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On Generalized Bivariate (p,q)-Bernoulli–Fibonacci Polynomials and Generalized Bivariate (p,q)-Bernoulli–Lucas Polynomials

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Abstract: Many properties of special polynomials, such as recurrence relations, sum formulas, and symmetric properties, have been studied in the literature with the help of generating functions and their functional equations. In this paper, we define the generalized (p,q)-Bernoulli–Fibonacci and generalized (p,q)-Bernoulli–Lucas polynomials and numbers by using the (p,q)-Bernoulli numbers, unified (p,q)-Bernoulli polynomials, $h(x)$-Fibonacci polynomials, and $h(x)$-Lucas polynomials. We also introduce the generalized bivariate (p,q)-Bernoulli–Fibonacci and generalized bivariate (p,q)-Bernoulli–Lucas polynomials and numbers. Then, we derive some properties of these newly established polynomials and numbers by using their generating functions with their functional equations. Finally, we provide some families of bilinear and bilateral generating functions for the generalized bivariate (p,q)-Bernoulli–Fibonacci polynomials.

Keywords: q-Bernoulli numbers; (p,q)-Bernoulli numbers; unified (p,q)-Bernoulli polynomials; $h(x)$-Fibonacci polynomials; generating functions

MSC: 05A19; 11B37; 11B39; 11B83

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1. Introduction

Special polynomials and special numbers are frequently used in many branches of mathematics, especially in areas such as mathematical physics, mathematical modeling, difference equations, and analytical number theory. With the help of the generating functions of these polynomials and numbers, some identities, sum formulas, and symmetric identities containing these polynomials have been obtained. Many special numbers and special polynomials including Fibonacci and Lucas numbers have been studied with interest by mathematicians from past to present. For $n \geq 2$, Fibonacci and Lucas numbers [1] are defined by

$$F_n = F_{n-1} + F_{n-2},$$

and

$$L_n = L_{n-1} + L_{n-2},$$

with the initial values $F_0 = 0$, $F_1 = 1$, $L_0 = 2$, and $L_1 = 1$. In [2], Nalli and Haukkanen defined the $h(x)$-Fibonacci polynomials and $h(x)$-Lucas polynomials, including the Fibonacci polynomials, Pell polynomials, Lucas polynomials, and Pell–Lucas polynomials. Let $h(x)$

be a polynomial with real coefficients. The recurrence relations of the $h(x)$-Fibonacci polynomials and $h(x)$-Lucas polynomials are defined by

$$F_{n,h}(x) = h(x)F_{n-1,h}(x) + F_{n-2,h}(x), \quad n \geq 2$$

and

$$L_{n,h}(x) = h(x)L_{n-1,h}(x) + L_{n-2,h}(x), \quad n \geq 2$$

where $F_{0,h}(x) = 0$, $F_{1,h}(x) = 1$, $L_{0,h}(x) = 2$, and $L_{1,h}(x) = h(x)$. They derived the generating functions of $h(x)$-Fibonacci polynomials and $h(x)$-Lucas polynomials as follows:

$$\sum_{n=0}^{\infty} F_{n,h}(x)\zeta^n = \frac{\zeta}{1 - h(x)\zeta - \zeta^2},$$

and

$$\sum_{n=0}^{\infty} L_{n,h}(x)\zeta^n = \frac{2 - h(x)\zeta}{1 - h(x)\zeta - \zeta^2}.$$

For more information on Fibonacci- and Lucas-type polynomials, numbers, and their applications, for example, in the theory of geometric functions, see [3–15]. On the other hand, the Bernoulli numbers B_n are defined with the help of the following generating function as [16]

$$\sum_{n=0}^{\infty} B_n \frac{\zeta^n}{n!} = \frac{\zeta}{e^{\zeta} - 1}, \quad |\zeta| < 2\pi.$$

In [17], Rahmani defined the p-Bernoulli numbers by means of the following generating function as

$$\sum_{n=0}^{\infty} B_{n,p} \frac{\zeta^n}{n!} = 2F1(1,1;p+2;1-e^{\zeta}), \tag{1}$$

where $p \geq -1$ integer and $2F1(a,b;c;u)$ denotes the Gaussian hypergeometric function defined by

$$2F1(a,b;c;u) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{u^n}{n!},$$

and $(x)_n$ denotes the shifted factorial defined by $(x)_n = x(x+1)(x+2)\ldots(x+n-1)$ for $n > 0$, $(x)_0 = 1$, and x any real or complex number. Substituting $p = 0$ into (1), $B_{n,0} = B_n$, ordinary Bernoulli numbers are obtained. Moreover, Rahmani gave some important properties of p-Bernoulli numbers. Rahmani also defined the p-Bernoulli polynomials as follows:

$$\sum_{n=0}^{\infty} B_{n,p}(x) \frac{\zeta^n}{n!} = e^{x\zeta}\, 2F1(1,1;p+2;1-e^{\zeta}). \tag{2}$$

Substituting $x = 0$ into (2), $B_{n,p}(0) = B_{n,p}$, p-Bernoulli numbers are obtained. The readers can also see [18]. After Rahmani, Pathan [19] generalized these numbers and polynomials called (p,q)-Bernoulli numbers and (p,q)-Bernoulli polynomials, respectively. The (p,q)-Bernoulli numbers are defined by means of the following generating function as

$$2F1(1,q+1;p+2;1-e^{\zeta}) = \sum_{n=0}^{\infty} B_{n,p,q} \frac{\zeta^n}{n!}. \tag{3}$$

For $q = 0$, (3) reduces to (1). Moreover, the author introduced the unified (p,q)-Bernoulli polynomials defined by

$$e^{x\zeta}\, 2F1(1,q+1;p+2;1-e^{\zeta}) = \sum_{n=0}^{\infty} B_{n,p,q}(x) \frac{\zeta^n}{n!}, \tag{4}$$

for every integer $p \geq -1$. For $x = 0$, (4) reduces to (3).

In the light of the above paper, with the help of the (p,q)-Bernoulli numbers, unified (p,q)-Bernoulli polynomials, $h(x)$-Fibonacci polynomials, and $h(x)$-Lucas polynomials, we define the generalized (p,q)-Bernoulli–Fibonacci and generalized (p,q)-Bernoulli–Lucas polynomials and numbers. We also introduce the generalized bivariate (p,q)-Bernoulli–Fibonacci and generalized bivariate (p,q)-Bernoulli–Lucas polynomials and numbers. Then, we derive some properties of these newly established polynomials and numbers by using their generating functions with their functional equations. Finally, we provide some families of bilinear and bilateral generating functions for the generalized bivariate (p,q)-Bernoulli–Fibonacci polynomials.

2. Generalized (p,q)-Bernoulli–Fibonacci and Generalized (p,q)-Bernoulli–Lucas Polynomials and Numbers

In this part of the paper, we introduce the generalized (p,q)-Bernoulli–Fibonacci polynomials and generalized (p,q)-Bernoulli–Lucas polynomials. Then, we derive some properties of these polynomials by using the their generating functions.

Definition 1. *The generalized (p,q)-Bernoulli–Fibonacci polynomials ${}_BF_{n,h,p,q}(x)$ are given by the following generating function:*

$$\zeta(1-h(x)\zeta-\zeta^2)^{-1}\,2F1(1,q+1;p+2;1-e^{\zeta})=\sum_{n=0}^{\infty}{}_BF_{n,h,p,q}(x)\frac{\zeta^n}{n!},\tag{5}$$

for every integer $p\geq -1$.

Some special cases of the generalized (p,q)-Bernoulli–Fibonacci polynomials ${}_BF_{n,h,p,q}(x)$ are as follows:

- Setting $q=0$ into (5), ${}_BF_{n,h,p,0}(x)={}_B\,F_{n,h,p}(x)$, generalized p-Bernoulli–Fibonacci polynomials are obtained.
- Setting $h(x)=x$ into (5), generalized (p,q)-Bernoulli–Fibonacci polynomials become (p,q)-Bernoulli–Fibonacci polynomials.
- Setting $h(x)=1$ into (5), generalized (p,q)-Bernoulli–Fibonacci polynomials become (p,q)-Bernoulli–Fibonacci numbers.
- Setting $h(x)=2x$ into (5), generalized (p,q)-Bernoulli–Fibonacci polynomials become (p,q)-Bernoulli–Pell polynomials.
- Setting $h(x)=2$ into (5), generalized (p,q)-Bernoulli–Fibonacci polynomials become (p,q)-Bernoulli–Pell numbers.

Definition 2. *The generalized (p,q)-Bernoulli–Lucas polynomials ${}_BL_{n,h,p,q}(x)$ are given by the following generating function:*

$$(2-h(x)\zeta)(1-h(x)\zeta-\zeta^2)^{-1}\,2F1(1,q+1;p+2;1-e^{\zeta})=\sum_{n=0}^{\infty}{}_BL_{n,h,p,q}(x)\frac{\zeta^n}{n!}.\tag{6}$$

Some special cases of the generalized (p,q)-Bernoulli–Lucas polynomials ${}_BL_{n,h,p,q}(x)$ are as follows:

- Setting $q=0$ into (6), ${}_BL_{n,h,p,0}(x)={}_B\,L_{n,h,p}(x)$, generalized p-Bernoulli–Lucas polynomials are obtained.
- Setting $h(x)=x$ into (6), generalized (p,q)-Bernoulli–Lucas polynomials become (p,q)-Bernoulli–Lucas polynomials.
- Setting $h(x)=1$ into (6), generalized (p,q)-Bernoulli–Lucas polynomials become (p,q)-Bernoulli–Lucas numbers.
- Setting $h(x)=2x$ into (6), generalized (p,q)-Bernoulli–Lucas polynomials become (p,q)-Bernoulli–Pell–Lucas polynomials.

- Setting $h(x) = 2$ into (6), generalized (p,q)-Bernoulli–Lucas polynomials become (p,q)-Bernoulli–Pell–Lucas numbers.

  We can rewrite (5) as

$$\sum_{n=0}^{\infty} {}_BF_{n,h,p,q}(x)\frac{\zeta^n}{n!} = \sum_{n=0}^{\infty} F_{n,h}(x)\zeta^n \sum_{j=0}^{\infty} B_{j,p,q}\frac{\zeta^j}{j!}.$$

Comparing the coefficients of ζ^n on both sides of the above equation, we have

$${}_BF_{n,h,p,q}(x) = n!\sum_{j=0}^{n} F_{n-j,h}(x)\frac{B_{j,p,q}}{j!}.$$

Similarly, we may reformulate (6) as

$$\sum_{n=0}^{\infty} {}_BL_{n,h,p,q}(x)\frac{\zeta^n}{n!} = \sum_{n=0}^{\infty} L_{n,h}(x)\zeta^n \sum_{m=0}^{\infty} B_{m,p,q}\frac{\zeta^m}{m!}.$$

Thus, we have

$${}_BL_{n,h,p,q}(x) = n!\sum_{m=0}^{n} L_{n-m,h}(x)\frac{B_{m,p,q}}{m!}.$$

Theorem 1. *The representation of (p,q)-Bernoulli numbers in terms of generalized (p,q)-Bernoulli–Fibonacci polynomials is*

$$\frac{B_{n,p,q}}{n!} = \frac{{}_BF_{n+1,h,p,q}(x)}{(n+1)!} - \frac{h(x)\,{}_BF_{n,h,p,q}(x)}{n!} - \frac{{}_BF_{n-1,h,p,q}(x)}{(n-1)!}, \quad n \geq 1.$$

Proof. By using (5), we have

$$2F1(1,q+1;p+2;1-e^{\zeta}) = (1-h(x)\zeta-\zeta^2)\sum_{n=0}^{\infty} {}_BF_{n,h,p,q}(x)\frac{\zeta^{n-1}}{n!}$$

$$\sum_{n=0}^{\infty} B_{n,p,q}\frac{\zeta^n}{n!} = (1-h(x)\zeta-\zeta^2)\sum_{n=0}^{\infty} {}_BF_{n,h,p,q}(x)\frac{\zeta^{n-1}}{n!}.$$

Comparing the coefficients of ζ^n, we obtain the desired result. □

Theorem 2. *For $n \geq 1$, we have*

$${}_BF_{n,h,p,q}(x) = n!\sum_{m=0}^{n}\sum_{i=0}^{[\frac{m-1}{2}]}\binom{m-i-1}{i}\frac{B_{n-m,p,q}}{(n-m)!}h^{m-2i-1}(x), \tag{7}$$

where $\left|h(x)\zeta+\zeta^2\right| < 1$.

Proof. Using (5), we obtain

$$\zeta(1-h(x)\zeta-\zeta^2)^{-1}\,2F1(1,q+1;p+2;1-e^{\zeta})$$

$$
\begin{aligned}
&= \zeta\, 2F1(1, q+1; p+2; 1-e^{\zeta}) \sum_{n=0}^{\infty} (h(x)\zeta + \zeta^2)^n \\
&= \zeta\, 2F1(1, q+1; p+2; 1-e^{\zeta}) \sum_{n=0}^{\infty} \sum_{i=0}^{n} \binom{n}{i} (h(x)\zeta)^{n-i} (\zeta^2)^i \\
&= 2F1(1, q+1; p+2; 1-e^{\zeta}) \sum_{n=0}^{\infty} \sum_{i=0}^{n} \binom{n}{i} (h(x)\zeta)^{n-i} (\zeta^{2i+1}).
\end{aligned}
$$

On writing $n+i+1=m$ in the right hand-side of the above equation, we have

$$
\zeta(1-h(x)\zeta-\zeta^2)^{-1}\, 2F1(1, q+1; p+2; 1-e^{\zeta})
$$

$$
= 2F1(1, q+1; p+2; 1-e^{\zeta}) \sum_{m=0}^{\infty} \left[\sum_{i=0}^{[\frac{m-1}{2}]} \binom{m-i-1}{i} h^{m-2i-1}(x) \right] \zeta^m
$$

$$
\sum_{n=0}^{\infty} {}_B F_{n,h,p,q}(x) \frac{\zeta^n}{n!} = \sum_{n=0}^{\infty} B_{n,p,q} \frac{\zeta^n}{n!} \sum_{m=0}^{\infty} \left[\sum_{i=0}^{[\frac{m-1}{2}]} \binom{m-i-1}{i} h^{m-2i-1}(x) \right] \zeta^m.
$$

Replace n with $n-m$ and compare the coefficients of ζ^n to obtain the result (7). □

Theorem 3. *For $n \geq 2$, we have*

$$
\begin{aligned}
B_{n,p,q} \frac{2}{n!} &= h(x)_B F_{n,h,p,q}(x) \frac{1}{n!} + {}_B L_{n,h,p,q}(x) \frac{1}{n!} \\
&\quad - h(x) \left[h(x)_B F_{n-1,h,p,q}(x) \frac{1}{(n-1)!} + {}_B L_{n-1,h,p,q}(x) \frac{1}{(n-1)!} \right] \\
&\quad - \left[h(x)_B F_{n-2,h,p,q}(x) \frac{1}{(n-2)!} + {}_B L_{n-2,h,p,q}(x) \frac{1}{(n-2)!} \right],
\end{aligned}
\tag{8}
$$

and

$$
{}_B L_{n,h,p,q}(x) \frac{1}{n!} = {}_B F_{n+1,h,p,q}(x) \frac{2}{(n+1)!} - h(x)_B F_{n,h,p,q}(x) \frac{1}{n!}.
\tag{9}
$$

Proof. Through the following equation, we have

$$
2(1-h(x)\zeta-\zeta^2)^{-1}\, 2F1(1, q+1; p+2; 1-e^{\zeta}) = h(x) \sum_{n=0}^{\infty} {}_B F_{n,h,p,q}(x) \frac{\zeta^n}{n!} + \sum_{n=0}^{\infty} {}_B L_{n,h,p,q}(x) \frac{\zeta^n}{n!}.
$$

So, we obtain

$$
\begin{aligned}
2\sum_{n=0}^{\infty} B_{n,p,q} \frac{\zeta^n}{n!} &= (1-h(x)\zeta-\zeta^2) \left[h(x) \sum_{n=0}^{\infty} {}_B F_{n,h,p,q}(x) \frac{\zeta^n}{n!} + \sum_{n=0}^{\infty} {}_B L_{n,h,p,q}(x) \frac{\zeta^n}{n!} \right] \\
&= h(x) \sum_{n=0}^{\infty} {}_B F_{n,h,p,q}(x) \frac{\zeta^n}{n!} + \sum_{n=0}^{\infty} {}_B L_{n,h,p,q}(x) \frac{\zeta^n}{n!} \\
&\quad - h(x)\zeta \left[h(x) \sum_{n=0}^{\infty} {}_B F_{n,h,p,q}(x) \frac{\zeta^n}{n!} + \sum_{n=0}^{\infty} {}_B L_{n,h,p,q}(x) \frac{\zeta^n}{n!} \right] \\
&\quad - \zeta^2 \left[h(x) \sum_{n=0}^{\infty} {}_B F_{n,h,p,q}(x) \frac{\zeta^n}{n!} + \sum_{n=0}^{\infty} {}_B L_{n,h,p,q}(x) \frac{\zeta^n}{n!} \right].
\end{aligned}
$$

Comparing the coefficients of ζ^n, we obtain the result (8). By virtue of (6), we obtain

$$
(2-h(x)\zeta)(1-h(x)\zeta-\zeta^2)^{-1}\, 2F1(1, q+1; p+2; 1-e^{\zeta}) = \sum_{n=0}^{\infty} {}_B L_{n,h,p,q}(x) \frac{\zeta^n}{n!}
$$

$$2\sum_{n=0}^{\infty} {}_{B}F_{n,h,p,q}(x)\frac{\zeta^{n-1}}{n!} - h(x)\sum_{n=0}^{\infty} {}_{B}F_{n,h,p,q}(x)\frac{\zeta^{n}}{n!} = \sum_{n=0}^{\infty} {}_{B}L_{n,h,p,q}(x)\frac{\zeta^{n}}{n!}.$$

Comparing the coefficients of ζ^n, we obtain the our assertion (9). □

3. Generalized Bivariate (p,q)-Bernoulli–Fibonacci and Generalized Bivariate (p,q)-Bernoulli–Lucas Polynomials and Numbers

Definition 3. *The generalized bivariate (p,q)-Fibonacci–Bernoulli polynomials are defined by the following generating function as*

$$\zeta(1-h(x)\zeta-\zeta^2)^{-1}e^{y\zeta}\,2F1(1,q+1;p+2;1-e^{\zeta}) = \sum_{n=0}^{\infty} {}_{B}F_{n,h,p,q}(x,y)\frac{\zeta^{n}}{n!}, \tag{10}$$

for every integer $p \geq -1$.

Letting $y = 0$ in (10), ${}_{B}F_{n,h,p,q}(x,0) = {}_{B}F_{n,h,p,q}(x)$, the generalized (p,q)-Bernoulli–Fibonacci polynomials are obtained.

Definition 4. *The generalized bivariate (p,q)-Lucas–Bernoulli polynomials are defined by the following generating function as*

$$(2-h(x)\zeta)(1-h(x)\zeta-\zeta^2)^{-1}e^{y\zeta}\,2F1(1,q+1;p+2;1-e^{\zeta}) = \sum_{n=0}^{\infty} {}_{B}L_{n,h,p,q}(x,y)\frac{\zeta^{n}}{n!}, \tag{11}$$

for every integer $p \geq -1$.

Letting $y = 0$ in (11), ${}_{B}L_{n,h,p,q}(x,0) = {}_{B}L_{n,h,p,q}(x)$, the generalized (p,q)-Bernoulli–Lucas polynomials are obtained.

Theorem 4. *The following summation formula holds true:*

$${}_{B}F_{n,h,p,q}(x) = \sum_{m=0}^{n}\binom{n}{m}(-y)^m\,{}_{B}F_{n-m,h,p,q}(x,y), \tag{12}$$

and

$${}_{B}L_{n,h,p,q}(x) = \sum_{m=0}^{n}\binom{n}{m}(-y)^m\,{}_{B}L_{n-m,h,p,q}(x,y). \tag{13}$$

Proof. By using (10), we have

$$\begin{aligned}
\zeta(1-h(x)\zeta-\zeta^2)^{-1}\,2F1(1,q+1;p+2;1-e^{\zeta}) &= e^{-y\zeta}\sum_{n=0}^{\infty} {}_{B}F_{n,h,p,q}(x,y)\frac{\zeta^{n}}{n!} \\
&= \sum_{m=0}^{\infty}(-y)^m\frac{\zeta^m}{m!}\sum_{n=0}^{\infty} {}_{B}F_{n,h,p,q}(x,y)\frac{\zeta^{n}}{n!} \\
\sum_{n=0}^{\infty} {}_{B}F_{n,h,p,q}(x)\frac{\zeta^{n}}{n!} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-y)^m\,{}_{B}F_{n-m,h,p,q}(x,y)\frac{\zeta^{n}}{m!(n-m)!}.
\end{aligned}$$

Using the Cauchy product and comparing the coefficients of ζ^n, we obtain (12). The proof of (13) is similar. □

Theorem 5. *Let $p \geq -1$. The following representations for generalized bivariate (p,q)-Bernoulli–Fibonacci polynomials and generalized bivariate (p,q)-Bernoulli–Lucas polynomials involving Euler polynomials $E_n(x)$ hold true:*

$$\begin{aligned} {}_BF_{n,h,p,q}(x,y) &= \frac{1}{2}\Bigg[\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}\binom{m}{k}E_{n-m}(y)\,{}_BF_{m-k,h,p,q}(x) \\ &\quad + \sum_{m=0}^{n}\binom{n}{m}E_{n-m}(y)\,{}_BF_{m,h,p,q}(x)\Bigg], \end{aligned} \tag{14}$$

and

$$\begin{aligned} {}_BL_{n,h,p,q}(x,y) &= \frac{1}{2}\Bigg[\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}\binom{m}{k}E_{n-m}(y)\,{}_BL_{m-k,h,p,q}(x) \\ &\quad + \sum_{m=0}^{n}\binom{n}{m}E_{n-m}(y)\,{}_BL_{m,h,p,q}(x)\Bigg]. \end{aligned} \tag{15}$$

Proof. The generating function for the Euler polynomials $E_n(y)$ gives

$$e^{y\zeta} = \frac{e^{\zeta}+1}{2}\sum_{n=0}^{\infty}E_n(y)\frac{\zeta^n}{n!}.$$

Substituting this values of $e^{y\zeta}$ in (10) gives

$$\frac{e^{\zeta}+1}{2}\sum_{n=0}^{\infty}E_n(y)\frac{\zeta^n}{n!}\zeta(1-h(x)\zeta-\zeta^2)^{-1}\,2F1(1,q+1;p+2;1-e^{\zeta})$$
$$= \sum_{n=0}^{\infty}{}_BF_{n,h,p,q}(x,y)\frac{\zeta^n}{n!}$$
$$\frac{1}{2}\left[\sum_{m=0}^{\infty}\frac{\zeta^m}{m!}\sum_{n=0}^{\infty}E_n(y)\frac{\zeta^n}{n!}\sum_{n=0}^{\infty}{}_BF_{n,h,p,q}(x)\frac{\zeta^n}{n!}+\sum_{n=0}^{\infty}E_n(y)\frac{\zeta^n}{n!}\sum_{n=0}^{\infty}{}_BF_{n,h,p,q}(x)\frac{\zeta^n}{n!}\right]$$
$$= \sum_{n=0}^{\infty}{}_BF_{n,h,p,q}(x,y)\frac{\zeta^n}{n!}.$$

Using the Cauchy product and comparing the coefficients of ζ^n, we obtain (14). The proof of (15) is similar. □

4. Some Families of Generating Functions for the Generalized Bivariate (p,q)-Bernoulli–Fibonacci and Generalized Bivariate (p,q)-Bernoulli–Lucas Polynomials

In this section, we derive bilinear and bilateral generating functions for the generalized bivariate (p,q)-Bernoulli–Fibonacci polynomials by using some methods that were used earlier in [20] (see also [21–24]).

Theorem 6. *Suppose that $\Lambda_\mu(\mathbf{T})$ is an identically non-vanishing function of m complex variables $t_1,\ldots,t_s$ $(m \in \mathbb{N})$ and of complex order μ. Additionally, let the function $\Lambda_\mu(\mathbf{T})$ have the following generating function:*

$$\begin{aligned} \Upsilon_{\mu,\nu}(\mathbf{T};w) &: = \sum_{k=0}^{\infty}a_k\Lambda_{\mu+\nu k}(\mathbf{T})w^k \\ &\quad (a_k \neq 0;\ \ \mu,\nu \in \mathbb{C};\ \mathbf{T} = (t_1,\ldots,t_s);\ s \in \mathbb{N}). \end{aligned}$$

Then, for $\Psi_{n,r,\mu,\nu}(x,y;\mathbf{T};h)$ *given by*

$$\Psi_{n,r,\mu,\nu}(x,y;\mathbf{T};h) := \sum_{k=0}^{[n/r]} a_k \frac{{}_BF_{n-rk,h,p,q}(x,y)}{(n-rk)!} \Lambda_{\mu+\nu k}(\mathbf{T})h^k, \tag{16}$$

$$\begin{aligned} &\sum_{n=0}^{\infty} \Psi_{n,r,\mu,\nu}\Big(x,y;\mathbf{T};\frac{\varpi}{v^r}\Big)v^n \\ = \quad & \Upsilon_{\mu,\nu}(\mathbf{T};\varpi)v(1-h(x)v-v^2)^{-1}e^{yv}\ 2F1(1,q+1;p+2;1-e^v), \end{aligned} \tag{17}$$

where $n,r \in \mathbb{N}$.

Proof. By substituting

$$\Psi_{n,r,\mu,\nu}\Big(x,y;\mathbf{T};\frac{\varpi}{v^r}\Big)$$

from the definition (16) into the left-hand side of (17), we can write the following form of the left-hand side of the equality (17) of Theorem 6:

$$\sum_{n=0}^{\infty} \Psi_{n,r,\mu,\nu}\Big(x,y;\mathbf{T};\frac{\varpi}{v^r}\Big)v^n = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/r]} a_k \frac{{}_BF_{n-rk,h,p,q}(x,y)}{(n-rk)!} \Lambda_{\mu+\nu k}(\mathbf{T})\varpi^k v^{n-rk},$$

which, upon replacing n with $n+rk$, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \Psi_{n,r,\mu,\nu}\Big(x,y;\mathbf{T};\frac{\varpi}{v^r}\Big)v^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_k \frac{{}_BF_{n,h,p,q}(x,y)}{n!} \Lambda_{\mu+\nu k}(\mathbf{T})\varpi^k v^n \\ &= \left(\sum_{k=0}^{\infty} a_k\Lambda_{\mu+\nu k}(\mathbf{T})\varpi^k\right)\left(\sum_{n=0}^{\infty} {}_BF_{n,h,p,q}(x,y)\frac{v^n}{n!}\right) \\ &= \Upsilon_{\mu,\nu}(\mathbf{T};\varpi)v(1-h(x)v-v^2)^{-1}e^{yv}\ 2F1(1,q+1;p+2;1-e^v), \end{aligned}$$

which is the right-hand side of the generating function (17) asserted by Theorem 6. □

To give some examples of the generating functions expressed by Theorem 6 above, we first set

$$s = 1 \text{ and } \Lambda_{\mu+\nu k}(\gamma) = B_{\mu+\nu k,p}(\gamma) \quad (k,\mu,\nu \in \mathbb{N}_0)$$

in Theorem 6. Here, $B_{n,p}(x)$ denotes the p-Bernoulli polynomials defined by (2). Thus, we deduce from Theorem 6 the following result, which provides a class of bilateral generating functions for the p-Bernoulli polynomials and the generalized bivariate (p,q)-Bernoulli–Fibonacci polynomials.

Corollary 1. *If* $\Upsilon_{\mu,\nu}(\gamma;w) := \sum_{k=0}^{\infty} a_k B_{\mu+\nu k,p}(\gamma)w^k, (a_k \neq 0,\ \ k,\mu,\nu \in \mathbb{N}_0)$, *and*

$$W_{n,r,\mu,\nu}(x,y;\gamma;\zeta) := \sum_{k=0}^{[n/r]} a_k \frac{{}_BF_{n-rk,h,p,q}(x,y)}{(n-rk)!} B_{\mu+\nu k,p}(\gamma)\zeta^k$$

where $n,\mu,\nu \in \mathbb{N}_0$; $r \in \mathbb{N}$, *then*

$$\begin{aligned} &\sum_{n=0}^{\infty} W_{n,r,\mu,\nu}\Big(x,y;\gamma;\frac{u}{t^r}\Big)t^n \\ = \quad & \Upsilon_{\mu,\nu}(\gamma;u)t(1-h(x)t-t^2)^{-1}e^{yt}\ 2F1(1,q+1;p+2;1-e^t). \end{aligned}$$

Remark 1. *Using (10) and taking $a_k = \frac{1}{k!}$, $\mu = 0$, $\nu = 1$, we have*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/r]} \frac{{}_BF_{n-rk,h,p,q}(x,y)}{(n-rk)!}\frac{B_{k,p}(\gamma)}{k!}u^k t^{n-rk}$$
$$= e^{\gamma u}\, 2F1(1,1;p+2;1-e^u)t(1-h(x)t-t^2)^{-1}e^{yt}\, 2F1(1,q+1;p+2;1-e^t).$$

Finally, in terms of the generalized bivariate (p,q)-Fibonacci–Bernoulli polynomials ${}_BF_{n,h,p,q}(x,y)$ generated by (10), we set

$$s = 2 \text{ and } \Lambda_{\mu+\nu k}(x_1,y_1) = {}_BF_{n,h,p,q}(x_1,y_1)$$

in Theorem 6. We find that the following class of bilinear generating functions for the bivariate polynomials ${}_BF_{n,h,p,q}(x,y)$.

Corollary 2. *If $\Upsilon_{\mu,\nu}(x_1,y_1;w) := \sum_{k=0}^{\infty} a_k\, {}_BF_{n,h,p,q}(x_1,y_1)w^k$, $(a_k \neq 0,\ \mu,\nu \in \mathbb{N}_0)$, and*

$$W_{n,r,\mu,\nu}(x,y;x_1,y_1;\zeta) := \sum_{k=0}^{[n/r]} a_k \frac{{}_BF_{n-rk,h,p,q}(x,y)}{(n-rk)!}\, {}_BF_{\mu+\nu k,h,p,q}(x_1,y_1)\zeta^k$$

where $n,\mu,\nu \in \mathbb{N}_0$ and $r \in \mathbb{N}$, then

$$\begin{aligned}&\sum_{n=0}^{\infty} W_{n,r,\mu,\nu}\left(x,y;x_1,y_1;\frac{u}{t^r}\right)t^n \\ =\ & \Upsilon_{\mu,\nu}(x_1,y_1;u)t(1-h(x)t-t^2)^{-1}e^{yt}\, 2F1(1,q+1;p+2;1-e^t).\end{aligned}$$

Remark 2. *By virtue of (10), and if we set $a_k = \frac{1}{k!}$, $\mu = 0$, $\nu = 1$, we have*

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{k=0}^{[n/r]} \frac{{}_BF_{n-rk,h,p,q}(x,y)}{(n-rk)!}\frac{{}_BF_{k,h,p,q}(x_1,y_1)}{k!}u^k t^{n-rk} \\ =\ & u(1-h(x_1)u-u^2)^{-1}e^{y_1 u}\, 2F1(1,q+1;p+2;1-e^u) \\ & \times t(1-h(x)t-t^2)^{-1}e^{yt}\, 2F1(1,q+1;p+2;1-e^t).\end{aligned}$$

5. Conclusions

In this paper, using the (p,q)-Bernoulli numbers, unified (p,q)-Bernoulli polynomials, $h(x)$-Fibonacci polynomials, and $h(x)$-Lucas polynomials, we define the generalized (p,q)-Bernoulli–Fibonacci and generalized (p,q)-Bernoulli–Lucas, generalized bivariate (p,q)-Bernoulli–Fibonacci, and generalized bivariate (p,q)-Bernoulli–Lucas polynomials and numbers, respectively. We obtain some important identities and relations of these newly established polynomials by using their generating functions and functional equations. Finally, we provide some generating functions for the generalized bivariate (p,q)-Bernoulli–Fibonacci polynomials. For the last section, every proper choice of the coefficients a_k $(k \in \mathbb{N}_0)$, if the multivariable function $\Lambda_{\mu+\nu k}(t_1,\ldots,t_s)$, $(s \in \mathbb{N})$, is expressed as a proper product of many ordinary functions, the allegations of Theorem 6, are able to be applied to obtain various families of bilinear and bilateral generating functions for the families of the polynomials ${}_BF_{n,h,p,q}(x,y)$. With the help of this article, different types of polynomial families can be defined. Different types of polynomial families can be defined by taking bivariate Fibonacci and bivariate Lucas polynomials instead of $h(x)$-Fibonacci and $h(x)$-Lucas polynomials, which we discussed in this article. Our work is to define a new polynomial family with the help of different types of polynomial families that differ from previous studies. For future studies, researchers can define different types of polynomials with the help of this study.

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