physical sciences forum

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Proceeding Paper

Cosmological Solutions of Integrable $F(R)$ Gravity Models with an Additional Scalar Field †

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† Presented at the 2nd Electronic Conference on Universe, 16 February–2 March 2023; Available online: https://ecu2023.sciforum.net/.

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Abstract: We consider $F(R)$ cosmological models with a scalar field. For the R^2 model in the spatially flat Friedmann–Lemaître–Robertson–Walker metric, the Ricci scalar R can smoothly change its sign during the evolution if and only if the scalar field is a phantom one. In the Bianchi I metric, the Ricci scalar cannot smoothly change its sign if the corresponding solution is anisotropic at $R = 0$. This result does not depend on the type of the scalar field. In the Bianchi I metric, the general solution of evolution equations has been obtained.

Keywords: modified gravity; cosmology; exact solutions

Citation: Vernov, S.; Ivanov, V. Cosmological Solutions of Integrable $F(R)$ Gravity Models with an Additional Scalar Field. *Phys. Sci. Forum* **2023**, *7*, 17. https://doi.org/10.3390/ECU2023-14039

Academic Editor: Gonzalo Olmo

Published: 16 Febuary 2023

1. Modified Gravity Models and GR Models with Scalar Fields

On the one hand, $F(R)$ gravity generalizes the General Relativity (GR). On the other hand, gravitational $F(R)$ models can be considered as models with a nonminimally coupled scalar field without kinetic term. These models can be transformed into GR models with a standard minimally coupled scalar field by the conformal metric transformation. Note that such a transformation is possible only if the first derivative $F_{,R} \equiv \frac{dF}{dR} > 0$. A GR model with an ordinary scalar field has is well-defined at any value of R, whereas for $F(R)$ models, the possible domain of values of R is restricted by the requirement the effective gravitational constant should be positive, hence, $F_{,R} > 0$. It is interesting to analyze whether $F_{,R}$ can smoothly change its sign during evolution or not. For models with a nonminimally coupled scalar field, this problem has been discussed in [1]. For $F(R)$ models it has been shown [2], that cosmological solutions have anisotropic instabilities associated with the crossing of the barrier $F_{,R}(R) = 0$. The solutions in the spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) metric can be smooth, whereas solutions in the Bianchi I metric should have singularities. These results have been obtained for $F(R)$ models without scalar fields.

The simplest modifications of GR that allow us to describe inflation are the $R + R^2$ Starobinsky model [3] and the Higss-driven inflation [4]. Both models are in good agreement with the Planck measurements of the Cosmic Microwave Background (CMB) radiation [5]. Inflationary R^2 models gravity with the Higgs-like boson are actively investigated in the context of the possible production of primordial black holes [6–8]. Note that the R^2 term arises as a quantum correction in inflationary models with scalar fields [9–12].

The Starobinsky inflationary model [3,13] includes both the R^2 term and the standard Hilbert–Einstein term. Adding to this model the cosmological constant, one can obtain a model with exact cosmological solutions [14]. The integrability of the Starobinsky model as well as the integrability of the pure R^2 model have been shown by the singularity analysis [15,16]. These models pass the so-called weak Painlevé test [16,17]. At the same time, exact cosmological solutions for the Starobinsky inflationary model are not known.

Most of the results of the explicit integration of cosmological models with scalar fields are related to one-field cosmological models and the spatially flat FLRW metric [18–22]. Anisotropic cosmological solutions in $F(R)$ gravity models have been actively investigated in Refs. [23–26].

In our papers [27,28], we analyze $F(R)$ gravity models with scalar fields. After the metric transformation of such models [29], one obtains models with a nonstandard kinetic part of the scalar field Lagrangian, the so-called chiral cosmological models [30–36]. We are looking for general solutions of evolutionary equations in the pure R^2 model with a scalar field in the FLRW and Bianchi I metrics, using the corresponding chiral cosmological model. The absence of the scalar field potential in the model considered allows us to get explicitly the Hubble parameter and the scalar field in the cosmic time [27,28]. The obtained solutions allow one to get solutions of the initial $F(R)$ gravity model with a scalar field in terms of elementary functions of the parametric time. Therefore, solutions in cosmic time can be found in quadratures. This method allows finding only solutions with $F_{,R} > 0$, so solutions with $F_{,R}$ changing its sign during the evolution cannot be found by this method and their existence should be checked additionally.

2. Equations and Solutions in the Bianchi I and FLRW Metrics

Let us consider a pure R^2 model, described by the following action:

$$S_R = \int d^4x\sqrt{-g}\Big[F_0R^2 - \frac{\varepsilon_\psi}{2}g^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi - V(\psi)\Big], \tag{1}$$

where $F_0 > 0$ is a constant, R is the Ricci scalar, ψ is a scalar field or a phantom scalar field in dependence of the sign of $\varepsilon_\psi = \pm 1$.

Let us consider the case of the Bianchi I metric with the following interval [23,26,37,38]:

$$ds^2 = -dt^2 + a^2(t)\Big[\mathrm{e}^{2\beta_1(t)}\,dx_1^2 + \mathrm{e}^{2\beta_2(t)}\,dx_2^2 + \mathrm{e}^{2\beta_3(t)}\,dx_3^2\Big]. \tag{2}$$

The functions $\beta_i(t)$ satisfy the relation

$$\beta_1(t) + \beta_2(t) + \beta_3(t) = 0. \tag{3}$$

It is useful to introduce a shear,

$$\sigma^2 \equiv \dot\beta_1^2 + \dot\beta_2^2 + \dot\beta_3^2 = 2\Big(\dot\beta_1^2 + \dot\beta_1\dot\beta_2 + \dot\beta_2^2\Big), \tag{4}$$

that measures a total amount of anisotropy, "dots" denote derivatives with respect to time t. The Ricci scalar is

$$R = \sigma^2 + 6\Big(\dot H + 2H^2\Big), \quad \text{where} \quad H = \frac{\dot a}{a}. \tag{5}$$

The evolution equations in the Bianchi I metric have the following form [26,28]:

$$4H\dot\sigma^2 - \Big(\sigma^2\Big)^2 - 4\Big(2\dot H + 3H^2\Big)\sigma^2 + 24H\ddot H - 12\dot H^2 + 72H^2\dot H = \frac{3\varepsilon_\psi}{2F_0}\Big(\dot\psi^2 + 2V(\psi)\Big), \tag{6}$$

$$\begin{aligned} &-\ddot\sigma^2 - 2H\dot\sigma^2 - \frac{1}{4}\Big(\sigma^2\Big)^2 - \Big(2\dot H + 3H^2\Big)\sigma^2 \\ &+ \Big(6\dot H + 12H^2 + \sigma^2\Big)\ddot\beta_i + \Big(6\ddot H + 42H\dot H + 36H^3 + 3H\sigma^2 + \dot\sigma^2\Big)\dot\beta_i \\ &- 3\Big(2\dddot H + 12H\ddot H + 9\dot H^2 + 18H^2\dot H\Big) = \frac{\varepsilon_\psi}{8F_0}\Big(\dot\psi^2 - 2V(\psi)\Big), \qquad i = 1,2,3. \end{aligned} \tag{7}$$

If $V(\psi) = 0$, then we can eliminate $\dot\psi$ and obtain the following equations

$$\frac{1}{6}\left(\ddot\sigma^2 + 5H\dot\sigma^2 - 2\Big(2\dot H + 3H^2\Big)\sigma^2 - \frac{(\sigma^2)^2}{2}\right) + \dddot H + 9H\ddot H + 3\dot H^2 + 18H^2\dot H = 0, \tag{8}$$

$$\dot{\sigma}^2 + \left(3H + \frac{\dot{R}}{R}\right)\sigma^2 = 0. \tag{9}$$

Therefore, the evolution equations can be presented in the form of the following fifth-order system [28]:

$$\dddot{H} = \frac{1}{2R}\left(r_1 - 2r_2\left[2\sigma^2 + R\right]\right), \tag{10}$$

$$\ddot{\sigma}^2 = \frac{3}{R}\left(4\sigma^2 r_2 - r_1\right), \tag{11}$$

where

$$r_1 = \left(\dot{\sigma}^2\right)^2 + \left(4H\sigma^2 + 6\ddot{H} + 36H\dot{H} + 24H^3\right)\dot{\sigma}^2 + 2\left(\dot{H}\sigma^2 + 14H\ddot{H} + 14\dot{H}^2 + 36H^2\dot{H}\right)\sigma^2,$$

$$r_2 = \frac{1}{12}\left(10H\dot{\sigma}^2 - 4\sigma^2\left(2\dot{H} + 3H^2\right) - \left(\sigma^2\right)^2\right) + 9H\ddot{H} + 3\dot{H}^2 + 18H^2\dot{H}\,.$$

Note that Equation (9) includes neither $\dddot{H}$, nor $\ddot{\sigma}^2$. The initial conditions for system (10)–(11) should satisfy the relation (9).

One can see, that the right-hand sides of Equations (10) and (11) are singular at $R = 0$ if $\sigma^2 \neq 0$. Therefore, we see an essential difference between anisotropic solutions, for which the sign of R is defined by the initial conditions and cannot be smoothly changed during evolution, and isotropic solutions in the spatially flat FLRW metric. The existence of isotropic solutions that can smoothly change their sign during evolution has been proved in [27], where the following explicit example has been found.

In the spatially flat FLRW metric, all $\beta_i(t) \equiv 0$ and, hence $\sigma^2(t) \equiv 0$. At $\sigma^2(t) \equiv 0$, Equations (9) and (11) are satisfied. The Hubble parameter $H(t)$ is a solution of Equation (10) that has the following form:

$$\dddot{H} = r_2 = 9H\ddot{H} + 3\dot{H}^2 + 18H^2\dot{H}\,. \tag{12}$$

This equation is equivalent to the following equation:

$$(\ddot{H} + 3H\dot{H})\frac{d}{dt}\left[\dot{H}^2 - 2H\ddot{H} - 6H^2\dot{H}\right] = \left(\dot{H}^2 - 2H\ddot{H} - 6H^2\dot{H}\right)\frac{d}{dt}\left[\ddot{H} + 3H\dot{H}\right]. \tag{13}$$

Equation (13) has been analyzed in detail in our paper [27], where two families of solutions have been found. The Hubble parameter $H(t)$ is either a solution of equation

$$\ddot{H} + 3H\dot{H} = 0\,, \tag{14}$$

or a solution of equation

$$\dot{H}^2 - 2H\ddot{H} - 6H^2\dot{H} = 0\,. \tag{15}$$

Equation (14) can be easily integrated and we get

$$2\dot{H} + 3H^2 = 2C,$$

where C is an integration constant.

If $C < 0$, then the following exact solution exists:

$$H(t) = -\frac{\sqrt{-6C}}{3}\tan\left[\frac{\sqrt{-6C}}{2}(t - t')\right], \quad \psi(t) = \frac{6C\sqrt{2F_0}}{\cos^2\left(\sqrt{\frac{-3C}{2}}(t - t')\right)}\,. \tag{16}$$

Note that solution (16) exists only if $\varepsilon_\psi = -1$, because

$$\dot{\psi}^2 = -72\varepsilon_\psi F_0 \dot{H}^2. \quad (17)$$

So, the scalar field ψ is a phantom one. Substituting $\sigma^2 = 0$ and $H(t)$ given by (16) into (5), we get that the scalar curvature R changes its sign at $t = t' \pm \frac{\pi}{3}\sqrt{-2/(3C)}$. Therefore the solution obtained has no analogue in the Einstein frame. It is one of main results of our paper [27].

3. Conclusions

Modified gravitational models are being actively investigated both to describe the accelerating expansion of the observed Universe and to link general relativity with quantum theory due to adding of string theory-inspired corrections [39,40]. It is an important issue to obtain cosmological solutions for such types of models. The standard way to explore $F(R)$ gravity models and models with the non-minimally coupled scalar field is to use the conformal metric transformation and to analyze the corresponding GR model in the Einstein frame [22,41]. However, one can lose some solutions in this way. For example, if $F_{,R}$ and, therefore, the effective gravitational constant change their signs during the evolution, then the corresponding smooth solutions for $F(R)$ gravity models have no analogs in the Einstein frame. If such solutions exist, they can be found in the original frame only.

In [27], we have found smooth particular solutions, in particular, solution (16), which have no analog in the Einstein frame. Note that these solutions do not admit any amount of anisotropy. In the Bianchi I metric, the evolution Equations (10) and (11) have singularities at $R = 0$ for anisotropic solutions. So, the Ricci scalar R does not change its sign during evolution if the corresponding solution is anisotropic at $R = 0$. General solutions in the spatially flat FLRW and Bianchi I metrics have been obtained both for the R^2 gravity model with a scalar field, and for the corresponding two-field chiral cosmological model [27,28].

Using the method proposed, we plan to seek the general solutions of other $F(R)$ gravity models with scalar fields as well as two-field models with non-minimally coupled scalar fields.

Author Contributions: The authors contributed equally to this work. Investigation and writing, V.I. and S.V. All authors have read and agreed to the published version of the manuscript.

Funding: S.V. was partially funded by the Russian Foundation for Basic Research grant No. 20-02-00411.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: There are no new data related to this study.

Conflicts of Interest: The authors declare no conflict of interest.

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