a complex field with many different definitions and interpretations, is challenging, and no general consensus has been reached yet regarding the geometrical meaning of these operators. For a discussion of this topic in the context of fractional derivatives, see [30].

Equation (5) shows that the fractional Laplacian is a pseudo-differential operator, in the sense that it acts in the spectral (Fourier) domain by multiplication, rather than in the spatial domain by differentiation. However, it can be related by a standard first-order derivative to a first-order fractional operator $\left(-\Delta_\lambda^{(1)}\right)^{\frac{\alpha}{2}}$, defined by the following property

$$(-\Delta_\lambda)^{\frac{\alpha}{2}} \equiv \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(-\Delta_\lambda^{(1)}\right)^{\frac{\alpha}{2}}. \tag{6}$$

It is easy to verify that the first-order operator that satisfies Equation (6) above has the following form

$$\left(-\Delta_\lambda^{(1)}\right)^{\frac{\alpha}{2}} f(\lambda) = c_\alpha^{(1)} \int_0^\infty \mathrm{d}\zeta \frac{f(\lambda+\zeta)-f(\lambda-\zeta)}{\zeta^\alpha}, \tag{7}$$

for $0 \leq \alpha < 2$, where

$$c_\alpha^{(1)} = \frac{2^{\alpha-1}}{\sqrt{\pi}} \frac{\Gamma\left(\frac{1+\alpha}{2}\right)}{\Gamma\left(\frac{2-\alpha}{2}\right)}, \qquad c_\alpha = \alpha c_\alpha^{(1)}. \tag{8}$$

The operator $\left(-\Delta_\lambda^{(1)}\right)^{\frac{\alpha}{2}}$ has the following spectral form

$$\left(-\Delta_\lambda^{(1)}\right)^{\frac{\alpha}{2}} f(\lambda) = -\frac{\mathrm{i}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}\omega\, \omega |\omega|^{\alpha-2} \mathcal{F}[f](\omega)\, \mathrm{e}^{-i\omega\lambda}. \tag{9}$$

Sometimes, it is useful to represent the operators $(-\Delta_\lambda)^{\frac{\alpha}{2}}$ and $\left(-\Delta_\lambda^{(1)}\right)^{\frac{\alpha}{2}}$ with the help of the Laplacian potential as follows. Given a function $f(\lambda)$, we can write

$$(-\Delta_\lambda)^{\frac{\alpha}{2}} f(\lambda) = \frac{\mathrm{d}^2\Phi_\alpha(\lambda)}{\mathrm{d}\lambda^2}, \tag{10}$$

$$\left(-\Delta_\lambda^{(1)}\right)^{\frac{\alpha}{2}} f(\lambda) = \frac{\mathrm{d}\Phi_\alpha(\lambda)}{\mathrm{d}\lambda}. \tag{11}$$

The object $\Phi_\alpha(\lambda)$ in the right-hand side of the above equation is the Laplacian potential associated to the function $f(\lambda)$ and the fractional Laplacian at order α, and it is defined as follows

$$\Phi_\alpha(\lambda) = \frac{c_\alpha}{(\alpha-1)\alpha} \int_0^\infty \mathrm{d}\zeta\, \frac{f(\lambda-\zeta) + f(\lambda+\zeta)}{\zeta^{\alpha-1}}. \tag{12}$$

The fractional potential of the fractional Laplacian is a concept that generalizes the standard potential theory to fractional orders. The fractional Laplacian can be seen as a non-local operator that takes into account a global or semi-global average rather than a local one, and it can be defined in terms of the Riesz potential.

The operators $(-\Delta_\lambda)^{\frac{\alpha}{2}}$ and $\left(-\Delta_\lambda^{(1)}\right)^{\frac{\alpha}{2}}$ can be used to generalize the standard Laplacian and the first-order derivative. For example, these generalizations are useful for describing systems in backgrounds that deviate from homogeneity and isotropy, exhibiting distinctive non-local and non-linear physical characteristics. Subsequently, these operators will be employed in the construction of the (FLSP) model in the forthcoming discussion.

3. Fractional Spinning Particle in Electromagnetic Field

In this section, we formulate the Lagrangian for the FLSP within the electromagnetic field framework. This derivation involves a generalization of the standard spinning particle action, transforming it into a non-local model. The kinetic term of this model is expressed in terms of the one-dimensional fractional Laplacian. A non-spinning fractional particle model was previously introduced along the same lines in [16].

3.1. Classical Relativistic Particle

In order to generalize the relativistic particle model to a FLSP, it is advantageous to begin with the covariant particle formulation, emphasizing the world-line reparametrization symmetry. This choice proves beneficial as it allows us to exploit the world-line symmetry for expressing kinetic energy in a polynomial form and introducing the second derivative. The subsequent transition to FLSP involves interpreting this second derivative as a one-dimensional Laplacian, subsequently replaced by the fractional Laplacian. This subsection briefly revisits the covariant formulation of the free relativistic particle, serving as the foundation for our generalization. For more in-depth insights, readers can refer to the original paper [26].

The action governing the motion of a free, spinless relativistic particle is given by

$$S_0[x] = \int_{\sigma_i}^{\sigma_f} \mathrm{d}\sigma L_0[x], \qquad L_0[x] = m\sqrt{\eta_{ab}\dot{x}^a\dot{x}^b}, \qquad \dot{x}^a = \frac{dx^a}{d\sigma}, \tag{13}$$

where the parameter σ specifies the position of the particle along its world-line. The action (13) maintains invariance under the reparametrization of its world-line, denoted as $\sigma \to f(\sigma)$, where f represents an arbitrary smooth function on σ. The action $S_0[x]$ is not polynomial in its variables and cannot be generalized to a massless particle. These drawbacks can be overcome by reformulating the Lagrangian in terms of a new field, $e(\sigma)$, called 'einbein' (note that technically, the einbein is $e^{-1}(\sigma)$, but this detail bears no consequence for our construction). The resulting equivalent Lagrangian takes the following form

$$L_0[x] = \frac{\dot{x}^2}{2e} + \frac{em^2}{2}\,. \tag{14}$$

The reparametrization of the Lagrangian from Equation (14) is defined by the following transformations

$$x^a(\sigma) \to x^a(f(\sigma))\,, \tag{15}$$

$$e^{-1}(\sigma) \to e^{-1}(f(\sigma))\big(\dot{f}(\sigma)\big)^{-1}\,, \tag{16}$$

$$f(\sigma_i) = \sigma_i\,, \qquad f(\sigma_f) = \sigma_f\,. \tag{17}$$

The equivalence between the Lagrangians from Equations (13) and (14) can be proved by solving the equation of motion for e, which gives $e = \sqrt{\dot{x}^2}/m$, and replacing the result back in the Lagrangian. As observed from the right-hand side of Equation (14), the revised Lagrangian becomes polynomial in derivatives, establishing a well-defined theory, even for the case of $m = 0$. Moreover, this model exhibits versatility, allowing for generalizations to include particle spin [26], curvature [31], or strings. The reparametrization invariance of the relativistic particle allows us to choose a specific world-line parameter. For instance, opting for the particle's proper time, denoted as τ, imposes constraints on the velocities $\dot{x}^2 = 1$. On the other hand, a direct consequence of reparametrization invariance is the vanishing of the Hamiltonian. This issue can be addressed by choosing a time coordinate, denoted as $T(x)$, instead of the proper time, τ, resulting in a model where Lorentz covariance is lost. In the subsequent discussion, we are going to exploit the reparametrization invariance to introduce the fractional Laplacian and construct a FLSP model.

3.2. FLSP Model

We begin our exploration with the standard spinning particle model, as introduced by Corben [23], offering the advantage of providing an intuitive description of the interaction between spin and the electromagnetic field.

In order to describe the interaction with an external electromagnetic field, we consider a particle of mass m, electric charge q, electric moment $\boldsymbol{\delta}$,and magnetic moment $\boldsymbol{\mu}$ moving in an external electromagnetic field defined in terms of the four-potential, A_a. Then, in Corben's model, the Lagrangian from (13) is generalized to the following Lagrangian [23]

$$L[x] = m\left(1 + \frac{q}{2m^2}S_{ab}F^{ab}\right)\sqrt{\dot{x}_c\dot{x}^c} + qA_a\dot{x}^a\,, \tag{18}$$

where $F_{ab} = \partial_a A_b - \partial_b A_a$ is the electromagnetic field. The spin degrees of freedom are encoded in the spin tensor, S_{ab}, that is related to the particle polarization tensor, $\Pi_{ab}(\boldsymbol{\delta}, \boldsymbol{\mu})$, by the following relation

$$\Pi_{ab}(\boldsymbol{\delta}, \boldsymbol{\mu}) = \frac{q}{m}S_{ab}\,. \tag{19}$$

The extension of the Lagrangian from Equation (18) to the fractional case presents a non-trivial challenge due to the non-polynomial dependence of $L[x]$ in $\dot{x}^a$. As in Section 3.1, we introduce the world-line field, $e(\sigma)$, resulting in an equivalent classical Lagrangian that maintains manifest invariance under world-line reparametrization. This standard procedure yields the following equivalent Lagrangian

$$L[x] = \frac{\dot{x}_a \dot{x}^a}{2e} + \frac{m}{2} e \Big(1 + \frac{q}{2m^2} S_{ab} F^{ab} \Big)^2 + q A_a \dot{x}^a \,. \tag{20}$$

The Lagrangian $L[x]$ generalizes the Lagrangian given by Equation (14) to a classical particle interacting with an electromagnetic field. Since we are interested in models with fractional Laplacian, we rewrite the kinetic term as

$$L_0[x] = -\frac{x_a}{2} \frac{\mathrm{d}}{\mathrm{d}\sigma} \left(\frac{1}{e} \frac{\mathrm{d}x^a}{\mathrm{d}\sigma} \right), \tag{21}$$

under the assumption that

$$\lim_{|\sigma| \to \infty} \frac{\mathrm{d}}{\mathrm{d}\sigma} \left(\frac{x_a}{e} \frac{\mathrm{d}x^a}{\mathrm{d}\sigma} \right) = 0 \,. \tag{22}$$

The fractional Lagrangian governing our proposed FLSP model is derived by replacing the first-order and second-order derivatives with respect to σ in the standard Lagrangian (20) with first-order fractional derivatives and the fractional Laplacian. Consequently, the expression for the FLSP Lagrangian becomes

$$L^{(\alpha)}[x] = -\frac{\gamma}{2} e x_a (-\Delta_\lambda)^{\frac{\alpha}{2}} x^a + \frac{m}{2} e \Big(1 + \frac{q}{2m^2} S_{ab} F^{ab} \Big)^2 + \beta q e A_a \Big(-\Delta_\lambda^{(1)} \Big)^{\frac{\alpha}{2}} x^a \,, \tag{23}$$

where we have used the world-line reparametrization invariance to define $\mathrm{d}\lambda = e(\sigma)\,\mathrm{d}\sigma$. Here, we have introduced the constants γ and β that give correct dimensions to the corresponding terms of $L^{(\alpha)}[x]$. The action corresponding to the Lagrangian (23) is given by

$$S^{(\alpha)}[x] = \int \mathrm{d}\lambda L^{(\alpha)}[x] \,. \tag{24}$$

Several points merit discussion in this context. Firstly, it is important to note that the variable λ is defined by both the einbein, e, and the world-line parameter, σ. While λ characterizes the particle's evolution, the proper time coordinate is $x^0(\lambda)$ in the global inertial frame specified by the x^a coordinates. Secondly, in comparison with classical particle actions where the einbein field can be treated as a dynamical variable or used to impose the constraint $e = \sqrt{\dot{x}^2}$, the action (24) no longer depends on e, which was initially utilized to define the world-line parameter, λ. Lastly, it is crucial to acknowledge that the generalization of the standard Lagrangian from Equation (20) to a fractional particle is not unique. Instead of following the steps outlined in Equations (21) and (22), we could have directly replaced the derivatives in λ with first-order fractional derivatives $\Big(-\Delta_\lambda^{(1)} \Big)^{\frac{\alpha}{2}}$. While this is a viable option, we opt for the fractional Laplacian in the kinetic term. This choice is motivated by the fact that $(-\Delta_\lambda)^{\frac{\alpha}{2}}$ has well-known properties, simplifying the interpretation of the kinetic term. Moreover, defining the non-local derivative as a bilinear form preserves a more straightforward mathematical interpretation, lacking in $\Big(-\Delta_\lambda^{(1)} \Big)^{\frac{\alpha}{2}}$. Another rationale for utilizing the fractional Laplacian in the kinetic term is its equivalence, under the extension problem by Caffarelli and Silvestre [32], to a local sigma model with a higher dimension [16]. In a formulation based on the first fractional derivatives, this mapping can be employed only after the integration of one of the first-order fractional derivatives that leads to a Lagrangian of the form (23).

The Lagrangian $L^{(\alpha)}[x]$ incorporates fractional derivatives in place of both first- and second-order time derivatives, as evidenced by the first and last terms on the right-hand side of Equation (23). The first-order fractional derivative in the last term characterizes a non-localized interaction between the particle and the electromagnetic field. Consequently, the model defined by Equation (23) is hybrid, featuring a local interaction between spin degrees of freedom and the electromagnetic field, while the interaction between charge and the electromagnetic potential remains non-localized. Alternatively, less general hybrid models can be derived by exclusively generalizing the second-order derivative while retaining the standard first-order derivative in λ. In such cases, the fractional model preserves the gauge symmetry.

3.3. Equations of Motion

The interaction between a spinning particle and an electromagnetic field presents two challenges. The first involves describing the particle's dynamics, dictated by the electric and magnetic moments $\boldsymbol{\delta}$ and $\boldsymbol{\mu}$, respectively, which is tied to finding the Euler–Lagrange equations. The second problem arises due to the dependence of $\boldsymbol{\delta}$ and $\boldsymbol{\mu}$ on motion, necessitating the determination of the precessions of these two moments. In this work, we primarily address the first problem, leaving the intricacies related to the second problem for discussion in the Section 5.

The equation of motion for the fractional spinning particle can be derived by applying the variational principle to the action $S^{(\alpha)}[x]$. To achieve this, we first consider arbitrary infinitesimal coordinate transformations

$$x^a(\lambda) \to x'^a(\lambda) = x^a(\lambda) + \delta x^a(\lambda)\,. \tag{25}$$

The first term from the right-hand side of Equation (23) varies under the transformations (25) as follows

$$\delta S_0^{(\alpha)}[x] = -\gamma \int \mathrm{d}\lambda\, \delta x_a (-\Delta_\lambda)^{\frac{\alpha}{2}} x^a\,. \tag{26}$$

The right-hand side of Equation (26) has been obtained by applying the inversion relation [3]

$$\int_{-\infty}^{+\infty} \mathrm{d}\lambda\, f(\lambda)(-\Delta_\lambda)^{\frac{\alpha}{2}} g(\lambda) = \int_{-\infty}^{+\infty} \mathrm{d}\lambda\, g(\lambda)(-\Delta_\lambda)^{\frac{\alpha}{2}} f(\lambda)\,, \tag{27}$$

that holds for any two suitable functions, f and g, and is a result of the formulation of the fractional Laplacian in terms of distributions. The mass term has the standard variation

$$\delta S_1^{(\alpha)} = \frac{q}{2m} \int \mathrm{d}\lambda \left(1 + \frac{q}{2m^2} S_{rs} F^{rs}\right) S_{bc} \partial_a F^{bc} \delta x^a\,. \tag{28}$$

The variation of the last term is given by

$$\delta S_2^{(\alpha)} = \beta q \int \mathrm{d}\lambda \left[\partial_a A_b \delta x^a \left(-\Delta_\lambda^{(1)}\right)^{\frac{\alpha}{2}} x^b + A_a \left(-\Delta_\lambda^{(1)}\right)^{\frac{\alpha}{2}} \delta x^a\right]. \tag{29}$$

The second term from the right-hand side of Equation (29) can be further transformed if we use the following equivalent expression form of the operator $\left(-\Delta_\lambda^{(1)}\right)^{\frac{\alpha}{2}}$

$$\left(-\Delta_\lambda^{(1)}\right)^{\frac{\alpha}{2}} f(\lambda) = c_\alpha^{(1)} \int_{-\infty}^{\infty} \mathrm{d}s \frac{s f(\lambda + s)}{|s|^{1+\alpha}}\,. \tag{30}$$

To calculate its variation, an inversion formula for the operator $\left(-\Delta_\lambda^{(1)}\right)^{\frac{\alpha}{2}}$ is required. Let us establish this formula. Consider two continuous functions, f and g. Utilizing the definition of $\left(-\Delta_\lambda^{(1)}\right)^{\frac{\alpha}{2}}$ given in Equation (7), we can express

$$\begin{aligned}
\int_{-\infty}^{+\infty} \mathrm{d}\lambda g(\lambda) \left(-\Delta_\lambda^{(1)}\right)^{\frac{\alpha}{2}} f(\lambda) &= \int_{-\infty}^{+\infty} \mathrm{d}\lambda g(\lambda) \left[c_\alpha^{(1)} \int_{-\infty}^{+\infty} \mathrm{d}\zeta \frac{\zeta f(\lambda + \zeta)}{|\zeta|^{1+\alpha}}\right] \\
&= \int_{-\infty}^{+\infty} \mathrm{d}\lambda c_\alpha^{(1)} \left[\int_{-\infty}^{+\infty} \mathrm{d}\zeta \frac{g(\lambda) f(\lambda + \zeta)\zeta}{|\zeta|^{1+\alpha}}\right].
\end{aligned} \tag{31}$$

Next, a change of variables, $\rho = \lambda - \xi$, transforms Equation (31) into

$$\begin{aligned} &\text{Boundary term} + \int_{-\infty}^{+\infty} \mathrm{d}\rho\, c_\alpha^{(1)} \left[\int_{+\infty}^{-\infty} (-)\mathrm{d}\xi \frac{g(\rho+\xi) f(\rho)(-\xi)}{|-\xi|^{1+\alpha}} \right] \\ &= - \int_{-\infty}^{+\infty} \mathrm{d}\rho\, c_\alpha^{(1)} \left[\int_{-\infty}^{+\infty} \frac{g(\rho+\xi) f(\rho)(\xi)}{|\xi|^{1+\alpha}} d\xi \right] \\ &= - \int_{-\infty}^{\infty} \mathrm{d}\rho f(\rho) \left[c_\alpha^{(1)} \int_{-\infty}^{+\infty} \mathrm{d}\xi \frac{\xi g(\rho+\xi)}{|\xi|^{1+\alpha}} \right] \\ &= - \int_{-\infty}^{+\infty} \mathrm{d}\rho f(\rho) \left(-\Delta_\rho^{(1)} \right)^{\frac{\alpha}{2}} g(\rho)\,, \end{aligned} \tag{32}$$

establishing the following inversion formula

$$\int_{-\infty}^{+\infty} \mathrm{d}\lambda\, f(\lambda) \left(-\Delta_\lambda^{(1)} \right)^{\frac{\alpha}{2}} g(\lambda) = - \int_{-\infty}^{+\infty} \mathrm{d}\lambda\, g(\lambda) \left(-\Delta_\lambda^{(1)} \right)^{\frac{\alpha}{2}} f(\lambda)\,, \tag{33}$$

for all f and g that satisfy

$$\lim_{\lambda \to \pm\infty} f(\lambda + \zeta) g(\lambda) = 0\,, \tag{34}$$

representing the vanishing condition for the boundary term in Equation (32). By substituting Equation (33) into Equation (29), we obtain

$$\delta S_2^{(\alpha)} = \beta q \int \mathrm{d}\lambda \left[\partial_a A_b \left(-\Delta_\lambda^{(1)} \right)^{\frac{\alpha}{2}} x^b - \left(-\Delta_\lambda^{(1)} \right)^{\frac{\alpha}{2}} A_a \right] \delta x^a\,. \tag{35}$$

From Equations (26), (28), and (35), we obtain the equations of motion

$$\gamma (-\Delta_\lambda)^{\frac{\alpha}{2}} x_a - \frac{q}{2m} \left(1 + \frac{q}{2m^2} S_{rs} F^{rs} \right) S_{bc} \partial_a F^{bc} - \beta q \left[\partial_a A_b \left(-\Delta_\lambda^{(1)} \right)^{\frac{\alpha}{2}} x^b - \left(-\Delta_\lambda^{(1)} \right)^{\frac{\alpha}{2}} A_a \right] = 0\,. \tag{36}$$

We can write the equations of motion (36) in terms of Laplacian potentials $\mathrm{X}_a(\lambda)$ and $\mathrm{A}_a(\lambda)$ associated to the functions $x_{\alpha,a}(\lambda)$ and $A_{\alpha,a}(\lambda)$ and defined by Equation (12). The result takes the following form

$$\gamma \frac{\mathrm{d}^2 \mathrm{X}_{\alpha,a}(\lambda)}{\mathrm{d}\lambda^2} - \frac{q}{2m} \left(1 + \frac{q}{2m^2} S_{rs} F^{rs} \right) S_{bc} \partial_a F^{bc} - \beta q \left[\partial_a A_b \frac{\mathrm{d}\mathrm{X}_\alpha^b(\lambda)}{\mathrm{d}\lambda} - \frac{\mathrm{d}\mathrm{A}_{\alpha,a}(\lambda)}{\mathrm{d}\lambda} \right] = 0\,. \tag{37}$$

Equation (36) describes the dynamics of particle coordinates $x^a(\lambda)$ with respect to the world-line parameter, λwhich, in turn, is determined by τ and the einbein, $e(\tau)$, as discussed earlier. Given that λ specifies the particle's location on the world-line, the fractional Lagrangian introduces a delocalization of the particle's coordinates, simultaneously affecting all time-like, x^0, and space-like, x^i, coordinates. This characteristic aligns with the relativistic interpretation of spacetime coordinates. In essence, the FLSP model depicts both a memory-like effect in time and a diffusion-like effect in space simultaneously. It is important to emphasize that there is no equation of motion for the auxiliary field, e, since the einbein has already been employed in defining $\mathrm{d}\lambda$.

For an arbitrary electromagnetic field, both Equations (36) and (37) lead to intricate integro-differential equations, necessitating the use of computer-assisted analysis. While analytical solutions are challenging, they can be obtained in specific electromagnetic field configurations. To gain insights into the optimal strategies for handling these equations, we will explore three simple examples.

4. Examples of FLSP in Simple External Fields

In this section, we explore the equations of motion for the FLSP with a constant spin tensor in three simple external electromagnetic fields. In the first example, we solve the equations of motion in a constant magnetic field. Moving on to the second example, we determine the Laplacian potentials in a constant electric field. Since neither of these solutions depend on the spin tensor, we turn our attention, in the third example, to the particle's motion in a quadratic potential. Here, we determine the solutions to the equations of motion as functions of the constant spin tensor.

4.1. FLSP in Constant Magnetic Field

As a first example, consider the fractional spinning particle moving in the constant magnetic field B, defined by the four-potential

$$A_a = (0, 0, 0, Bx^1)\,, \tag{38}$$

where B is positive. Since the magnetic field is constant, the second term in the equation of motion (36) vanishes, and the equivalent set of equations on components is

$$\gamma(-\Delta_\lambda)^{\frac{\alpha}{2}} x_0 = 0\,, \tag{39}$$
$$\gamma(-\Delta_\lambda)^{\frac{\alpha}{2}} x_1 - \beta q B\left(-\Delta_\lambda^{(1)}\right)^{\frac{\alpha}{2}} x_3 = 0\,, \tag{40}$$
$$\gamma(-\Delta_\lambda)^{\frac{\alpha}{2}} x_2 = 0\,, \tag{41}$$
$$\gamma(-\Delta_\lambda)^{\frac{\alpha}{2}} x_3 + \beta q B\left(-\Delta_\lambda^{(1)}\right)^{\frac{\alpha}{2}} x_1 = 0\,. \tag{42}$$

The movement of the fractional spinning particle along the directions x^0 and x^2 is free. For a discussion of the solutions with this property in the context of fractional particles, see [16]. Let us solve the equations of motion for x^1 and x^3. To this end, we introduce the new functions

$$y_a(\lambda) = \left(-\Delta_\lambda^{(1)}\right)^{\frac{\alpha}{2}} x_a(\lambda)\,, \quad a = 1, 3\,. \tag{43}$$

By using the relation (6), we can write Equations (40) and (42) as follows

$$\gamma \frac{\mathrm{d}y_1(\lambda)}{\mathrm{d}\lambda} - \beta q B y_3(\lambda) = 0\,, \tag{44}$$
$$\gamma \frac{\mathrm{d}y_3(\lambda)}{\mathrm{d}\lambda} + \beta q B y_1(\lambda) = 0\,. \tag{45}$$

The set of ordinary differential Equations (44) and (45) can be solved by elementary methods. The result gives

$$\left(-\Delta_\lambda^{(1)}\right)^{\frac{\alpha}{2}} x_a(\lambda) = C_a\, \mathrm{e}^{i\left|\frac{\beta q B}{\gamma}\right|\lambda}\,, \quad a = 1, 3, \tag{46}$$

where C_a are real integration constants that are not independent of each other. The relations between C_1 and C_3 can be obtained by substituting the right-hand side of Equation (46) into the system (44) and (45), which gives $\gamma = +1$ and $iC_1 = C_3$. In order to find $x^a(\lambda)$, we need to invert the operator $\left(-\Delta_\lambda^{(1)}\right)^{\frac{\alpha}{2}}$. This can be achieved as follows. We perform a Fourier transform on $y_a(\lambda)$ using the formula (4), resulting in

$$\mathcal{F}[y_a(\lambda)](\omega) = \sqrt{2\pi} C_a \delta(\left|\frac{\beta q B}{\gamma}\right| - \omega)\,, \quad a = 1, 3. \tag{47}$$

On the other hand, from the definition (4), we can show that

$$\mathcal{F}[\left(-\Delta_\lambda^{(1)}\right)^{\frac{\alpha}{2}} f(\lambda)](\omega) = -i\omega|\omega|^{\alpha-2}\mathcal{F}[f(\lambda)](\omega)\,. \tag{48}$$

If we apply formula (48) to $y_a(\lambda)$ and combine the result with Equation (47), we obtain

$$\mathcal{F}[x_a(\lambda)](\omega) = \frac{i\sqrt{2\pi} C_a}{\omega|\omega|^{\alpha-2}} \delta(\left|\frac{\beta q B}{\gamma}\right| - \omega)\,. \tag{49}$$

Finally, by applying the inverse Fourier transform to Equation (49), we obtain

$$x_a(\lambda) = iC_a \int_{-\infty}^{+\infty} d\omega \frac{\delta(\left|\frac{\beta q B}{\gamma}\right| - \omega)\mathrm{e}^{i\omega\lambda}}{\omega|\omega|^{\alpha-2}} = iC_a \left(\left|\frac{\beta q B}{\gamma}\right|\right)^{1-\alpha} \mathrm{e}^{i\left|\frac{\beta q B}{\gamma}\right|\lambda}. \tag{50}$$

The preceding example illustrates that the model defined by the Lagrangian $L^{(\alpha)}[x]$ can result in standard differential equations. However, in more intricate scenarios, one may encounter integro-differential equations, as demonstrated earlier, with integrals primarily arising from terms explicitly containing A_a. It is worth noting that, in the presence of a constant magnetic field, there is an absence of interaction between spinning degrees of freedom and the magnetic field. Also, the result obtained in this case reduces to the standard one, with the parameter α contributing just to a constant factor.

4.2. FLSP in Constant Electric Field

The second example we are going to discuss here is the fractional spinning particle subject to an electric field with the following four-potential

$$A^0 = -\sum_{i=1}^{3} \mathcal{E}_i x^i\,, \qquad A^i = 0\,. \qquad i = 1,2,3. \tag{51}$$

Here, $\mathcal{E}^i$ are the real constants that play the role of electric field components. In this example, we are going to calculate the Laplacian potential $\mathrm{X}_{\alpha,a}(\lambda)$, which satisfies Equation (37). Firstly, we note that Equation (51) implies the following

$$\partial_a \mathrm{A}_\alpha^b = \begin{cases} -\mathcal{E}_i, & \text{if } a = i = 1,2,3 \text{ and } b = 0\,, \\ 0, & \text{otherwise}\,. \end{cases} \tag{52}$$

Then, by plugging Equations (51) and (52) into Equation (37), we obtain the following set of ordinary differential equations

$$\frac{\mathrm{d}^2 \mathrm{X}_{\alpha,0}}{d\lambda^2} = 0\,, \tag{53}$$

$$\frac{\mathrm{d}^2 \mathrm{X}_{\alpha,i}}{d\lambda^2} = \frac{\beta q}{\gamma} \mathcal{E}_i \frac{\mathrm{d}\mathrm{X}_{\alpha,0}}{d\lambda}\,, \qquad i = 1,2,3\,. \tag{54}$$

The general solutions to Equations (53) and (54) can be written as

$$\mathrm{X}_{\alpha,0}(\lambda) = \mathcal{U}_\alpha \lambda + \mathcal{V}_\alpha\,, \tag{55}$$

$$\mathrm{X}_{\alpha,i}(\lambda) = \frac{\beta q}{2\gamma} \mathcal{E}_i \mathcal{U}_\alpha \lambda^2 + \mathcal{W}_{\alpha,i}\lambda + \mathcal{C}_{\alpha,i}\,, \tag{56}$$

where $\mathcal{U}_\alpha$, $\mathcal{V}_\alpha$, $\mathcal{W}_{\alpha,i}$, and $\mathcal{C}_{\alpha,i}$ are integration constants. The solutions above depict a quadratic evolution in the world-line parameter for the Laplacian potentials, corresponding to the space-like directions, and a linear evolution for the Laplacian potential associated with the time-like direction.

The above examples highlight that the spin-field interaction does not play a significant role in the presence of constant magnetic and electric fields. Nevertheless, novel effects emerge in comparison with the classical case, arising from the particle's fractionality. This distinction is evident in the particle's coordinates, as seen in Equation (50), where the power of the cyclotron frequency depends on α in the case of a magnetic field.

Conversely, the Laplacian potentials corresponding to the particle's coordinates in a constant electric field, as revealed by Equations (55) and (56), are solutions to ordinary differential equations that do not explicitly exhibit dependence on α. According to the definition provided in Equation (12), the exchange between fractional and ordinary differential equations occurs at the expense of encapsulating all information about fractionality within the definition of the Laplacian potentials. Consequently, Equations (55) and (56) bear a resemblance to the classical particle's trajectory. However, it is crucial to note that the nature and interpretation of functions $x^a(t)$ and the variable t for the classical particle, and $\mathrm{X}^a_\alpha(\lambda)$ and variable λ for the FLSP, differ between the two cases.

4.3. FLSP in Quadratic Electromagnetic Potential

The interaction between spin and the electromagnetic field becomes relevant only when the external fields exhibit at least quadratic dependence on the particle's coordinates. However, deriving the equations of motion under such circumstances tends to result in very complex equations, primarily due to the dependency of the electromagnetic four-potential on the particle's coordinates. In situations where the interaction between the electromagnetic field and the spin tensor is considerably stronger than the interaction between the particle's charge and the four-potential, that is when $|\beta| << 1$, we can simplify Equations (36) or (37) by neglecting the last term. This simplification allows for obtaining analytical solutions to the corresponding problems. The exploration of two examples illustrating this scenario and with a constant spin tensor follows below.

Consider the four-potential given by

$$A^a = \frac{\mathcal{D}^a}{2}\left(\mathcal{C}_b x^b\right)^2, \tag{57}$$

where $\mathcal{D}^a$ and $\mathcal{C}^a$ are real constants. Substituting the corresponding electromagnetic field into the approximate equation of motion yields the following expression

$$\gamma(-\Delta_\lambda)^{\frac{\alpha}{2}} x_a - \mathcal{U}_a \mathcal{C}_b x^b - \mathcal{V}_a = 0\,, \tag{58}$$

where we have introduced the following short-hand notations

$$\mathcal{G}_{ab} = \mathcal{C}_a \mathcal{D}_b - \mathcal{C}_b \mathcal{D}_a\,, \tag{59}$$

$$\mathcal{U}_a = \frac{q^2}{4m^3} S^{rs} S^{bc} \mathcal{G}_{rs} \mathcal{G}_{bc} \mathcal{C}_a\,, \tag{60}$$

$$\mathcal{V}_a = \frac{q}{2m} S^{bc} \mathcal{G}_{bc} \mathcal{C}_a\,. \tag{61}$$

Under the assumption of a constant spin tensor, $\mathcal{G}_{ab}$, $\mathcal{U}_a$, and $\mathcal{V}_a$ are all constants. With this observation, let us proceed to solve fractional differential Equation (58). To begin, we contract it with $\mathcal{C}^a$ and obtain the following equation

$$\gamma(-\Delta_\lambda)^{\frac{\alpha}{2}} y - \mathcal{Z} y - \mathcal{W} = 0\,, \tag{62}$$

where

$$y = \mathcal{C}^a x_a\,, \qquad \mathcal{Z} = \mathcal{C}^a \mathcal{U}_a\,, \qquad \mathcal{W} = \mathcal{C}^a \mathcal{V}_a\,. \tag{63}$$

Next, we make the following change of variable

$$u = \mathcal{Z} y + \mathcal{W}\,. \tag{64}$$

Substituting the u from (64) into Equation (62), we obtain

$$(-\Delta_\lambda)^{\frac{\alpha}{2}} u = \frac{\mathcal{Z}}{\gamma} u\,. \tag{65}$$

Lastly, we identify that Equation (65) is the eigenfunction and eigenvalue equation for the fractional Laplacian. This equation is solved by the exponential function with the fractional frequency ω_α, taking the form

$$u(\lambda) = u_0 e^{i\omega_\alpha \lambda}, \qquad |\omega_\alpha| = \left(-\frac{\mathcal{Z}}{\gamma}\right)^{\frac{1}{\alpha}}, \tag{66}$$

and u_0 is an arbitrary real constant. It follows that the solution to Equation (58) is given by

$$x^a(\lambda) = \frac{4m^3 \mathcal{C}^a}{q^2 S^{rs} S^{lp} \mathcal{G}_{rs} \mathcal{G}_{lp} \mathcal{C}^b \mathcal{C}_b \mathcal{C}^c \mathcal{C}_c} \exp\left[-i\left(\frac{q^2 S^{rs} S^{bc} \mathcal{G}_{rs} \mathcal{G}_{bc} \mathcal{C}^a \mathcal{C}_a}{4\gamma m^3}\right)^{\frac{1}{\alpha}} \lambda\right] - \frac{2m^2 S^{bc} \mathcal{G}_{bc} \mathcal{C}^a}{q S^{rs} S^{bc} \mathcal{G}_{rs} \mathcal{G}_{bc} \mathcal{C}^a \mathcal{C}_a}, \tag{67}$$

where we have fixed the phase of ω_α such that the frequency is positive. In formula (67), we have undone the short-hand notation for the constants in order to display the non-polynomial dependency of the fractional particle trajectory on the spin tensor, S^{ab}. This dependency arises from the interaction between the particle spin and the electromagnetic field. Despite its non-linear and non-polynomial nature, the solution exhibits oscillatory behavior with respect to the world-line parameter, λ. Notably, the interaction between the spin tensor and the electromagnetic field plays a pivotal role in determining both the amplitude of oscillation in λ and the initial position along all four space-time directions.

5. Discussion

In this paper, we have introduced a model of a fractional Laplacian spinning particle in interaction with an external electromagnetic field. The FLSP generalizes the covariant Corben model and is characterized by a non-local kinetic term constructed from the one-dimensional fractional Laplacian with respect to the particle's world-line parameter, as well as a non-local interaction term between the particle's charge and the four-potential constructed from a first-order fractional derivative, where the ordinary first-order derivative is the fractional Laplacian. Additionally, the model contains a standard local interaction term between the spin tensor and electromagnetic tensor.

Studying the dynamics of this model is met with two challenges. The first one is to determine the particle's equations of motion in the presence of the spin tensor, or, equivalently, the electric and magnetic moments. The second one is to describe the dynamics of the spin tensor or, in the three-vectorial description, the torque of the moments. These are the same problems encountered in the study of the standard charged particle in the electromagnetic field.

In this paper, we have addressed only the first problem; that is, we have determined the equations of motion of the particle with a constant spin tensor from the variational principle. The task of calculating the extrema of the fractional action meets its own challenges, connected to the mathematical properties of the fractional derivatives, making the direct application of the Euler–Lagrange equations unsuitable. Instead, we have employed the inversion formulas for the fractional Laplacian and demonstrated a new inversion formula for the first-order fractional derivative. Also, we have given the equations of motion in terms of the Laplacian potentials of particle coordinates. The equations for Laplacian potentials have the advantage of containing ordinary derivatives instead of fractional derivatives. However, the Laplacian potentials do not belong to the same space of functions as the coordinates; instead, they are defined in terms of integrals along the world-lines that depend on the fractional parameter.

In general, the equations of motion obtained here are non-linear and non-local fractional differential equations in the world-line variable, making their study extremely difficult and more suitable for computer-assisted analysis than for analytical investigations. Under the assumption of a constant spin tensor and in some simple electromagnetic field configurations, such as constant magnetic and electric fields, and quadratic four-potential fields, it is possible to obtain concrete solutions to the equations of motion and determine the coordinates and Laplacian potentials. We have analyzed these cases and obtained general analytic solutions to the fractional differential equations. To display the effect of the interaction between the spinning degrees of freedom and the electromagnetic field on the particle coordinates, we made the approximation of a weaker interaction between the charge and the four-potential when compared with the interaction between the constant spin tensor and the electromagnetic field. Particular solutions can be obtained by imposing boundary conditions relevant to phenomena of interest, as in the case of standard differential equations.

There are several important open problems in the FLSP model proposed in this paper that deserve attention. The first one is the description of the spin tensor dynamics or the precessions of the moments. The electric and magnetic moments are three-dimensional vectors embedded in the spin tensor. Therefore, their orientation depends on the particle's trajectory. However, relating the components of these three-dimensional vectors with the components of S_{ab} involves the introduction of the four-dimensional moment, p^a, for example by defining the four vector S^a dual to the spin tensor as $S^a = \varepsilon^{abcd} p_b S_{cd}/2$. One major difficulty in reproducing this construction in the fractional models is that there is no coherent definition of the linear or canonical momenta. In the case of the FLSP model, this translates into the difficulty of describing the relation between the kinetic energy and momenta within the algebra of operators $\{(-\Delta_\lambda)^{\frac{\alpha}{2}}, \left(-\Delta_\lambda^{(1)}\right)^{\frac{\alpha}{2}}, \mathrm{d}/\mathrm{d}\lambda\}$. There are several avenues that could be explored to address the problem of momenta. Nevertheless, calculating the spin tensor dynamics, in any fractional model, depends on a better understanding of the physical interpretation of the fractional derivatives and the mathematical structures behind them.

Another interesting problem in the FLSP model from both mathematical and physical points of view is to solve the fractional differential equations with a constant spin tensor for particular models. Both analytical and computer-assisted methods are required to shed light on the properties of these equations. In particular, it is important to understand the role played by the boundary conditions and their effect on the fractional particle.

In constructing the FLSP model, we have started from the world-line formulation of the relativistic particle, since it allows us to introduce the fractional Laplacian in the kinetic term. Therefore, the natural variable of the problem is the world-line parameter, λ. However, it is important to discuss the problem of the model in particular reference frames, such as the proper reference frame. Solving this problem implies the study of different frame-dependent formulations of the fractional operators and their transformation under the Poincaré group.

The FLSP generalization of the classical particle models to fractional models is not unique. Several aspects can be emphasized in each construction, such as the order of fractional derivative to be used, the locality of the interacting terms, the locality of the external field, etc. The exploration of these directions and the physical properties of each model, for example, the symmetries, represents an important line of research. A distinct system of interest for applications is the non-relativistic FLSP, which deserves attention. Another important problem is the actual derivation of the fractional properties of the fractional particle models from first principles. However, we expect that the general procedure to construct the fractional spinning model that we have proposed here applies verbatim to other classical spinning models that can be used as a starting point for different fractional particles, either free or in interaction with the electromagnetic or other fields. These constructions are motivated by the necessity of understanding better the mathematical–physical properties of the fractional differential equations and their physical interpretation, as well as their applications to concrete mathematical and physical problems.

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References

1. Beghin, L.; Mainardi, F.; Garrappa, R. (Eds.) *Nonlocal and Fractional Operators*; Springer: Cham, Switzerland, 2021.
2. Treeby, B.E.; Cox, B.T. Modeling power law absorption and dispersion in viscoelastic solids using a split-field and the fractional Laplacian. *J. Acoust. Soc.* **2014**, *136*, 1499–1510. [CrossRef] [PubMed]
3. Ciaurri, O.; Roncal, L.; Stinga, P.; Torrea, J.; Varona, J. Nonlocal discrete diffusion equations and the fractional discrete Laplacian, regularity and applications. *Adv. Math.* **2018**, *330*, 688–738. [CrossRef]
4. Giusti, A. MOND-like fractional Laplacian theory. *Phys. Rev.* **2020**, *D101*, 124029. [CrossRef]
5. Giusti A.; Garrappa R.; Vachon G. On the Kuzmin model in fractional Newtonian gravity. *Eur. Phys. J. Plus* **2020**, *135*, 798. [CrossRef]
6. Sorensen, T.O. The large-Z behavior of pseudorelativistic atoms. *J. Math. Phys.* **2005**, *46*, 052307. [CrossRef]
7. Fournais, S.; Lewin, M.; Triay, A. The Scott Correction in Dirac–Fock Theory. *Commun. Math. Phys.* **2020**, *378*, 569–600. [CrossRef]
8. La Nave, G.; Limtragool, K.; Phillips, P. W. Fractional Electromagnetism in Quantum Matter and High-Energy Physics. *Rev. Mod. Phys.* **2019**, *91*, 021003. [CrossRef]
9. Li, Z.; Tang, H.; Yuan, S.; Zhang, H.; Kong, L.; Sun, H. Modeling Long-Distance Forward and Backward Diffusion Processes in Tracer Transport Using the Fractional Laplacian on Bounded Domains. *Fractal Fract.* **2023**, *7*, 823. [CrossRef]
10. Heydeman, M.; Jepsen, C. B.; Ji, Z.; Yarom, A. Polyakov's confinement mechanism for generalized Maxwell theory. *JHEP* **2023**, *4*, 119. [CrossRef]
11. Stephanovich, V.A.; Olchawa, W. Stabilization of 1D solitons by fractional derivatives in systems with quintic nonlinearity. *Sci. Rep.* **2022**, *12*, 384. [CrossRef]
12. El-Nabulsi, R.A.; Anukool, W. A family of nonlinear Schrodinger equations and their solitons solutions. *Chaos Solitons Fractals* **2023**, *166*, 112907. [CrossRef]
13. Frassino, A.M.; Panella, O. Quantization of nonlocal fractional field theories via the extension problem. *Phys. Rev.* **2019**, *D100*, 116008. [CrossRef]
14. Paulos, M.F.; Rychkov, S.; van Rees, B.C.; Zan, B. Conformal Invariance in the Long-Range Ising Model. *Nucl. Phys.* **2016**, *B902*, 246–291. [CrossRef]
15. Rajabpour, M.A. Conformal symmetry in non-local field theories. *JHEP* **2011**, *6*, 076. [CrossRef]
16. Vancea, I.V. Fractional Particle and Sigma Model. *arXiv* **2023**, arXiv:2309.03054.
17. Samiee, M.; Akhavan-Safaei, A.; Zayernouri, M. A fractional subgrid-scale model for turbulent flows: Theoretical formulation and a priori study. *Phys. Fluids* **2020**, *32*, 055102. [CrossRef]
18. Suzuki, J.L.; Gulian, M.; Zayernouri, M.; D'Elia, M. Fractional Modeling in Action: A Survey of Nonlocal Models for Subsurface Transport, Turbulent Flows, and Anomalous Materials. *J. Peridyn. Nonlocal Model* **2023**, *5*, 392–459. [CrossRef]
19. Lin, Z.; Liu, F.; Wang, D.; Gu, Y. Reproducing kernel particle method for two-dimensional time-space fractional diffusion equations in irregular domains. *Eng. Anal. Bound. Elem.* **2018**, *97*, 131–143. [CrossRef]
20. Laskin, N. Fractional quantum mechanics and Lévy path integrals. *Phys. Lett.* **2000**, *A268*, 298–305. [CrossRef]
21. Bertoin, J. *Lévy Processes*; Cambridge University Press: Cambridge, MA, USA, 1996.
22. Frydryszak, A. *Lagrangian Models of Particles with Spin: The First Seventy Years*. In *From Field Theory to Quantum Groups*; Jancewicz, B., Sobczyk, J., Eds.; World Scientific: Singapore, 1996; pp. 151–172.
23. Corben, H.C. *Classical and Quantum Theories of Spinning Particles*; Holden-Day: San Francisco, CA, USA, 1968.
24. Ellis, J. Motion of a classical particle with spin. *Math. Proc. Camb. Philos. Soc.* **1975**, *78*, 145–156. [CrossRef]
25. Deriglazov, A. *Classical Mechanics: Hamiltonian and Lagrangian Formalism*; Springer: Cham, Switzerland, 2017.
26. Brink, L.; Di Vecchia, P.; Howe, P. A Lagrangian formulation of the classical and quantum dynamics of spinning particles. *Nuc. Phys.* **1997**, *B118*, 76–94. [CrossRef]
27. Chen, W.; Li, Y.; Ma, P. *The Fractional Laplacian*; World Scientific: Singapore, 2020.
28. D'Ovidio, M.; Garra, R. Fractional gradient and its application to the fractional advection equation. *arXiv* **2013**, arXiv:1305.4400.
29. Case, J.S.; Chang, S.Y. On Fractional GJMS Operators. *Commun. Pur. Appl. Math.* **2016**, *69*, 1017–1061. [CrossRef]
30. Tarasov, V.E. Geometric Interpretation of Fractional-Order Derivative. *FCAA* **2016**, *19*, 1200–1221. [CrossRef]

31. Pisarski, R.D. Field theory of paths with a curvature-dependent term. *Phys. Rev.* **1986**, *D34* 670. [CrossRef]
32. Caffarelli, L.; Silvestre, L. An Extension Problem Related to the Fractional Laplacian. *Commun. Partial. Differ. Equations* **2007**, *32*, 1245–1260. [CrossRef]