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A Blow-Up Criterion for the Density-Dependent Incompressible Magnetohydrodynamic System with Zero Viscosity

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Abstract: In this paper, we provide a blow-up criterion for the density-dependent incompressible magnetohydrodynamic system with zero viscosity. The proof uses the L^p-method and the Kato–Ponce inequalities in the harmonic analysis. The novelty of our work lies in the fact that we deal with the case in which the resistivity η is positive.

Keywords: magnetohydrodynamic system; incompressible; blow-up criterion

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1. Introduction

Magnetohydrodynamics (MHD) is concerned with the study of applications between magnetic fields and fluid conductors of electricity. The application of magnetohydrodynamics covers a very wide range of physical objects, from liquid metals to cosmic plasmas.

We consider the following 3D density-dependent incompressible magnetohydrodynamic system:

$$\partial_t\rho + u\cdot\nabla\rho = 0, \tag{1}$$

$$\rho\partial_t u + \rho(u\cdot\nabla)u + \nabla\pi = \operatorname{rot} b \times b, \tag{2}$$

$$\partial_t b + u\cdot\nabla b - b\cdot\nabla u = \eta\Delta b, \tag{3}$$

$$\operatorname{div} u = 0,\ \ \operatorname{div} b = 0 \ \text{ in } \mathbb{R}^3\times(0,\infty), \tag{4}$$

$$\lim_{|x|\to\infty}(\rho,u,b) = (1,0,0), \tag{5}$$

$$(\rho,u,b)(\cdot,0) = (\rho_0,u_0,b_0) \ \text{ in } \mathbb{R}^3. \tag{6}$$

The unknowns are the fluid velocity field $u = u(x,t)$, the pressure $\pi = \pi(x,t)$, the density $\rho = \rho(x,t)$, and the magnetic field $b = b(x,t)$. $\eta > 0$ is the resistivity coefficient. The term $\operatorname{rot} b \times b$ in (2) is the Lorentz force with low regularity, and thus it is the difficult term.

For the case of $b = 0$, there are many studies. Beirão da Veiga and Valli [1,2] and Valli and Zajaczkowski [3] proved the unique solvability, local in time, in some supercritical Sobolev spaces and Hölder spaces in bounded domains. It is worth pointing out that, in 1995, Berselli [4] discussed the standard ideal flow. Danchin [5] and Danchin and Fanelli [6] (see also [7,8]) proved the unique solvability, local in time, in some critical Besov spaces. Recently, Bae et al [9] showed a regularity criterion:

$$\nabla u \in L^1(0,T;L^\infty(\mathbb{R}^3)). \tag{7}$$

This refined the previous blow-up criteria [5–7]:

$$\omega := \operatorname{rot} u \in L^1(0,T;\dot{B}^{\frac{d}{2}}_{2,1}(\mathbb{R}^d)), \tag{8}$$

$$\nabla u \in L^1(0,T;L^\infty) \ \text{ and } \ \nabla\pi \in L^1(0,T;B^{s-1}_{\infty,r}), s\geq 1, 1\leq r\leq\infty. \tag{9}$$

In [10], the authors proved the local well-posedness of smooth solutions in Sobolev spaces. The aim of this article is to prove (7) for the system (1)–(6). We will prove the following.

Theorem 1. *Let* $0 < \inf \rho_0 \leq \rho_0 \leq C, \nabla\rho_0 \in H^2, u_0, b_0 \in H^3$ *with* $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$ *in* $\mathbb{R}^3$. *Let* (ρ, u, b) *be the unique solution to the problem* (1)–(6). *If* (7) *holds true with some* $0 < T < \infty$, *then the solution* (ρ, u, b) *can be extended beyond* $T > 0$.

Remark 1. *In [8], Zhou, Fan and Xin showed the same blow-up criterion (8), which is refined by (7) for the ideal MHD system.*

Remark 2. *When* $\eta = 0$, *we are unable to show a similar result.*

In the following proofs, we will use the bilinear commutator and product estimates due to Kato–Ponce [11]:

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \leq C(\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1}g\|_{L^{q_1}} + \|g\|_{L^{p_2}}\|\Lambda^s f\|_{L^{q_2}}), \tag{10}$$
$$\|\Lambda^s(fg)\|_{L^p} \leq C(\|f\|_{L^{p_1}}\|\Lambda^s g\|_{L^{q_1}} + \|\Lambda^s f\|_{L^{p_2}}\|g\|_{L^{q_2}}), \tag{11}$$

with $s > 0, \Lambda := (-\Delta)^{\frac{1}{2}}$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$.

2. Proof of Theorem 1

We only need to prove a priori estimates.

First, thanks to the maximum principle, it is easy to see that

$$\frac{1}{C} \leq \rho \leq C. \tag{12}$$

We will use the identity

$$b \cdot \nabla b + b \times \operatorname{rot} b = \frac{1}{2}\nabla|b|^2. \tag{13}$$

Testing (2) by u, using (1), (4) and (13), we find that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|u|^2\mathrm{d}x = \int (b \cdot \nabla)b \cdot u\mathrm{d}x. \tag{14}$$

Testing (3) by b and using (4), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |b|^2\mathrm{d}x + \eta\int |\nabla b|^2\mathrm{d}x = \int (b \cdot \nabla)u \cdot b\mathrm{d}x. \tag{15}$$

Summing up (14) and (15), we have the well-known energy identity

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (\rho|u|^2 + |b|^2)\mathrm{d}x + \eta\int |\nabla b|^2\mathrm{d}x = 0,$$

and hence

$$\int (|u|^2 + |b|^2)\mathrm{d}x + \int_0^T\int |\nabla b|^2\mathrm{d}x\mathrm{d}t \leq C. \tag{16}$$

It is easy to deduce that

$$\|\nabla\rho\|_{L^\infty} \leq \|\nabla\rho_0\|_{L^\infty}\exp\left(\int_0^t \|\nabla u(s)\|_{L^\infty}\mathrm{d}s\right) \leq C. \tag{17}$$

Testing (3) by $|b|^{q-2}b$ $(2<q<\infty)$ and using (4), we derive

$$\begin{aligned}&\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\|b\|_{L^q}^q+\eta\int|b|^{q-2}|\nabla b|^2\mathrm{d}x+\eta\int\frac{1}{2}\nabla|b|^2\cdot\nabla|b|^{q-2}\mathrm{d}x\\ =&\int b\cdot\nabla u\cdot|b|^{q-2}b\mathrm{d}x\le\|\nabla u\|_{L^\infty}\|b\|_{L^q}^q,\end{aligned}$$

and therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\|b\|_{L^q}\le\|\nabla u\|_{L^\infty}\|b\|_{L^q},$$

which gives

$$\|b\|_{L^q}\le\|b_0\|_{L^q}\exp\left(\int_0^t\|\nabla u(s)\|_{L^\infty}\mathrm{d}s\right). \tag{18}$$

Taking $q\to\infty$, one has

$$\|b\|_{L^\infty}\le C. \tag{19}$$

(2) can be rewritten as

$$\partial_t u+u\cdot\nabla u+\frac{1}{\rho}\nabla\pi=\frac{1}{\rho}\mathrm{rot}\,b\times b. \tag{20}$$

Testing (20) by $\nabla\pi$ and using (4), it follows that

$$\begin{aligned}\int\frac{1}{\rho}|\nabla\pi|^2\mathrm{d}x &= -\int u\cdot\nabla u\cdot\nabla\pi\mathrm{d}x+\int\left(\frac{1}{\rho}\mathrm{rot}\,b\times b\right)\cdot\nabla\pi\mathrm{d}x\\ &\le C(\|u\cdot\nabla u\|_{L^2}+\|\mathrm{rot}\,b\times b\|_{L^2})\|\nabla\pi\|_{L^2}\\ &\le C(\|u\|_{L^2}\|\nabla u\|_{L^\infty}+\|b\|_{L^\infty}\|\nabla b\|_{L^2})\|\nabla\pi\|_{L^2},\end{aligned}$$

whence

$$\|\nabla\pi\|_{L^2}\le C\|\nabla u\|_{L^\infty}+C\|\nabla b\|_{L^2}. \tag{21}$$

Taking rot to (20) and denoting the vorticity $\omega:=\mathrm{rot}\,u$, we obtain

$$\partial_t\omega+u\cdot\nabla\omega=\omega\cdot\nabla u-\nabla\frac{1}{\rho}\times\nabla\pi+\mathrm{rot}\left(\frac{\mathrm{rot}\,b}{\rho}\times b\right). \tag{22}$$

Testing (22) by ω and using (4), (17) and (19), we compute

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\omega|^2\mathrm{d}x\\ =&\int\omega\cdot\nabla u\cdot\omega\mathrm{d}x-\int\left(\nabla\frac{1}{\rho}\times\nabla\pi\right)\omega\mathrm{d}x+\int\mathrm{rot}\left(\frac{\mathrm{rot}\,b}{\rho}\times b\right)\cdot\omega\mathrm{d}x\\ \le&\|\nabla u\|_{L^\infty}\int|\omega|^2\mathrm{d}x+\left\|\nabla\frac{1}{\rho}\right\|\|\nabla\pi\|_{L^2}\|\omega\|_{L^2}\\ &+C\|\nabla\rho\|_{L^\infty}\|b\|_{L^\infty}\|\nabla b\|_{L^2}\|\omega\|_{L^2}+C\|b\|_{L^\infty}\|\Delta b\|_{L^2}\|\omega\|_{L^2}+C\|\nabla b\|_{L^4}^2\|\omega\|_{L^2}\\ \le&C\|\nabla u\|_{L^\infty}\int|\omega|^2\mathrm{d}x+C\|\nabla\pi\|_{L^2}\|\omega\|_{L^2}+C\|\nabla b\|_{L^2}\|\omega\|_{L^2}\\ &+\frac{\eta}{4}\|\Delta b\|_{L^2}^2+C\|\omega\|_{L^2}^2.\end{aligned} \tag{23}$$

Here, we have used the Gagliardo–Nirenberg inequality

$$\|\nabla b\|_{L^4}^2\le C\|b\|_{L^\infty}\|\Delta b\|_{L^2}. \tag{24}$$

On the other hand, testing (3) by $-\Delta b$ and using (4) and (19), we achieve

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla b|^2\mathrm{d}x+\eta\int|\Delta b|^2\mathrm{d}x \\ = &\sum_j\int(u\cdot\nabla)b\partial_j^2 b\mathrm{d}x-\int(b\cdot\nabla)u\cdot\Delta b\mathrm{d}x \\ = &-\sum_j\int\partial_j u\cdot\nabla b\cdot\partial_j b\mathrm{d}x-\int b\cdot\nabla u\cdot\Delta b\mathrm{d}x \\ \le &C\|\nabla u\|_{L^\infty}\|\nabla b\|_{L^2}^2+\|b\|_{L^\infty}\|\nabla u\|_{L^2}\|\Delta b\|_{L^2} \\ \le &\frac{\eta}{4}\|\Delta b\|_{L^2}^2+C\|\nabla u\|_{L^\infty}\|\nabla b\|_{L^2}^2+C\|\omega\|_{L^2}^2. \end{aligned} \tag{25}$$

Here, we have used the inequality

$$\|\nabla u\|_{L^r}\le C\|\omega\|_{L^r}\ \text{ with }\ 1<r<\infty. \tag{26}$$

Summing up (23) and (25) and using (21) and the Gronwall inequality, we reach

$$\|\omega\|_{L^2}+\|\nabla b\|_{L^2}\le C. \tag{27}$$

Taking div to (20) and using (4), we observe that

$$-\Delta\pi=f:=\rho\mathrm{div}\,(u\cdot\nabla u)+\rho\nabla\frac{1}{\rho}\cdot\nabla\pi-\rho\mathrm{div}\left(\frac{\mathrm{rot}\,b}{\rho}\times b\right), \tag{28}$$

from which, with (17), (19), (21) and (27), we have

$$\begin{aligned} \|\Delta\pi\|_{L^4} \le\ & \|f\|_{L^4}\le C\|\nabla u\|_{L^\infty}\|\nabla u\|_{L^4}+C\|\nabla\rho\|_{L^\infty}\|\nabla\pi\|_{L^4} \\ & +C\|\nabla\rho\|_{L^\infty}\|b\|_{L^\infty}\|\mathrm{rot}\,b\|_{L^4}+C\|\Delta b\|_{L^4}\|b\|_{L^\infty}+C\|\nabla b\|_{L^8}^2 \\ \le\ & C\|\nabla u\|_{L^\infty}\|\omega\|_{L^4}+C\|\nabla\pi\|_{L^4}+C\|\mathrm{rot}\,b\|_{L^4}+C\|\Delta b\|_{L^4} \\ \le\ & C\|\nabla u\|_{L^\infty}\|\omega\|_{L^4}+C\|\nabla\pi\|_{L^2}^{\frac{4}{7}}\|\Delta\pi\|_{L^4}^{\frac{3}{7}}+C\|\mathrm{rot}\,b\|_{L^4}+C\|\Delta b\|_{L^4} \\ \le\ & \frac{1}{2}\|\Delta\pi\|_{L^4}+C\|\nabla u\|_{L^\infty}\|\omega\|_{L^4}+C\|\nabla u\|_{L^\infty}+C+C\|\mathrm{rot}\,b\|_{L^4}+C\|\Delta b\|_{L^4}, \end{aligned}$$

which yields

$$\|\Delta\pi\|_{L^4}\le C\|\nabla u\|_{L^\infty}\|\omega\|_{L^4}+C\|\nabla u\|_{L^\infty}+C+C\|\mathrm{rot}\,b\|_{L^4}+C|\Delta b\|_{L^4}. \tag{29}$$

Here, we have used the Gagliardo–Nirenberg inequalities

$$\|\nabla b\|_{L^8}^2 \le C\|b\|_{L^\infty}\|\Delta b\|_{L^4}, \tag{30}$$

$$\|\nabla\pi\|_{L^4} \le C\|\nabla\pi\|_{L^2}^{\frac{4}{7}}\|\Delta\pi\|_{L^4}^{\frac{3}{7}}. \tag{31}$$

Testing (22) by $|\omega|^2\omega$ and using (4), (17), (19), (29), (30) and (31), we have

$$\begin{aligned} \frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{L^4}^4 \le\ & C\|\nabla u\|_{L^\infty}\|\omega\|_{L^4}^4+C\left\|\nabla\frac{1}{\rho}\right\|_{L^\infty}\|\nabla\pi\|_{L^4}\|\omega\|_{L^4}^3 \\ & +C(\|b\|_{L^\infty}\|\Delta b\|_{L^4}+\|\nabla b\|_{L^8}^2)\|\omega\|_{L^4}^3+C\left\|\nabla\frac{1}{\rho}\right\|_{L^\infty}\|b\|_{L^\infty}\|\mathrm{rot}\,b\|_{L^4}\|\omega\|_{L^4}^3 \\ \le\ & C\|\nabla u\|_{L^\infty}\|\omega\|_{L^4}^4+C\|\nabla\pi\|_{L^4}\|\omega\|_{L^4}^3+C(\|\Delta b\|_{L^4}+\|\mathrm{rot}\,b\|_{L^4})\|\omega\|_{L^4}^3, \end{aligned}$$

which implies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{L^4}^2 &\leq C\|\nabla u\|_{L^\infty}\|\omega\|_{L^4}^2 + C(\|\nabla u\|_{L^\infty} + 1 + \|\mathrm{rot}\, b\|_{L^4} + \|\Delta b\|_{L^4})\|\omega\|_{L^4} \\ &\leq C\|\nabla u\|_{L^\infty}\|\omega\|_{L^4}^2 + C(\|\nabla u\|_{L^\infty} + 1 + \|\mathrm{rot}\, b\|_{L^4})\|\omega\|_{L^4} + C\|\omega\|_{L^4}^2 + C\|\Delta b\|_{L^4}^2,\end{aligned}$$

and thus

$$\begin{aligned}\|\omega\|_{L^4}^2 \quad \leq \quad & \|\omega_0\|_{L^4}^2 + C\int_0^t [(\|\nabla u\|_{L^\infty} + 1)\|\omega\|_{L^4}^2 \\ & +(\|\nabla u\|_{L^\infty} + 1 + \|\mathrm{rot}\, b\|_{L^4})\|\omega\|_{L^4}]\mathrm{d}s + \int_0^t \|\Delta b\|_{L^4}^2 \mathrm{d}s. \end{aligned} \tag{32}$$

On the other hand, using the $L^2(0,T;W^{2,4})$-theory of the heat equation, it follows that

$$\begin{aligned}\int_0^t \|\Delta b\|_{L^4}^2 \mathrm{d}s \quad &\leq \quad C + C\int_0^t \|u\cdot\nabla b - b\cdot\nabla u\|_{L^4}^2 \mathrm{d}s \\ &\leq \quad C + C\int_0^t (\|u\|_{L^\infty}^2\|\nabla b\|_{L^4}^2 + \|b\|_{L^\infty}^2\|\nabla u\|_{L^4}^2)\mathrm{d}s \\ &\leq \quad C + C\int_0^t (\|u\|_{L^6}^{\frac{4}{3}}\|\nabla u\|_{L^\infty}^{\frac{2}{3}}\|\mathrm{rot}\, b\|_{L^4}^2 + \|\omega\|_{L^4}^2)\mathrm{d}s \\ &\leq \quad C + C\int_0^t (\|\nabla u\|_{L^\infty}^{\frac{2}{3}}\|\mathrm{rot}\, b\|_{L^4}^2 + \|\omega\|_{L^4}^2)\mathrm{d}s. \end{aligned} \tag{33}$$

Here, we have used the inequality

$$\|\nabla b\|_{L^p} \leq C\|\mathrm{rot}\, b\|_{L^p} \ \text{ with } \ 1 < p < \infty. \tag{34}$$

Inserting (33) into (32), we have

$$\begin{aligned}\|\omega\|_{L^4}^2 \quad \leq \quad & C + C\int_0^t (\|\nabla u\|_{L^\infty} +)\|\omega\|_{L^4}^2 \mathrm{d}s + C\int_0^t (\|\nabla u\|_{L^\infty} + \|\mathrm{rot}\, b\|_{L^4})\|\omega\|_{L^4}\mathrm{d}s \\ & +C\int_0^t \|\nabla u\|_{L^\infty}^{\frac{2}{3}}\|\mathrm{rot}\, b\|_{L^4}^2 \mathrm{d}s. \end{aligned} \tag{35}$$

Taking rot to (3) and denoting the current $J := \mathrm{rot}\, b$, we infer that

$$\partial_t J - \eta\Delta J + \mathrm{rot}\,(u\cdot\nabla b - b\cdot\nabla u) = 0. \tag{36}$$

Testing (36) by $|J|^2 J$, using (17), (19) and (34), we derive

$$\begin{aligned}& \frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|J\|_{L^4}^4 + \eta\int |J|^2|\nabla J|^2 \mathrm{d}x + \eta\int \frac{1}{2}\nabla|J|^2\cdot\nabla|J|^2 \mathrm{d}x \\ = \quad & \int (b\cdot\nabla u - u\cdot\nabla b)\mathrm{rot}\,(|J|^2 J)\mathrm{d}x \\ = \quad & \int (b\cdot\nabla u - u\cdot\nabla b)(|J|^2\mathrm{rot}\, J + \nabla|J|^2\times J)\mathrm{d}x \\ \leq \quad & \|b\cdot\nabla u - u\cdot\nabla b\|_{L^4}\|J\|_{L^4}\||J|\cdot|\nabla J|\|_{L^2} \\ \leq \quad & C(\|b\|_{L^\infty}\|\nabla u\|_{L^4} + \|u\|_{L^\infty}\|\nabla b\|_{L^4})\|J\|_{L^4}\||J|\cdot|\nabla J|\|_{L^2} \\ \leq \quad & \frac{\eta}{2}\int |J|^2|\nabla J|^2 \mathrm{d}x + C(\|\omega\|_{L^4}^2 + \|u\|_{L^\infty}^2\|J\|_{L^4}^2)\|J\|_{L^4}^2,\end{aligned}$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\|J\|_{L^4}^2 \leq C\|\omega\|_{L^4}^2 + C\|\nabla u\|_{L^\infty}^{\frac{2}{3}}\|J\|_{L^4}^2.$$

Integrating the above inequality, one has

$$\|\mathrm{rot}\, b\|_{L^4}^2 \leq C + C\int_0^t (\|\omega\|_{L^4}^2 + \|\nabla u\|_{L^\infty}^{\frac{2}{3}} \|\mathrm{rot}\, b\|_{L^4}^2)\mathrm{d}s. \tag{37}$$

Summing up (35) and (37), using the Gronwall inequality, we arrive at

$$\|\nabla u\|_{L^4} + \|\nabla b\|_{L^4} + \int_0^T \|\Delta b\|_{L^4}^2 \mathrm{d}t \leq C, \tag{38}$$

$$\|u\|_{L^\infty} + \int_0^T \|\nabla b\|_{L^\infty}^2 \mathrm{d}t \leq C. \tag{39}$$

Applying Λ^3 to (1), testing by $\Lambda^3\rho$ and using (4) and (10), we obtain

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (\Lambda^3\rho)^2 \mathrm{d}x &= -\int (\Lambda^3(u\cdot\nabla\rho) - u\cdot\nabla\Lambda^3\rho)\Lambda^3\rho \mathrm{d}x \\ &\leq C(\|\nabla u\|_{L^\infty}\|\Lambda^3\rho\|_{L^2} + \|\nabla\rho\|_{L^\infty}\|\Lambda^3 u\|_{L^2})\|\Lambda^3\rho\|_{L^2}. \end{aligned} \tag{40}$$

Applying Λ^3 to (2), testing by $\Lambda^3 u$ and using (1) and (4), we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|\Lambda^3 u|^2 \mathrm{d}x = \int (\Lambda^3(b\cdot\nabla b) - b\cdot\nabla\Lambda^3 b)\Lambda^3 u \mathrm{d}x + \int b\cdot\nabla\Lambda^3 b\cdot\Lambda^3 u\mathrm{d}x \\ &-\int (\Lambda^3(\rho\partial_t u) - \rho\Lambda^3\partial_t u)\Lambda^3 u\mathrm{d}x - \int (\Lambda^3(\rho u\cdot\nabla u) - \rho u\cdot\nabla\Lambda^3 u)\Lambda^3 u\mathrm{d}x \\ = &: I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{41}$$

Applying Λ^3 to (3), testing by $\Lambda^3 b$ and using (4), we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\Lambda^3 b|^2\mathrm{d}x + \eta\int |\Lambda^4 b|^2 \mathrm{d}x \\ = &\int (\Lambda^3(b\cdot\nabla u) - b\cdot\nabla\Lambda^3 u)\Lambda^3 b\mathrm{d}x + \int b\cdot\Lambda^3 u\cdot\Lambda^3 b\mathrm{d}x \\ &-\int (\Lambda^3(u\cdot\nabla b) - u\cdot\nabla\Lambda^3 b)\Lambda^3 b\mathrm{d}x =: I_5 + I_6 + I_7. \end{aligned} \tag{42}$$

Summing up (41) and (42) and noting that $I_2 + I_6 = 0$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (\rho|\Lambda^3 u|^2 + |\Lambda^3 b|^2)\mathrm{d}x + \eta\int |\Lambda^4 b|^2\mathrm{d}x = I_1 + I_3 + I_4 + I_5 + I_7. \tag{43}$$

Using (10) and (11), we bound I_1, I_4, I_5 and I_7 as follows.

$$\begin{aligned} I_1 &\leq C\|\nabla b\|_{L^\infty}(\|\Lambda^3 b\|_{L^2}^2 + \|\Lambda^3 u\|_{L^2}^2); \\ I_4 &\leq C(\|\nabla(\rho u)\|_{L^\infty}\|\Lambda^3 u\|_{L^2} + \|\nabla u\|_{L^\infty}\|\Lambda^3(\rho u)\|_{L^2})\|\Lambda^3 u\|_{L^2} \\ &\leq C(\|\nabla u\|_{L^\infty} + 1)\|\Lambda^3 u\|_{L^2}^2 + C\|\nabla u\|_{L^\infty}(\|\Lambda^3 u\|_{L^2} + \|u\|_{L^\infty}\|\Lambda^3\rho\|_{L^2})\|\Lambda^3 u\|_{L^2} \\ &\leq C(\|\nabla u\|_{L^\infty} + 1)\|\Lambda^3 u\|_{L^2}^2 + C\|\nabla u\|_{L^\infty}\|\Lambda^3\rho\|_{L^2}^2; \\ I_5 + I_7 &\leq C(\|\nabla b\|_{L^\infty}\|\Lambda^3 u\|_{L^2} + \|\nabla u\|_{L^\infty}\|\Lambda^3 b\|_{L^2})\|\Lambda^3 b\|_{L^2}. \end{aligned}$$

To bound I_3, we proceed as follows.

$$\begin{aligned}
I_3 &\leq C(\|\partial_t u\|_{L^\infty}\|\Lambda^3\rho\|_{L^2}+\|\nabla\rho\|_{L^\infty}\|\Lambda^2\partial_t u\|_{L^2})\|\Lambda^3 u\|_{L^2}\\
&\leq C\left\|u\cdot\nabla u+\frac{1}{\rho}\nabla\pi-\frac{\operatorname{rot} b}{\rho}\times b\right\|_{L^\infty}\|\Lambda^3\rho\|_{L^2}\|\Lambda^3 u\|_{L^2}\\
&\quad +C\left\|\Delta\left(u\cdot\nabla u+\frac{1}{\rho}\nabla\pi-\frac{\operatorname{rot} b}{\rho}\times b\right)\right\|_{L^2}\|\Lambda^3 u\|_{L^2}\\
&\leq C(\|\nabla u\|_{L^\infty}+\|\nabla\pi\|_{L^\infty}+\|\nabla b\|_{L^\infty})\|\Lambda^3\rho\|_{L^2}\|\Lambda^3 u\|_{L^2}\\
&\quad +C\Bigg(\|u\|_{L^\infty}\|\Lambda^3 u\|_{L^2}+\|\Lambda^2\nabla\pi\|_{L^2}+\|\nabla\pi\|_{L^3}\left\|\Lambda^2\frac{1}{\rho}\right\|_{L^6}\\
&\quad +\left\|\Delta\frac{1}{\rho}\right\|_{L^6}\|\operatorname{rot} b\|_{L^3}\|b\|_{L^\infty}+\|b\|_{L^\infty}\|\Lambda^3 b\|_{L^2}\Bigg)\|\Lambda^3 u\|_{L^2}\\
&\leq C(\|\nabla u\|_{L^\infty}+\|\nabla\pi\|_{L^\infty}+\|\nabla b\|_{L^\infty})\|\Lambda^3\rho\|_{L^2}\|\Lambda^3 u\|_{L^2}\\
&\quad +C(\|\Lambda^3 u\|_{L^2}+\|\Lambda^2\nabla\pi\|_{L^2}+\|\nabla\pi\|_{L^3}\|\Delta\rho\|_{L^6}+\|\Delta\rho\|_{L^6}+\|\Lambda^3 b\|_{L^2})\|\Lambda^3 u\|_{L^2}\\
&\leq C(\|\nabla u\|_{L^\infty}+\|\nabla\pi\|_{L^\infty}+\|\nabla b\|_{L^\infty})\|\Lambda^3\rho\|_{L^2}\|\Lambda^3 u\|_{L^2}\\
&\quad +C(\|\Lambda^3 u\|_{L^2}+\|\nabla f\|_{L^2}+\|\nabla\pi\|_{L^3}\|\Lambda^3\rho\|_{L^2}+\|\Lambda^3\rho\|_{L^2}+\|\Lambda^3 b\|_{L^2})\|\Lambda^3 u\|_{L^2}. \qquad (44)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\|\nabla\pi\|_{L^\infty} &\leq C\|\nabla\pi\|_{L^2}+C\|\Delta\pi\|_{L^4}\\
&\leq C\|\nabla u\|_{L^\infty}+C+C\|\Delta b\|_{L^4}; \qquad (45)
\end{aligned}$$

$$\begin{aligned}
\|\nabla\Delta\pi\|_{L^2} &= \|\nabla f\|_{L^2}\\
&\leq C\|\nabla\rho\|_{L^\infty}\|\nabla u\|_{L^4}^2+C\|u\|_{L^\infty}\|\Lambda^3 u\|_{L^2}+C\|\nabla\rho\|_{L^\infty}\|\nabla^2\pi\|_{L^2}\\
&\quad +C\|\Delta\rho\|_{L^6}\|\nabla\pi\|_{L^3}+C\|\nabla\rho\|_{L^\infty}(\|\Delta b\|_{L^2}\|b\|_{L^\infty}+\|\nabla b\|_{L^4}^2+\|\nabla\rho\|_{L^\infty}\|\nabla b\|_{L^2})\\
&\quad +C\|b\|_{L^\infty}\|\Lambda^3 b\|_{L^2}+C\left\|\Delta\frac{1}{\rho}\right\|_{L^6}\|b\|_{L^\infty}\|\nabla b\|_{L^3}\\
&\leq C+C\|\Lambda^3 u\|_{L^2}+C\|\nabla^2\pi\|_{L^2}+C\|\Lambda^3\rho\|_{L^2}\|\nabla\pi\|_{L^3}+C\|\Lambda^3 b\|_{L^2}+C\|\Lambda^3\rho\|_{L^2}\\
&\leq \frac{1}{2}\|\nabla f\|_{L^2}+C\|\nabla\pi\|_{L^2}+C+C\|\Lambda^3 u\|_{L^2}\\
&\quad +C\|\Lambda^3\rho\|_{L^2}+C\|\Lambda^3 b\|_{L^2}+C\|\Lambda^3\rho\|_{L^2}\|\nabla\pi\|_{L^3},
\end{aligned}$$

which gives

$$\begin{aligned}
\|\nabla f\|_{L^2} &\leq C+C\|\nabla u\|_{L^\infty}+C\|\Lambda^3\rho\|_{L^2}+C\|\Lambda^3 u\|_{L^2}\\
&\quad +C\|\Lambda^3 b\|_{L^2}+C\|\Lambda^3\rho\|_{L^2}(\|\nabla\pi\|_{L^2}+\|\Delta\pi\|_{L^4})\\
&\leq C+C\|\nabla u\|_{L^\infty}+C\|\Lambda^3(\rho,u,b)\|_{L^2}+C\|\Lambda^3\rho\|_{L^2}(\|\nabla u\|_{L^\infty}+1+\|\Delta b\|_{L^4}).
\end{aligned}$$

Inserting the above estimates into (44), we obtain

$$\begin{aligned}
I_3 &\leq C(\|\nabla u\|_{L^\infty}+1+\|\nabla b\|_{L^\infty}+\|\Delta b\|_{L^4})(\|\Lambda^3\rho\|_{L^2}^2+\|\Lambda^3 u\|_{L^2}^2)+C\|\Lambda^3(\rho,u,b)\|_{L^2}^2\\
&\leq C(\|\nabla u\|_{L^\infty}+1+\|\nabla b\|_{L^\infty}+\|\Delta b\|_{L^4})\|\Lambda^3(\rho,u,b)\|_{L^2}^2.
\end{aligned}$$

Inserting the above estimates of I_1, I_3, I_4, I_5 and I_7 into (43), we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(\rho|\Lambda^3 u|^2+|\Lambda^3 b|^2)\mathrm{d}x+\eta\int|\Lambda^4 b|^2\mathrm{d}x\\
&\leq C(\|\nabla u\|_{L^\infty}+1+\|\nabla b\|_{L^\infty}+\|\Delta b\|_{L^4})\|\Lambda^3(\rho,u,b)\|_{L^2}^2. \qquad (46)
\end{aligned}$$

Summing up (40) and (46) and using the Gronwall inequality, we conclude that

$$\|\Lambda^3(\rho,u,b)\|_{L^2}+\int_0^T\int|\Lambda^4 b|^2\mathrm{d}x\mathrm{d}t\leq C.$$

This completes the proof. □

3. Conclusions

In this paper, we prove a refined blow-up criterion for the inhomogeneous incompressible MHD system with zero viscosity, which is important and can be used for the simulation of MHD. For $\rho = 1$ and $\eta = 0$, Caflisch et al. [12] showed the following regularity criterion:

$$\operatorname{rot} u, \operatorname{rot} b \in L^1(0, T; L^\infty). \tag{47}$$

Since the problem is very challenging, we are unable to present further developments.

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