mathematics

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The Adjoint of α-Times-Integrated C-Regularized Semigroups

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Abstract: We consider an operator $\{S(t)\}_{t\geq 0}$ on a Banach space X with generator A, characterized by being an α-times-integrated C-regularized semigroup. The adjoint family $S^*(t) : X^* \to X^*$ is introduced for analysis. $\{S^*(t)\}_{t\geq 0}$ maintains the characteristics of an α-times-integrated C-regularized semigroup, though with strong continuity and Bochner integrals being substituted by weak* continuity and weak* integrals, respectively. Our investigation focuses on the closed subspace $X^{\odot}$, where $\{S^*(t)\}_{t\geq 0}$ exhibits strong continuity. Additionally, a comparison between the adjoint A^* of A and the generator of the adjoint family is conducted.

Keywords: α-times-integrated C-regularized semigroup; adjoint of α-times-integrated C-regularized semigroup; semigroup generator

MSC: 34K05; 47D06; 47D62

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Citation: Bachar, M. The Adjoint of α-Times-Integrated C-Regularized Semigroups. *Mathematics* **2024**, *12*, 1561. https://doi.org/10.3390/math12101561

Academic Editors: Vladimir P. Maksimov and Alexander Domoshnitsky

Received: 8 April 2024
Revised: 11 May 2024
Accepted: 13 May 2024
Published: 16 May 2024

1. Introduction

If A is the infinitesimal generator of a linear, strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ in a Banach space X, then for all $f \in \mathbf{L}^1([0,\infty), X)$, there exists a unique, strongly continuous solution to the integral equation

$$u(t) = A\int_0^t u(s)\,ds + \int_0^t f(s)\,ds. \tag{1}$$

Of course, this equation can (at least formally) be considered as the integrated version of the differential equation

$$u'(t) = Au(t) + f(t), u(0) = 0. \tag{2}$$

There are cases when (1) admits a solution only if f is sufficiently regular. One may require regularity in space, for instance:

$$f(t) = Cg(t), \;\; g \in \mathbf{L}^1([0,\infty), X), \tag{3}$$

where $C : X \to X$ is a bounded linear operator. In the context of partial differential equations, one may think of an operator C whose range consists of functions that are sufficiently regular in space. On the other hand, one may require time regularity, such as:

$$f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s)\,ds, \quad g \in \mathbf{L}^1([0,\infty), X), \tag{4}$$

which means that f is the fractional integral of order α of an $\mathbf{L}^1$-function g.

In the case of spatial regularity given by Equation (3), one arrives at the concept of a C-regularized semigroup (see, e.g., [1]). In the case of time regularity described by Equation (4), we obtain an α-times-integrated semigroup (see, e.g., [2,3] for integer α and [4–7] for fractional α). If both types of regularization are to be combined, we finally obtain an α-times-integrated, C-regularized semigroup, see [8–12]. For deeper insights into the properties of the resolvent families of the semigroup, we recommend exploring the works on resolvent families and abstract Volterra equations in locally convex spaces [13,14]. These studies offer particularly relevant and insightful perspectives on the corresponding resolvent families. Now, let X^* denote the dual space of X, and let $A^* : D(A^*) \to X^*$ be the adjoint operator of A. The dual operators $\{T^*(t)\}_{t\geq 0}$ of a strongly continuous linear semigroup generated by A satisfy the semigroup property again, but $T^*(t)x^*$ depends on t continuously only with respect to the weak* topology on X^*. The properties of such dual semigroups are well established [15–19]. In particular, there is a weakly* dense, closed subspace $X^{\odot} \subset X^*$ such that the restriction of $\{T^*(t)\}_{t\geq 0}$ to $X^{\odot}$ is strongly continuous in t. The generator of this semigroup is simply the part of A^* with values in $X^{\odot}$. Moreover, $X^{\odot}$ is the closure in the norm of X^* of the domain $D(A^*)$. If X is reflexive, then $X^{\odot}$ and X^* coincide, and $\{T^*(t)\}_{t\geq 0}$ is a strongly continuous semigroup on X^*, generated by A^*. Dual semigroups play a crucial role when numerics and control problems involving semigroups are considered.

In this paper, we generalize this concept to α-times-integrated C-regularized semigroups $\{S(t)\}_{t\geq 0}$. It is not surprising that this is possible. The interesting part is which additional assumptions are needed to make the machinery work. In order to define a single-valued generator of the α-times-integrated C-regularized semigroups, we require that $S(t)$ be nondegenerate (i.e., $S(t)x \equiv 0$ only if $x = 0$). The adjoint family $\{S^*(t)\}_{t\geq 0}$ is nondegenerate if and only if both $D(A)$ and $Rg(C)$ are dense subspaces of X. We can define the subspace of strong continuity $X^{\odot}$. Again, $X^{\odot}$ contains the closure of $D(A^*)$, and also we have $\overline{D(A^*)} = X^{\odot}$. If $A^{\odot}$ is the part of A^* in $X^{\odot}$, and $S^{\odot}(t) = S^*(t)_{|_{X^{\odot}}}$, then $A^{\odot}$ is a subset of the generator of $\{S^{\odot}(t)\}_{t\geq 0}$. To prove equality, we require the additional assumption that $D(A) \cap Rg(C)$ be dense in $D(A)$ with respect to the graph norm of A. This condition, of course, holds always for strongly continuous semigroups. We do not know whether this condition is necessary for equality.

The following sections of this paper provide a comprehensive exploration of these topics. Section 2 introduces the definition and basic properties of the adjoint family $\{S^*(t)\}_{t\geq 0}$, as well as the properties of α-times-integrated C-regularized semigroups in terms of the weak* topology. Section 3 explores whether the adjoint family can become nondegenerate. In Section 4, we discuss the relations between the generator of $\{S^*(t)\}_{t\geq 0}$ and the adjoint A^* of A. Finally, the theory of the subspace of strong continuity $X^{\odot}$ and its implications for reflexive spaces are given in Section 5.

2. Strongly Continuous α-Times-Integrated C-Regularized Semigroups

We begin by introducing the definition and properties of α-times-integrated C-regularized semigroups. In this paper, X will be a Banach space, and the space $B(X)$ will denote the space of bounded linear operators on X. This definition has been introduced by several investigators; for further details, see [8,9,20].

Definition 1 ([8,9,20])**.** *Let $\alpha \geq 0$ and $C \in B(X)$. A linear family of operators $\{S(t)\}_{t\geq 0} \subset B(X)$ is called an α-times-integrated C-regularized semigroup on X if it satisfies:*

(1) For all $x \in X$, $S(0)x = \begin{cases} Cx & \text{if } \alpha = 0, \\ 0 & \text{otherwise.} \end{cases}$

(2) $S(t)C = CS(t)$ for $t \geq 0$.

(3) $S(\cdot)x : [0, \infty) \to X$ is continuous for each $x \in X$.

(4) $S(t)S(s)x = \begin{cases} S(t+s)Cx & \text{if } \alpha = 0 \text{ and } x \in X, \\ \frac{1}{\Gamma(\alpha)}\left(\int_t^{s+t} - \int_0^s\right)(s+t-r)^{\alpha-1}S(r)Cxdr & \text{otherwise} \end{cases}$
for all $x \in X$ and $t, s \geq 0$.

Moreover, $\{S(t)\}_{t\geq 0}$ is said to be nondegenerate if $S(t)x = 0$ for all $t > 0$ implies $x = 0$.

The lemma referenced in Theorem 5 [8], Proposition 2.2 [21], and in the work by [10] can be found below.

Lemma 1 ([8,10,21])**.** *Suppose $\{S(t)\}_{t\geq 0}$ is a nondegenerate α-times-integrated C-regularized semigroup. Then, C is injective. Furthermore, for $\{S(t)\}_{t\geq 0}$ to be nondegenerate, it is necessary (and sufficient in the case of $\alpha = 0$) for C to be injective.*

The next definition outlines the characterization of the generator of the nondegenerate α-times-integrated C-regularized semigroup as presented in Definition 6 [8].

Definition 2 ([8])**.** *Let $\alpha \geq 0$, and $\{S(t)\}_{t\geq 0}$ be a nondegenerate α-times-integrated C-regularized semigroup. The generator A of $S(t)$ is defined by the following property: $x \in D(A)$ and $Ax = y$ if and only if*

$$S(t)x = \frac{t^{\alpha}}{\Gamma(\alpha+1)}Cx + \int_0^t S(s)y, ds \tag{5}$$

holds for all $t \geq 0$.

The assumption that $\{S(t)\}_{t\geq 0}$ is nondegenerate ensures that the operator A is well defined. The well-known properties of the generator of a nondegenerate α-times-integrated C-regularized semigroup $\{S(t)\}_{t\geq 0}$ can be found in Theorems 7, 8 [8].

Lemma 2 ([8])**.** *Let A be the generator of a nondegenerate α-times-integrated C-regularized semigroup $\{S(t)\}_{t\geq 0}$. Then,*

(a) A is a closed linear operator.
(b) For any $x \in D(A)$ and $t \geq 0$, $S(t)x \in D(A)$ and $AS(t)x = S(t)Ax$.
(c) $C^{-1}AC = A$.

3. Nondegeneracy of the Adjoint Family

Now, we turn to the adjoint family. In the subsequent analysis, X^* will denote the dual space of X. We will utilize the concept of the weak*-integral: if $f^* : [a,b] \to X^*$ is a function such that $\langle f^*, x\rangle$ is integrable for all $x \in X$, then the weak*-integral of f^* is defined by the property

$$\langle \text{weak}^* \int_a^b f^*(s)\, ds, x\rangle = \int_a^b \langle f^*(s), x\rangle\, ds \ \text{ for all } \ x \in X.$$

If $T : D(T) \to X$ is a closed, densely defined operator on X, then $T^* : D(T^*) \to X^*$ will denote the adjoint operator. The following properties of the adjoint operator are well known, see, for example, [19,22].

Lemma 3 ([19,22])**.** *Let $T : D(T) \subset X \to X$ be a closed, densely defined operator, and let $T^* : D(T^*) \subset X^* \to X^*$ be its adjoint. Then,*

(a) T^* is weakly*-closed.
(b) T^* is closed with respect to the norm topology in X^*.
(c) $D(T^*)$ is dense with respect to the weak*-topology in X^*.
(d) If X is reflexive, then $D(T^*)$ is dense with respect to the norm topology in X^*.

In the forthcoming discussion, we will explain the details of finding the adjoint family for the semigroup $\{S(t)\}_{t\geq 0}$. We will carefully look at its properties and explain why they are important for our mathematical analysis.

Definition 3. *Let $\{S(t)\}_{t\geq 0}$ be an α-times-integrated C-regularized semigroup on a Banach space X. The family $\{S^*(t)\}_{t\geq 0}$ is called the adjoint family of $\{S(t)\}_{t\geq 0}$.*

The following lemma can be easily proven through straightforward calculation.

Lemma 4. *Let $\alpha \geq 0$, $\{S(t)\}_{t\geq 0}$ be an α-times-integrated C-regularized semigroup on a Banach space X and let $\{S^*(t)\}_{t\geq 0}$ be the adjoint family. Then,*

(a) *For all* $x^* \in X^*$, $S^*(0)x^* = \begin{cases} C^*x^* \text{ if } \alpha = 0 \\ 0 \text{ else.} \end{cases}$

(b) $S^*(t)C^* = C^*S^*(t)$ *for all* $t \geq 0$.

(c) *For each* $x^* \in X^*$, *the map* $S^*(\cdot)x^* : [0, \infty) \to X^*$ *is continuous with respect to the weak***-topology in* X^*.

(d) *For* $x^* \in X^*$ *and* $t, s \geq 0$,

$$S^*(t)S^*(s)x^* =$$
$$\begin{cases} S^*(s+t)C^*x^* & \text{if } \alpha = 0, \\ weak^*[\int_t^{s+t} - \int_0^s] \frac{1}{\Gamma(\alpha)}(s+t-r)^{\alpha-1}S^*(r)C^*x^*dr & \text{else.} \end{cases}$$

Let us define the nondegenerate adjoint of an α-times-integrated C-regularized semigroup.

Definition 4. *Consider an* α*-times-integrated* C*-regularized semigroup* $\{S(t)\}_{t\geq 0}$ *on a Banach space* X*, and let* $\{S^*(t)\}_{t\geq 0}$ *be its adjoint family. We say that* $\{S^*(t)\}_{t\geq 0}$ *is nondegenerate if* $S^*(t)x^* = 0$ *for all* $t > 0$ *implies that* $x^* = 0$.

However, it is worth noting that the adjoint of a nondegenerate α-times-integrated C-regularized semigroup may not always be nondegenerate, as illustrated in the following example:

Example 1. *Let* $X = \ell_1 = \{(x_n) \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n| < \infty\}$*. For* $x = (x_n) \in X$*, we define*

$$(S(t)x)_i = \begin{cases} tx_{i/2} & , \quad \text{even } i \\ 0 & , \quad \text{odd } i. \end{cases}$$

and

$$(Cx)_i = \begin{cases} x_{i/2} & , \quad \text{even } i \\ 0 & , \quad \text{odd } i. \end{cases} \tag{6}$$

Then, $\{S(t)\}_{t\geq 0}$ *forms a nondegenerate one-time-integrated* C*-regularized semigroup. Moreover,* $X^* = \ell_\infty = \{(x_n^*) \subset \mathbb{R} : \sup_{n\in\mathbb{N}} |x_n^*| < \infty\}$ *and for* $x^* = (x_n^*) \in X^*$,

$$(S^*(t)x^*)_i = (tx_{2i}^*)$$

and

$$(C^*x^*)_i = (x_{2i}^*).$$

In this case, $\{S^*(t)\}_{t\geq 0}$ *is a* 1*-times-integrated* C^**-regularized semigroup on the Banach space* X^**. However, it is degenerate because there exists* $x^* = (1, 0, 0, \dots)' \neq 0$ *in* X^* *such that* $S^*(t)x^* = 0$ *for all* $t > 0$.

Remark 1. *This example can be extended to the case where* $\alpha \neq 0$ *as follows:* $S(t) = \frac{t^\alpha}{\Gamma(\alpha+1)}C$*, where the operator* C *is defined by (6).*

To characterize integrated regularized semigroups with nondegenerate adjoints, we need to introduce the following lemma.

Lemma 5. *Suppose* $\alpha \geq 0$*, and* $\{S(t)\}_{t\geq 0}$ *is a nondegenerate* α*-times-integrated* C*-regularized semigroup on a Banach space* X *with generator* A*. Let* $\epsilon \in (0, \infty]$ *and define*

$$W_\epsilon = span(\{S(t)x \mid t \in (0, \epsilon), x \in X\}).$$

Then, W_ϵ is a dense subspace of X if and only if both the domain of A and the range of C are dense in X.

Proof. Suppose W_ϵ is dense for any fixed $\epsilon \in (0, \infty]$. Let $x \in X$ and $\delta > 0$ be arbitrary. Then, there exist $n \in \mathbb{N}$, $t_1, t_2, \ldots, t_n \in (0, \epsilon)$, $y_1, y_2, \ldots, y_n \in X$ such that

$$\|x - \sum_{i=1}^{n} S(t_i) y_i\| \leq \frac{\delta}{2}.$$

Now, for each $i = 1, 2, \ldots, n$, let $M = sup(\|S(t_i)\|, 1)$. Then, there exist

$$m_i \in \mathbb{N}; s_{i,1}, s_{i,2}, \ldots, s_{i,m_i} \in (0, \epsilon); z_{i,1}, z_{i,2}, \ldots, z_{i,m_i} \in X$$

such that

$$\|y_i - \sum_{j=1}^{m_i} S(s_{i,j}) z_{i,j}\| \leq \frac{\delta}{2Mn}.$$

Therefore,

$$\|x - \sum_{i=1}^{n} \sum_{j=1}^{m_i} S(t_i) S(s_{i,j}) z_{i,j}\| \leq \quad \|x - \sum_{i=1}^{n} S(t_i) y_i\| + \sum_{i=1}^{n} \|S(t_i)\| \|y_i - \sum_{j=1}^{m_i} S(s_{i,j}) z_{i,j}\| \leq \delta.$$

However, each

$$S(t_i) S(s_{i,j}) z_{i,j} = \frac{1}{\Gamma(\alpha)} C\{\int_{t_i}^{t_i + s_{i,j}} - \int_0^{s_{i,j}}\} (t_i + s_{i,j} - r)^{\alpha - 1} S(r) z_{i,j} dr \in Rg(C).$$

Thus, we conclude that x can be approximated by a sequence in $Rg(C)$.

To prove that $D(A)$ is dense, it is sufficient to show that for $x \in X$ and $t > 0$, the vector $S(t)x$ can be approximated by elements in $D(A)$. This implies that every vector in the dense subspace W_ϵ can be approximated by elements in $D(A)$. We choose a sequence of functions $\rho_n \in \mathcal{C}^\infty([0, \infty), [0, \infty))$ with supports contained in $(0, 1/n)$ such that $\int_0^\infty \rho_n(s) ds = 1$. By the strong continuity of $S(t)x$ with respect to t, we obtain

$$y_n := \int_0^t \rho_n(t - s) S(s) x \, ds \to S(t)x \text{ as } n \to \infty.$$

All we have to show is that $y_n \in D(A)$. If we define

$$h_n(t) = \int_0^t \rho_n'(t - s) \frac{s^{\alpha - 1}}{\Gamma(\alpha)} ds, f_n(t) = C h_n(t), u_n(t) = \int_0^t \rho_n'(t - s) S(s) ds,$$

we notice that

$$\rho_n(t) = \int_0^t \frac{(t - s)^{-\alpha}}{\Gamma(1 - \alpha)} h_n(s) ds$$
$$y_n = \int_0^t u_n(s) \, ds.$$

By utilizing the properties of convolution and Laplace transforms, as demonstrated in Theorem 10 [8], we conclude that u_n solves the equation

$$u_n(t) = A \int_0^t u_n(s) \, ds + \int_0^t f_n(s) \, ds = A y_n + \int_0^t f_n(s) \, ds.$$

In particular, $y_n \in D(A)$.

Conversely, assuming that $Rg(C)$ and $D(A)$ are dense, let $x \in X$ and $\delta > 0$. Let $\epsilon > 0$. Pick $0 < t < \epsilon$, and $y \in D(A), z = Ay$ such that

$$\|x - Cy\| < \delta.$$

Then, we have

$$S(t)y = \frac{t^\alpha}{\Gamma(\alpha+1)}Cy + \int_0^t S(s)zds.$$

Thus,

$$Cy = \frac{\Gamma(\alpha+1)}{t^\alpha}\{S(t)y - \int_0^t S(s)zds\} \in \overline{W_\epsilon}.$$

□

In the upcoming theorem, we provide a necessary and sufficient condition for the adjoints of α-times-integrated C-regularized semigroups on a Banach space X to be nondegenerate.

Theorem 1. *Let $\alpha \geq 0$ and $\{S(t)\}_{t\geq 0}$ be a nondegenerate C-regularized α-times-integrated semigroup on a Banach space X with generator A. Let $\{S^*(t)\}_{t\geq 0}$ be its adjoint. Then, $\{S^*(t)\}_{t\geq 0}$ is nondegenerate if and only if both the domain of A and the range of C are dense.*

Proof. Let $x^* \in X^*$. Then, $S^*(t)x^* = 0$ for all $t > 0$ if and only if $\langle x^*, S(t)y\rangle = 0$ for all $t > 0$, and $y \in X$, which is equivalent to $\langle x^*, z\rangle = 0$ for all $z \in W_\infty$, where W_∞ is taken from Lemma 5. Therefore, $\{S^*(t)\}_{t\geq 0}$ is nondegenerate if and only if $x^* = 0$ is the only functional annihilating all of W_∞. This is equivalent to the assertion that W_∞ is dense, and by using Lemma 5, the result follows. □

4. The Adjoint of the Generator

In this section, we will examine the relationship between the adjoint of the generator of an α-times-integrated C-regularized semigroup and the weak* generator of the adjoint family. It is important to note that the adjoint operator A^* of the generator operator A of $\{S(t)\}_{t\geq 0}$ is well defined because the domain of A is densely defined, given our assumption that the adjoint semigroup $\{S^*(t)\}_{t\geq 0}$ is nondegenerate.

Theorem 2. *Let $\alpha \geq 0$ and $\{S(t)\}_{t\geq 0}$ be a nondegenerate α-times-integrated C-regularized semigroup, such that the adjoint $\{S^*(t)\}_{t\geq 0}$ is also nondegenerate. Let A be the generator of $\{S(t)\}_{t\geq 0}$ and A^* be its adjoint. Then,*

(a) If $x^* \in D(A^*)$ and $t > 0$, then $S^*(t)x^* \in D(A^*)$ and $A^*S^*(t)x^* = S^*(t)A^*x^*$.
(b) If $x^* \in D(A^*)$, then $C^*x^* \in D(A^*)$ and $A^*C^*x^* = C^*A^*x^*$.
Moreover, if $D(A) \cap Rg(C)$ is dense in $D(A)$ with respect to the graph norm of A, then
(c) If $C^*x^* \in D(A^*)$ and $A^*C^*x^* = C^*y^*$, then $x^* \in D(A^*)$ and $A^*x^* = y^*$.

Proof.

(a) Let $x^* \in D(A^*)$ and $x \in D(A)$ be arbitrary. Then, for any fixed $t > 0$, we have

$$\begin{aligned}\langle S^*(t)x^*, Ax\rangle &= \langle x^*, S(t)Ax\rangle = \langle x^*, AS(t)x\rangle \\ &= \langle S^*(t)A^*x^*, x\rangle\end{aligned}$$

This implies $S^*(t)x^* \in D(A^*)$ and $A^*S^*(t)x^* = S^*(t)A^*x^*$.

(b) Similarly as (a).

(c) Let $x \in D(A)$. Choose a sequence $x_n \in X$ such that $Cx_n \to x$ and $ACx_n \to Ax$. Note that $x_n \in D(A)$ and $CAx_n = ACx_n$, as shown in [8]. We have

$$\langle x^*, CAx_n\rangle = \langle C^*x^*, Ax_n\rangle = \langle A^*C^*x^*, x_n\rangle = \langle C^*y^*, x_n\rangle = \langle y^*, Cx_n\rangle.$$

In the limit, $\langle x^*, Ax\rangle = \langle y^*, x\rangle$, implying $A^*x^* = y^*$.
□

Theorem 3. *Let $\alpha \geq 0$ and $\{S(t)\}_{t\geq 0}$ be a nondegenerate α-times-integrated C-regularized semigroup, with its adjoint $\{S^*(t)\}_{t\geq 0}$ also being nondegenerate. Let A denote the generator of $\{S(t)\}_{t\geq 0}$ and A^* its adjoint.*

(a) If $x^* \in D(A^*)$ and $A^*x^* = y^*$, then for all $t > 0$,

$$S^*(t)x^* = \frac{t^\alpha}{\Gamma(\alpha+1)}C^*x^* + weak^* \int_0^t S^*(s)y^*\, ds. \tag{7}$$

(b) Suppose $D(A) \cap Rg(C)$ is dense in $D(A)$ with respect to the graph norm of A. If $x^*, y^* \in X^*$ such that (7) holds for all $t > 0$, then $x^* \in D(A^*)$ with $A^*x^* = y^*$.

Proof. First, let $y^* = A^*x^*$. Take any $x \in D(A)$. Then,

$$\begin{aligned}
\langle S^*(t)x^*, x\rangle &= \langle x^*, S(t)x\rangle \\
&= \langle x^*, \frac{t^\alpha}{\Gamma(\alpha+1)}Cx + A\int_0^t S(s)x\, ds\rangle \\
&= \frac{t^\alpha}{\Gamma(\alpha+1)}\langle C^*x^*, x\rangle + \int_0^t \langle S^*(s)A^*x^*, x\rangle\, ds \\
&= \langle \frac{t^\alpha}{\Gamma(\alpha+1)}C^*x^* + weak^* \int_0^t S^*y^*\, ds, x\rangle.
\end{aligned}$$

Since this holds for all x in the dense subspace $D(A)$, Equation (7) follows.

To prove (b), assume that $Rg(C) \cap D(A)$ is dense in $D(A)$ with respect to the graph norm of A. Let x^* and y^* satisfy (7). If $x \in D(A)$, we have

$$\begin{aligned}
&\langle y^*, \int_0^t S(s)x\, ds\rangle = \langle weak^* \int_0^t S^*(s)y^*\, ds, x\rangle \\
&= \langle S^*(t)x^* - \frac{t^\alpha}{\Gamma(\alpha+1)}C^*x^*, x\rangle = \langle x^*, S(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}Cx\rangle \\
&= \langle x^*, \int_0^t S(s)Ax\, ds\rangle.
\end{aligned}$$

Consequently, for all $x \in D(A)$ and $s > 0$, we have $\langle y^*, S(s)x\rangle = \langle x^*, S(s)Ax\rangle$. Now, let $x \in D(A)$ be arbitrary. Take a sequence $x_n \in D(A)$ such that $Cx_n \to x$ and $ACx_n \to Ax$. Fix some $t > 0$. Then,

$$\begin{aligned}
&\frac{t^\alpha}{\Gamma(\alpha+1)}\langle y^*, Cx_n\rangle = \langle y^*, S(t)x_n - \int_0^t S(s)Ax_n\, ds\rangle \\
&= \langle x^*, S(t)Ax_n - A\int_0^t S(s)Ax_n\, ds\rangle = \frac{t^\alpha}{\Gamma(\alpha+1)}\langle x^*, CAx_n\rangle.
\end{aligned}$$

Taking the limit for $n \to \infty$, we obtain $\langle y^*, x\rangle = \langle x^*, Ax\rangle$. Therefore, $y^* = A^*x^*$. □

5. The Subspace of Strong Continuity

The adjoint of a semigroup, which combines two mathematical operations, is typically continuous over time only in relation to a specific type of topology. We introduce the concept of a special subspace, known as the "*sun space*", to address this in the context of semigroup adjoints. The adjoint of an α-times-integrated C-regularized semigroup typically exhibits continuity over time solely concerning the weak* topology in X^*. To address this, we incorporate the concept of the subspace of strong continuity, denoted as $X^\odot$ or sometimes referred to as the "*sun space*", from the theory of adjoint of strongly continuous semigroups.

Definition 5. *Let $\{S(t)\}_{t\geq 0}$ be a nondegenerate, C-regularized, α-times-integrated semigroup with generator A. Assume that $D(A)$ and $Rg(C)$ are dense in X. Let $\{S(t)\}_{t\geq 0}$ be the adjoint family. We define*

$$X^{\odot} := \{x^* \in X^* \mid S^*(t)x^* \text{ is strongly continuous in } t\}. \tag{8}$$

Moreover, $\{S^{\odot}(t)\}_{t\geq 0}$ denotes the restriction $\{S^*(t)\mid_{X^{\odot}}$, and $A^{\odot}$ denotes the part of A^* in $X^{\odot}$, where $\overline{D(A^{\odot})} = X^{\odot}$, i.e., $y^* = A^{\odot}x^*$ if $x^*, y^* \in X^{\odot}$ and $y^* = A^*x^*$.

The following theorem explains important properties of nondegenerate semigroups that are α-times-integrated C-regularized, along with their adjoints. It introduces a special space called $X^{\odot}$, which shows how the adjoint family $\{S^*(t)\}_{t\geq 0}$ remains continuous over time. This theorem also shows that $X^{\odot}$ stays the same under specific operations and describes how the generator of the adjoint semigroup, denoted as B, equals the adjoint of the generator A, denoted as $A^{\odot}$. Furthermore, it clarifies the conditions when $X^{\odot}$ matches the weak*-closure of the domain of A^*. Overall, this theorem provides a thorough understanding of how adjoint semigroups behave and their structure concerning α-times-integrated C-regularized semigroups in Banach spaces, where domains and ranges are dense.

Theorem 4. *Let $\alpha \geq 0$ and $\{S(t)\}_{t\geq 0}$ be a nondegenerate, α-times-integrated C-regularized semigroup with generator A. Assume that $D(A)$ and $Rg(C)$ are dense in X, where $C \in B(X)$. Let $\{S^*(t)\}_{t\geq 0}$ be the adjoint family, and let A^* and C^* be the adjoints of A and C, respectively. Then,*

(a) $X^{\odot}$ is (norm-)closed and $D(A^*)$ is a weakly-dense, linear subspace of X^*.*
(b) $X^{\odot}$ is invariant under $S^*(t)$ and C^*.
(c) The restriction $\{S^{\odot}(t)\}_{t\geq 0}$ is a strongly continuous, α-times-integrated, C^*-regularized semigroup. If B is the generator of $\{S^{\odot}(t)\}_{t\geq 0}$, then $A^{\odot} = B$ in the sense that for all $x^* \in D(A^{\odot})$ we have $x^* \in D(B)$ and $Bx^* = A^{\odot}x^*$.
(d) If $D(A) \cap Rg(C)$ is dense in $D(A)$ with respect to the graph norm of A, then $A^{\odot}$ is the generator of $\{S^{\odot}(t)\}_{t\geq 0}$, and we have $D(A^{\odot}) \subset D(A^*)$. Moreover, $\overline{D(A^*)} = X^{\odot}$

Proof.

(a) It is clear that $X^{\odot}$ is a linear subspace of X^*. The closedness of $X^{\odot}$ follows easily from the uniform boundedness of the operators $S^*(t)$ for t in compact intervals. The weak* density will follow from $D(A^*) \subset X^{\odot}$ (to be proven in (d)) and the weak* density of $D(A^*)$ by using Lemma 3(c).
(b) To prove invariance under C^*, note that $S^*(t)C^*x = C^*S^*(t)x$, which is continuous in t if $S^*(t)x$ is continuous. Invariance under $S^*(t)$ follows similarly by using Lemma 4(d).
(c) Since $S^*(t)x$ is continuous in t for $x \in X^{\odot}$, the weak* integrals in Lemma 4 and in (7) are in fact Bochner integrals. Lemma 4 implies then that $\{S^{\odot}\}_{t\geq 0}$ is an α-times-integrated, C-regularized semigroup. If $y^* = A^{\odot}x^*$, then by Theorem 3, the pair (x^*, y^*) satisfies (7) with a Bochner integral. This is the defining equation for the generator of $\{S^{\odot}(t)\}_{t\geq 0}$, so that $y^* = Bx^*$.
(d) Now, let $D(A) \cap Rg(C)$ be dense in $D(A)$ with respect to the graph norm of A. Then, by using Theorem 3(b), we will have that, if $x^*, y^* \in X^{\odot}$, $y^* = A^*x^*$ (i.e., $y^* = A^{\odot}x^*$), if and only if (7) holds. The latter is equivalent to $y^* = Bx^*$, and we have $D(A^{\odot}) \subset D(A^*)$.
To prove that $\overline{D(A^*)} = X^{\odot}$. Let us consider $x^* \in D(A^*)$ and $C \in B(X)$. Then, there exists constants $c > 0$ and $M > 0$ such that, for any $x \in D(A)$, such that

$$\mid \langle x, Ax\rangle \mid \& \leq c \parallel x \parallel \quad \text{and} \mid \langle x, Cx\rangle \mid \& \leq M \parallel x \parallel .$$

Then, for $0 < s < t$, we have

$$
\begin{aligned}
& \mid \langle S^*(t)x^* - S^*(s)x^*, x \rangle \mid = \mid \langle \frac{t^\alpha - s^\alpha}{\Gamma(\alpha+1)} C^*x^* + weak^* \int_s^t S^*(r)A^*x^* dr, x \rangle \mid \\
\leq \quad & \frac{t^\alpha - s^\alpha}{\Gamma(\alpha+1)} \mid \langle x^*, Cx \rangle \mid + \int_s^t \mid \langle x^*, S(r)Ax \rangle \mid \; dr \\
\leq \quad & \frac{t^\alpha - s^\alpha}{\Gamma(\alpha+1)} M\|x\| + c(t-s) \sup_{0 \leq r \leq t} \|S(r)\| \|x\|.
\end{aligned}
$$

As $t \to s$, the estimate above goes to 0. Thus, $x^* \in X^\odot$ and hence $D(A^*) \subset X^\odot$. Since $X^\odot$ is closed in X^*, then $\overline{D(A^*)} \subset X^\odot$. By using the first part of (a), the fact that $D(A^\odot)$ is dense in $X^\odot$, and $D(A^\odot) \subset D(A^*) \subset X^\odot$, then $\overline{D(A^*)} = X^\odot$.

□

Corollary 1. *Let $\alpha \geq 0$ and $\{S(t)\}_{t \geq 0}$ be an α-times-integrated, C-regularized semigroup on a reflexive Banach space X with a densely defined generator A and with dense range $Rg(C)$. Then, $X^\odot = X^*$; in particular, the adjoint family $\{S^*(t)\}_{t \geq 0}$ is a strongly continuous, α-times-integrated, C^*-regularized semigroup on X^*. Moreover, let A^* be the adjoint operator of A and let B denote the generator of $\{S^*(t)\}_{t \geq 0}$. Then, for all $x^* \in D(A^*)$, we have $x^* \in D(B)$ with $Bx^* = A^*x^*$. If $D(A) \cap Rg(C)$ is dense in $D(A)$ with respect to the graph norm of A, then $A^* = B$.*

Proof. For reflexive spaces, the weak and weak* topologies are the same. Hence, $X^\odot$ is a weakly dense subspace. However, for convex sets, the weak and the norm closures are the same, and $X^\odot$ is closed in the norm topology. Thus, $X^\odot = X^*$. The remaining part of the corollary is a direct application of Theorem 4. □

Remark 2. *We have the following remark:*

- *If A has a nonempty resolvent, the hypothesis that $D(A) \cap Rg(C)$ is dense can be replaced by the weakest hypothesis that $C(D(A))$ is dense. In fact, let us say $\lambda \in \rho(A)$; it would follow that for $x \in D(A)$, take $x_n \in Rg(C)$ converging to $(\lambda - A)x$ and then:*

$$C(\lambda - A)^{-1}C^{-1}x_n = (\lambda - A)^{-1}x_n \to x$$

and also we will have

$$
\begin{aligned}
A(C(\lambda - A)^{-1}C^{-1}x_n) &= (A - \lambda + \lambda)(C(\lambda - A)^{-1}C^{-1}x_n) \\
&= -x_n + \lambda(C(\lambda - A)^{-1}C^{-1}x_n) \to -(\lambda - A)x + \lambda x = Ax,
\end{aligned}
$$

and we can note that

$$C(\lambda - A)^{-1}C^{-1}x_n \in C(D(A)) \subseteq D(A) \cap Rg(C).$$

- *The condition $C(D(A))$ is dense can be replaced by the condition that the range of C is dense. This can be immediate by the fact that C is bounded and $D(A)$ is dense. In fact, for any $x \in X$, one has $\int_0^t S(s)xds \in D(A)$ and $A \int_0^t S(s)xds = S(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}Cx$. By using the strong continuity, we have $\lim_{t \downarrow 0} \frac{1}{t} \int_0^t S(s)xds = x$, and then the result follows.*

- *Let* $\alpha \geq 0$ *and* $\{S(t)\}_{t\geq 0}$ *be a nondegenerate* α*-times-integrated* C*-regularized semigroup such that the adjoint* $\{S^*(t)\}_{t\geq 0}$ *is also nondegenerate. Let* A *be the generator of* $S(t)$ *and* A^* *be its adjoint. If* $x^* \in X^*$ *and* $t > 0$*, then*

$$\begin{aligned} &weak^* \int_0^t S^*(s)x^*\, ds \in D(A^*) \quad with \\ &A^*\left(weak^* \int_0^t S^*(s)x^*\, ds\right) = S^*(t)x^* - \frac{t^\alpha}{\Gamma(\alpha+1)} C^* x^*. \end{aligned}$$

In fact, if we pick an arbitrary $x \in D(A)$*, then*

$$\begin{aligned} &\langle weak^* \int_0^t S^*(s)x^*\, ds, Ax\rangle \\ &= \langle x^*, \int_0^t S(s)Ax\, ds\rangle = \langle x^*, S(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}Cx\rangle \\ &= \langle S^*(t)x^* - \frac{t^\alpha}{\Gamma(\alpha+1)}C^*x^*, x\rangle. \end{aligned}$$

- *Let* $\beta \geq 0$ *and* $\{S(t)\}_{t\geq 0}$ *be a* β*-times integrated,* C*-regularized semigroup on a reflexive Banach space* X *with a densely defined generator* A *and with dense range* $Rg(C)$*. For any* $\alpha > 0$*, we define*

$$T^*(t)x^* := D_t^{-\alpha} S^*(t)x^* = weak^* \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} S^*(\tau)x^* d\tau, \qquad for\ all\ x^* \in X^*, \tag{9}$$

where $D_t^{-\alpha}S^*(t)$ *is the fractional integral of* S^* *of order* α *(see, for instance, [8,23]). Then, we have that* $\{T^*(t)\}_{t\geq 0}$ *is an* $(\alpha+\beta)$*-times-integrated* C*-regularized semigroup on Banach space* X^* *with generator* A^**. In fact, from Theorem 15 [8], we have*

$$\begin{aligned} &\langle T^*(t)x^*, x\rangle = \langle x^*, T(t)x\rangle = \langle x^*, \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}Cx + A\int_0^t T(s)x\, ds\rangle \\ &= \langle x^*, A\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} S(s)x\, ds\rangle \\ &= \langle weak^* \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} S^*(\tau)x^* d\tau, x\rangle \end{aligned}$$

and also:

$$\begin{aligned} &\langle S^*(t)x^*, x\rangle = \langle x^*, S(t)x\rangle = \langle x^*, \frac{t^\beta}{\Gamma(\beta+1)}Cx + A\int_0^t S(s)x\, ds\rangle \\ &= \langle \frac{t^\beta}{\Gamma(\beta+1)}C^*x^* + weak \int_0^t S^*(s)A^*x^* ds, x\rangle \end{aligned}$$

Then, by using the fractional integral definition, the result follows.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Acknowledgments: The author would like to extend sincere appreciation to the Researchers Supporting Program for funding this work under Researchers Supporting Project number RSPD2024R963, King Saud University, Riyadh, Saudi Arabia.

Conflicts of Interest: The author declare no conflicts of interest.

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