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Applications of Symmetric Identities for Apostol–Bernoulli and Apostol–Euler Functions

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Abstract: In this paper, we perform a further investigation on the Apostol–Bernoulli and Apostol–Euler functions introduced by Luo. By using the Fourier expansions of the Apostol–Bernoulli and Apostol–Euler polynomials, we establish some symmetric identities for the Apostol–Bernoulli and Apostol–Euler functions. As applications, some known results, for example, Raabe's multiplication formula and Hermite's identity, are deduced as special cases.

Keywords: Apostol–Bernoulli functions; Bernoulli functions; Apostol–Euler functions; quasi-periodic Euler functions; combinatorial identity

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1. Introduction

Throughout this paper, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ will denote the set of natural numbers, the set of non-negative integers, the set of integers, the set of real numbers, and the set of complex numbers, respectively. Let $n \in \mathbb{N}, \omega \in \mathbb{C}$, and let x be a variable. The Apostol–Bernoulli polynomials $\mathcal{B}_n(x;\omega)$ and the Apostol–Euler polynomials $\mathcal{E}_n(x;\omega)$ are usually defined by the following generating functions (see, e.g., [1,2]):

$$\frac{te^{xt}}{\omega e^t - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(x;\omega)\frac{t^n}{n!} \quad (|t + \log \omega| < 2\pi), \tag{1}$$

and

$$\frac{2e^{xt}}{\omega e^t + 1} = \sum_{n=0}^{\infty} \mathcal{E}_n(x;\omega)\frac{t^n}{n!} \quad (|t + \log \omega| < \pi). \tag{2}$$

In particular, $B_n(x) = \mathcal{B}_n(x;1)$ and $E_n(x) = \mathcal{E}_n(x;1)$ are called the Bernoulli polynomials and the Euler polynomials, respectively. It is well known that the Apostol–Bernoulli numbers $\mathcal{B}_n(\omega)$ and the Apostol–Euler numbers $\mathcal{E}_n(\omega)$ are given by $\mathcal{B}_n(\omega) = \mathcal{B}_n(0;\omega)$ and $\mathcal{E}_n(\omega) = 2^n\mathcal{E}_n(1/2;\omega)$, which are the corresponding generalizations of the Bernoulli numbers $B_n = B_n(0)$ and the Euler numbers $E_n = 2^nE_n(1/2)$. We here mention that the Apostol–Bernoulli polynomials can be used to evaluate the values of the Lerch zeta function at zero and negative integers, see the paper of Apostol [3] for details. For the values of the Apostol–Bernoulli polynomials and the Lerch zeta function at rational numbers, the interested reader may consult the paper of Srivastava [4].

In the present paper, we will be concerned with some identities for the above-mentioned polynomials. Perhaps the best known results are the following multiplication formulas for the Bernoulli polynomials and the Euler polynomials (see, e.g., [5–7]):

$$a^{n-1}\sum_{j=0}^{a-1} B_n\left(x + \frac{j}{a}\right) = B_n(ax) \quad (a \in \mathbb{N}, n \in \mathbb{N}_0), \tag{3}$$

$$a^n \sum_{j=0}^{a-1} (-1)^j E_n\left(x + \frac{j}{a}\right) = E_n(ax) \quad (a \in \mathbb{N}, 2 \nmid a, n \in \mathbb{N}_0), \tag{4}$$

$$-\frac{2a^{n-1}}{n} \sum_{j=0}^{a-1} (-1)^j B_n\left(x + \frac{j}{a}\right) = E_{n-1}(ax) \quad (a, n \in \mathbb{N}, 2 \mid a). \tag{5}$$

We remark that Formula (3) is usually called Raabe's [8] multiplication formula for the Bernoulli polynomials. In particular, Howard [9] used (3) to obtain this for $a, n \in \mathbb{N}$ with $a \geq 2$,

$$B_n = \frac{1}{a(1-a^n)} \sum_{j=0}^{n-1} \binom{n}{j} a^j B_j \sum_{l=1}^{a-1} l^{n-j}, \tag{6}$$

from which he showed that the famous Staudt–Clausen theorem, Carlitz's congruence, Frobenius's congruence, and Ramanujan's congruence are deduced as easy consequences of (6). In the year 2001, Tuenter [10] found that (6) is a special case of the following symmetric identity for the Bernoulli numbers:

$$\sum_{j=0}^{n} \binom{n}{j} a^{j-1} B_j b^{n-j} S_{n-j}(a-1) = \sum_{j=0}^{n} \binom{n}{j} b^{j-1} B_j a^{n-j} S_{n-j}(b-1), \tag{7}$$

where $a, b \in \mathbb{N}$, $n \in \mathbb{N}_0$, $S_k(n)$ is the power sum given for $k, n \in \mathbb{N}_0$ by

$$S_k(n) = 0^k + 1^k + 2^k + \cdots + n^k. \tag{8}$$

Yang [11], in 2008, used the method of generating functions to establish two identities of symmetry for the higher-order Bernoulli polynomials, by virtue of which he extended (7) to the situation for the Bernoulli polynomials

$$\sum_{j=0}^{n} \binom{n}{j} a^{j-1} B_j(bx) b^{n-j} S_{n-j}(a-1) = \sum_{j=0}^{n} \binom{n}{j} b^{j-1} B_j(ax) a^{n-j} S_{n-j}(b-1), \tag{9}$$

and obtained the general form of (3):

$$a^{n-1} \sum_{j=0}^{a-1} B_n\left(bx + \frac{bj}{a}\right) = b^{n-1} \sum_{j=0}^{b-1} B_n\left(ax + \frac{aj}{b}\right), \tag{10}$$

where $a, b \in \mathbb{N}$, $n \in \mathbb{N}_0$, $S_k(n)$ is as in (8). In the same year, Kim [12] used the properties of symmetry for p-adic invariant integrals on $\mathbb{Z}_p$ to prove (9) and (10), demonstrating that for $a, b \in \mathbb{N}, n \in \mathbb{N}_0$ with $a \equiv b \pmod 2$,

$$\sum_{j=0}^{n} \binom{n}{j} a^j E_j(bx) b^{n-j} T_{n-j}(a-1) = \sum_{j=0}^{n} \binom{n}{j} b^j E_j(ax) a^{n-j} T_{n-j}(b-1), \tag{11}$$

and

$$a^n \sum_{j=0}^{a-1} (-1)^j E_n\left(bx + \frac{bj}{a}\right) = b^n \sum_{j=0}^{b-1} (-1)^j E_n\left(ax + \frac{aj}{b}\right), \tag{12}$$

where $T_k(n)$ is the alternate power sum given for $k, n \in \mathbb{N}_0$ by

$$T_k(n) = 0^k - 1^k + 2^k - \cdots + (-1)^n n^k. \tag{13}$$

After that, Liu and Wang [13] developed the method of generating functions used in [11], and established various identities for the higher-order Bernoulli polynomials, the higher-order Euler polynomials, and the higher-order degenerate Bernoulli polynomials, some of which extend (9)–(12). Further, Wang and Wang [14] obtained various identities between the Apostol–Bernoulli polynomials, the Apostol–Euler polynomials, and the power sums with respect to ω, showing that Yang's [11], Liu and Wang's [13], and Zhang and Yang's [15] results are deduced as special cases. The author and Zhang [16,17] in 2013 explored the identities of symmetry for the q-zeta functions and the q-Lerch Euler zeta functions, and gave the corresponding q-extensions of (9)–(12). It should be noted that the identities (10) and (12) can easily lead to the identities (9) and (11) when using the familiar addition theorems for the Bernoulli polynomials and the Euler polynomials described in [5,6]. For some new developments of identities of symmetry on these topics, one is referred to [18–24].

In this paper, we perform further investigation on the Apostol–Bernoulli functions $\overline{\mathcal{B}}_n(x;\omega)$ and the Apostol–Euler functions $\overline{\mathcal{E}}_n(x;\omega)$ introduced by Luo ([25] Equations (2.15) and (2.17)), which are defined for $n \in \mathbb{N}, x \in \mathbb{R}, \omega \in \mathbb{C} \setminus \{0\}$ by

$$\overline{\mathcal{B}}_n(x;\omega) = \omega^{-[x]} B_n(\{x\};\omega), \tag{14}$$

and

$$\overline{\mathcal{E}}_n(x;\omega) = (-1)^{[x]} \omega^{-[x]} \mathcal{E}_n(\{x\};\omega), \tag{15}$$

where $[x]$ is the floor function (also called the greatest integer function), $\{x\}$ denotes the fractional part of x satisfying that for $x \in \mathbb{R}$,

$$\{x\} = x - [x]. \tag{16}$$

By using the Fourier expansions of the Apostol–Bernoulli polynomials and the Apostol–Euler polynomials shown in [25,26], we establish some symmetric identities for the Apostol–Bernoulli functions and the Apostol–Euler functions. It turns out that some known results, for example, Raabe's [8] multiplication formula for the Bernoulli functions, Bayad's [27] multiplication formula for the Euler functions, and Hermite's [28] identity for the floor function, are obtained as special cases. Moreover, we also show that a relation for the number of lattice points in the case of triangles, a symmetric identity for the sums considered by Cetin et al. [29], are established as easy consequences.

2. An Auxiliary Lemma

Before giving our main results, we first present the following auxiliary lemma.

Lemma 1. *Let $a, b \in \mathbb{N}$. Then, for $x \in \mathbb{R}$,*

$$\sum_{j=0}^{a-1} \delta_{\mathbb{Z}}\left(bx + \frac{bj}{a}\right) = \sum_{j=0}^{b-1} \delta_{\mathbb{Z}}\left(ax + \frac{aj}{b}\right), \tag{17}$$

where $\delta_{\mathbb{Z}}(x) = 1$ or 0 according to $x \in \mathbb{Z}$ or $x \in \mathbb{R} \setminus \mathbb{Z}$.

Proof. We write $a = a_1 d$ and $b = b_1 d$, where $d = (a,b)$, we have $(a_1, b_1) = 1$. Hence, we obtain from the familiar division algorithm stated in ([30] Theorem 1.14) that for each $j \in \{0, 1, \ldots, a_1 - 1\}$, there exists $q_j \in \mathbb{Z}$ and unique $r_j \in \{0, 1, \ldots, a_1 - 1\}$ such that

$$b_1 j = a_1 q_j + r_j. \tag{18}$$

It follows from (18) that

$$\begin{aligned} \sum_{j=0}^{a-1} \delta_{\mathbb{Z}}\left(bx + \frac{bj}{a}\right) &= d \sum_{j=0}^{a_1-1} \delta_{\mathbb{Z}}\left(b_1 dx + \frac{b_1 j}{a_1}\right) \\ &= d \sum_{j=0}^{a_1-1} \delta_{\mathbb{Z}}\left(b_1 dx + \frac{j}{a_1}\right) \\ &= \begin{cases} d, & a_1 b_1 dx \in \mathbb{Z}, \\ 0, & a_1 b_1 dx \in \mathbb{R} \setminus \mathbb{Z}. \end{cases} \end{aligned} \tag{19}$$

Similarly, we have

$$\sum_{j=0}^{b-1} \delta_{\mathbb{Z}}\left(ax+\frac{aj}{b}\right)=\begin{cases} d, & a_1b_1dx\in\mathbb{Z}, \\ 0, & a_1b_1dx\in\mathbb{R}\setminus\mathbb{Z}. \end{cases} \tag{20}$$

Thus, by equating (19) and (20), we obtain (17) immediately. □

3. Statement of Main Results

For convenience, in the following we always denote by i the square root of -1 such that $\mathrm{i}^2=-1$. For the sake of convergence, the sum

$$\sum_{k=-\infty}^{+\infty} \frac{1}{k+a} \quad (a\notin\mathbb{Z})$$

is interpreted as

$$\lim_{N\to\infty}\sum_{k=-N}^{N}\frac{1}{k+a}.$$

We now give the symmetric identity for the Apostol–Bernoulli functions as follows.

Theorem 1. *Let $a,b,n\in\mathbb{N},\omega\in\mathbb{C}\setminus\{0\}$. Then, for $x\in\mathbb{R}$,*

$$a^{n-1}\sum_{j=0}^{a-1}\omega^{bj}\overline{\mathcal{B}}_n\left(bx+\frac{bj}{a};\omega^a\right)=b^{n-1}\sum_{j=0}^{b-1}\omega^{aj}\overline{\mathcal{B}}_n\left(ax+\frac{aj}{b};\omega^b\right). \tag{21}$$

Proof. We know from (14) and the Fourier expansions of the Apostol–Bernoulli polynomials shown in ([25] Theorem 2.1) or ([26] Theorem 1.1) that the Apostol–Bernoulli functions can be defined for $n\in\mathbb{N}$, $x\in\mathbb{R},\omega\in\mathbb{C}\setminus\{0\}$ by

$$\overline{\mathcal{B}}_n(x;\omega)=-\frac{n!}{\omega^x(2\pi\mathrm{i})^n}\sum_{k\in\mathbb{Z}}^{*}\frac{e^{2\pi\mathrm{i}kx}}{(k-\frac{\log\omega}{2\pi\mathrm{i}})^n}, \tag{22}$$

where $\sum_{k\in\mathbb{Z}}^{*}=\sum_{k\in\mathbb{Z}\setminus\{0\}}$ if $\omega=1$, and $\sum_{k\in\mathbb{Z}}^{*}=\sum_{k\in\mathbb{Z}}$ if $\omega\neq1$. By replacing x with $bx+\frac{bj}{a}$ and ω by ω^a in (22), we have

$$\overline{\mathcal{B}}_n\left(bx+\frac{bj}{a};\omega^a\right)=-\frac{n!}{\omega^{abx+bj}(2\pi\mathrm{i})^n}\sum_{k\in\mathbb{Z}}^{*}\frac{e^{2\pi\mathrm{i}bkx}\cdot e^{\frac{2\pi\mathrm{i}bkj}{a}}}{(k-a\frac{\log\omega}{2\pi\mathrm{i}})^n}. \tag{23}$$

It follows from (23) and the familiar geometric sums stated in ([30] Theorem 8.1) that

$$\begin{aligned} &a^{n-1}\sum_{j=0}^{a-1}\omega^{bj}\overline{\mathcal{B}}_n\left(bx+\frac{bj}{a};\omega^a\right) \\ &=-\frac{n!\cdot a^{n-1}}{\omega^{abx}(2\pi\mathrm{i})^n}\sum_{k\in\mathbb{Z}}^{*}\frac{e^{2\pi\mathrm{i}bkx}}{(k-a\frac{\log\omega}{2\pi\mathrm{i}})^n}\sum_{j=0}^{a-1}e^{\frac{2\pi\mathrm{i}bkj}{a}} \\ &=-\frac{n!\cdot a^{n}}{\omega^{abx}(2\pi\mathrm{i})^n}\sum_{\substack{k\in\mathbb{Z}\\ a|bk}}^{*}\frac{e^{2\pi\mathrm{i}bkx}}{(k-a\frac{\log\omega}{2\pi\mathrm{i}})^n}. \end{aligned} \tag{24}$$

Let $d=(a,b)$. If we write $a=a_1d$ and $b=b_1d$, then we can rewrite (24) as

$$
\begin{aligned}
&a^{n-1}\sum_{j=0}^{a-1}\omega^{bj}\overline{\mathcal{B}}_n\left(bx+\frac{bj}{a};\omega^a\right)\\
&=-\frac{n!\cdot a^n}{\omega^{abx}(2\pi\mathrm{i})^n}\sideset{}{^*}\sum_{\substack{k\in\mathbb{Z}\\ a_1|k}}\frac{e^{2\pi\mathrm{i}bkx}}{(k-a\frac{\log\omega}{2\pi\mathrm{i}})^n}\\
&=-\frac{n!\cdot a^n}{\omega^{abx}(2\pi\mathrm{i})^n}\sideset{}{^*}\sum_{k\in\mathbb{Z}}\frac{e^{2\pi\mathrm{i}a_1bkx}}{(a_1k-a_1d\frac{\log\omega}{2\pi\mathrm{i}})^n}\\
&=-\frac{n!\cdot(a,b)^n}{\omega^{abx}(2\pi\mathrm{i})^n}\sideset{}{^*}\sum_{k\in\mathbb{Z}}\frac{e^{\frac{2\pi\mathrm{i}abkx}{(a,b)}}}{(k-(a,b)\frac{\log\omega}{2\pi\mathrm{i}})^n}.
\end{aligned}
\tag{25}
$$

Replacing a with b and b with a in (25), we have

$$
b^{n-1}\sum_{j=0}^{b-1}\omega^{aj}\overline{\mathcal{B}}_n\left(ax+\frac{aj}{b};\omega^b\right)=-\frac{n!\cdot(a,b)^n}{\omega^{abx}(2\pi\mathrm{i})^n}\sideset{}{^*}\sum_{k\in\mathbb{Z}}\frac{e^{\frac{2\pi\mathrm{i}abkx}{(a,b)}}}{(k-(a,b)\frac{\log\omega}{2\pi\mathrm{i}})^n}. \tag{26}
$$

Therefore, we obtain (21) immediately when equating (25) and (26). This completes the proof of Theorem 1. □

It follows that we show some special cases of Theorem 1. We have the following results.

Corollary 1. *Let $a,n\in\mathbb{N},\omega\in\mathbb{C}\setminus\{0\}$. Then, for $x\in\mathbb{R}$,*

$$
a^{n-1}\sum_{j=0}^{a-1}\omega^{j}\overline{\mathcal{B}}_n\left(x+\frac{j}{a};\omega^a\right)=\overline{\mathcal{B}}_n(ax;\omega). \tag{27}
$$

Proof. Taking $b=1$ in Theorem 1, we obtain the desired result. □

In particular, the case $\omega=1$ in Corollary 1 gives Raabe's [8] multiplication formula for the Bernoulli functions, namely,

$$
a^{n-1}\sum_{j=0}^{a-1}\overline{B}_n\left(x+\frac{j}{a}\right)=\overline{B}_n(ax), \tag{28}
$$

where $a,n\in\mathbb{N}$, $x\in\mathbb{R}$, $\overline{B}_n(x)$ is the n-th Bernoulli function given for $n\in\mathbb{N},x\in\mathbb{R}$ by

$$
\overline{B}_1(x)=\begin{cases}0, & x\in\mathbb{Z},\\ B_1(\{x\}), & x\in\mathbb{R}\setminus\mathbb{Z},\end{cases}\qquad \overline{B}_n(x)=B_n(\{x\})\quad(n\geq 2,x\in\mathbb{R}). \tag{29}
$$

We note that Carlitz [31] and Bayad and Raouj [32] used Raabe's multiplication Formula (28) to establish some reciprocity formulas for the generalized Dedekind sums.

Corollary 2. *Let $a,b,n\in\mathbb{N}$. Then, for $x\in\mathbb{R}$,*

$$
a^{n-1}\sum_{j=0}^{a-1}\overline{B}_n\left(bx+\frac{bj}{a}\right)=b^{n-1}\sum_{j=0}^{b-1}\overline{B}_n\left(ax+\frac{aj}{b}\right). \tag{30}
$$

Proof. By setting $\omega=1$ in Theorem 1, the desired result follows immediately. □

It becomes obvious that the case $b=1$ in Corollary 2 leads to Raabe's multiplication Formula (28). We next use Corollary 2 to deduce the following result.

Corollary 3. *Let $a,b\in\mathbb{N}$. Then, for $x\in\mathbb{R}$,*

$$
\sum_{j=0}^{a-1}\left[bx+\frac{bj}{a}\right]=\sum_{j=0}^{b-1}\left[ax+\frac{aj}{b}\right]. \tag{31}
$$

Proof. Denote by $\delta_{l,k}$ the Kronecker delta function given for $l, k \in \mathbb{N} \cup \{0\}$ by $\delta_{l,k} = 1$ or 0 according to $l = k$ or $l \neq k$. It is easy to see from (29) that for $n \in \mathbb{N}, x \in \mathbb{R}$,

$$\overline{B}_n(x) = B_n(\{x\}) + \frac{1}{2}\delta_{1,n}\delta_{\mathbb{Z}}(x), \tag{32}$$

where $\delta_{\mathbb{Z}}(x)$ is as in (17). Since $B_1(x) = x - \frac{1}{2}$, by taking $n = 1$ in Corollary 2, in light of (16) and Lemma 1, we prove Corollary 3. □

If we take $b = 1$ in Corollary 3, then we obtain Hermite's [28] identity for the floor function, namely,

$$\sum_{j=0}^{a-1}\left[x + \frac{j}{a}\right] = [ax], \tag{33}$$

where $a \in \mathbb{N}, x \in \mathbb{R}$. For another generalizations of Hermite's identity (33), the interested reader may consult [33]. As another application of Corollary 3, we here give the following relation for the number of lattice points in the case of triangles.

Corollary 4. *Let $a, b \in \mathbb{N}, c \in \mathbb{R}$. Assume that $T_1(a,b,c)$ represents the triangle in $\mathbb{R}^2$ with vertices*

$$(-a\{c\}, 0), \quad (a, 0), \quad (a, b + b\{c\}),$$

$T_2(a,b,c)$ represents the triangle in $\mathbb{R}^2$ with vertices

$$(-b\{c\}, 0), \quad (b, 0), \quad (b, a + a\{c\}).$$

Then,

$$\#(T_1(a,b,c) \cap \mathbb{Z}^2) = \#(T_2(a,b,c) \cap \mathbb{Z}^2), \tag{34}$$

where # denotes the cardinality of a set S.

Proof. Since the number of lattice points on the acute angles of $T_1(a,b,c)$ is

$$[b + b\{c\}] + a + 1 + [a\{c\}] = a + b + 1 + [a\{c\}] + [b\{c\}], \tag{35}$$

the number of lattice points on the acute angles of $T_2(a,b,c)$ is

$$[a + a\{c\}] + b + 1 + [b\{c\}] = a + b + 1 + [a\{c\}] + [b\{c\}], \tag{36}$$

by taking $x = 0$ in Corollary 3, in view of (35) and (36), we see that Corollary 4 holds true in the case when $c = 0$. We now consider the case $c \neq 0$. Let $T_3(a,b,c)$ be the triangle in $\mathbb{R}^2$ with vertices

$$(-a\{c\}, 0), \quad (0, 0), \quad (0, b\{c\}),$$

and $T_4(a,b,c)$ be the triangle in $\mathbb{R}^2$ with vertices

$$(-b\{c\}, 0), \quad (0, 0), \quad (0, a\{c\}).$$

Obviously, the graph of $T_3(a,b,c)$ and the graph of $T_4(a,b,c)$ are symmetric about the line $y = -x$. This indicates that the number of lattice points in the interior of $T_3(a,b,c)$ is equal to the number of lattice points in the interior of $T_4(a,b,c)$, and the number of lattice points on the hypotenuse of $T_3(a,b,c)$ is equal to the number of lattice points on the hypotenuse of $T_4(a,b,c)$. Therefore, by taking $x = \{c\}$ in Corollary 3, we say from (35) and (36) that Corollary 4 holds true in the case when $c \neq 0$. This completes the proof of Corollary 4. □

We next present the symmetric identity for the Apostol–Euler functions, as follows.

Theorem 2. *Let $a, b, n \in \mathbb{N}, \omega \in \mathbb{C} \setminus \{0\}$ with $a \equiv b \ (mod\ 2)$. Then, for $x \in \mathbb{R}$,*

$$a^n \sum_{j=0}^{a-1} (-1)^j \omega^{bj} \overline{\mathcal{E}}_n\left(bx + \frac{bj}{a}; \omega^a\right) = b^n \sum_{j=0}^{b-1} (-1)^j \omega^{aj} \overline{\mathcal{E}}_n\left(ax + \frac{aj}{b}; \omega^b\right). \tag{37}$$

Proof. We see from (15) and the Fourier expansions of the Apostol–Euler polynomials shown in ([25] Theorem 2.2) or ([26] Theorem 1.2) that the Apostol–Euler functions can be defined for $n \in \mathbb{N}, x \in \mathbb{R}, \omega \in \mathbb{C} \setminus \{0\}$ by

$$\overline{\mathcal{E}}_n(x; \omega) = \frac{2 \cdot n!}{\omega^x (2\pi \mathrm{i})^{n+1}} \sum_{k \in \mathbb{Z}}^{**} \frac{e^{2\pi \mathrm{i}(k - \frac{1}{2})x}}{(k - \frac{1}{2} - \frac{\log \omega}{2\pi \mathrm{i}})^{n+1}}, \tag{38}$$

where $\sum_{k \in \mathbb{Z}}^{**} = \sum_{k \in \mathbb{Z} \setminus \{0\}}$ if $\omega = -1$, and $\sum_{k \in \mathbb{Z}}^{**} = \sum_{k \in \mathbb{Z}}$ if $\omega \neq -1$. If we replace x with $bx + \frac{bj}{a}$ and ω with ω^a in (38), then we have

$$\overline{\mathcal{E}}_n\left(bx + \frac{bj}{a}; \omega^a\right) = \frac{2 \cdot n!}{\omega^{abx+bj} (2\pi \mathrm{i})^{n+1}} \sum_{k \in \mathbb{Z}}^{**} \frac{e^{2\pi \mathrm{i}(k - \frac{1}{2})bx} \cdot e^{\frac{2\pi \mathrm{i}(bk - \frac{b}{2})j}{a}}}{(k - \frac{1}{2} - a\frac{\log \omega}{2\pi \mathrm{i}})^{n+1}}. \tag{39}$$

Hence, we discover from (39) and the geometric sums stated in ([30] Theorem 8.1) that

$$\begin{aligned}
&a^n \sum_{j=0}^{a-1} (-1)^j \omega^{bj} \overline{\mathcal{E}}_n\left(bx + \frac{bj}{a}; \omega^a\right) \\
&= \frac{2 \cdot n! \cdot a^n}{\omega^{abx} (2\pi \mathrm{i})^{n+1}} \sum_{k \in \mathbb{Z}}^{**} \frac{e^{2\pi \mathrm{i}(k - \frac{1}{2})bx}}{(k - \frac{1}{2} - a\frac{\log \omega}{2\pi \mathrm{i}})^{n+1}} \sum_{j=0}^{a-1} e^{\frac{2\pi \mathrm{i}(bk - \frac{b-a}{2})j}{a}} \\
&= \frac{2 \cdot n! \cdot a^{n+1}}{\omega^{abx} (2\pi \mathrm{i})^{n+1}} \sum_{\substack{k \in \mathbb{Z} \\ a | (bk - \frac{b-a}{2})}}^{**} \frac{e^{2\pi \mathrm{i}(k - \frac{1}{2})bx}}{(k - \frac{1}{2} - a\frac{\log \omega}{2\pi \mathrm{i}})^{n+1}}.
\end{aligned} \tag{40}$$

It is easily seen from $a \mid (bk - \frac{b-a}{2})$ for $k \in \mathbb{Z}$ that there exists $q \in \mathbb{Z}$ such that

$$b(2k-1) = a(2q-1). \tag{41}$$

If we write $a = a_1 d$ and $b = b_1 d$, where $d = (a, b)$, then we conclude from (41) that

$$a_1 \mid (2k-1) \quad (k \in \mathbb{Z}). \tag{42}$$

It follows from (42) that (40) can be rewritten as

$$\begin{aligned}
&a^n \sum_{j=0}^{a-1} (-1)^j \omega^{bj} \overline{\mathcal{E}}_n\left(bx + \frac{bj}{a}; \omega^a\right) \\
&= \frac{2 \cdot n! \cdot a^{n+1}}{\omega^{abx} (2\pi \mathrm{i})^{n+1}} \sum_{\substack{k \in \mathbb{Z} \\ a_1 | (2k-1)}}^{**} \frac{e^{2\pi \mathrm{i}(k - \frac{1}{2})bx}}{(k - \frac{1}{2} - a\frac{\log \omega}{2\pi \mathrm{i}})^{n+1}} \\
&= \frac{2 \cdot n! \cdot a^{n+1}}{\omega^{abx} (2\pi \mathrm{i})^{n+1}} \sum_{k \in \mathbb{Z}}^{**} \frac{e^{\pi \mathrm{i} a_1 bkx}}{(\frac{a_1 k}{2} - a\frac{\log \omega}{2\pi \mathrm{i}})^{n+1}} \\
&= \frac{2 \cdot n! \cdot (a,b)^{n+1}}{\omega^{abx} (2\pi \mathrm{i})^{n+1}} \sum_{k \in \mathbb{Z}}^{**} \frac{e^{\frac{\pi \mathrm{i} abkx}{(a,b)}}}{(\frac{k}{2} - (a,b)\frac{\log \omega}{2\pi \mathrm{i}})^{n+1}}.
\end{aligned} \tag{43}$$

Replacing a with b and b with a in (43) gives

$$
\begin{aligned}
& b^n \sum_{j=0}^{b-1} (-1)^j \omega^{aj} \overline{\mathcal{E}}_n \left(ax + \frac{aj}{b}; \omega^b \right) \\
&= \frac{2 \cdot n! \cdot (a,b)^{n+1}}{\omega^{abx} (2\pi \mathrm{i})^{n+1}} \sum_{k\in\mathbb{Z}}^{**} \frac{e^{\frac{\pi \mathrm{i} abkx}{(a,b)}}}{(\frac{k}{2} - (a,b)\frac{\log \omega}{2\pi \mathrm{i}})^{n+1}}. \qquad (44)
\end{aligned}
$$

Thus, by equating (43) and (44), we obtain (37) immediately and finish the proof of Theorem 2. □

We next discuss some special cases of Theorem 2. We have the following results.

Corollary 5. *Let $a, n \in \mathbb{N}, \omega \in \mathbb{C} \setminus \{0\}$ with $2 \nmid a$. Then, for $x \in \mathbb{R}$,*

$$
a^n \sum_{j=0}^{a-1} (-\omega)^j \overline{\mathcal{E}}_n \left(x + \frac{j}{a}; \omega^a \right) = \overline{\mathcal{E}}_n(ax; \omega). \qquad (45)
$$

Proof. Setting $b = 1$ in Theorem 2, we obtain the desired result. □

Trivially, the case $\omega = 1$ in Corollary 5 gives that for $a, n \in \mathbb{N}, x \in \mathbb{R}$ with $2 \nmid a$,

$$
a^n \sum_{j=0}^{a-1} (-1)^j \overline{E}_n \left(x + \frac{j}{a} \right) = \overline{E}_n(ax), \qquad (46)
$$

where $\overline{E}_n(x)$ is the n-th quasi-periodic Euler function given for $n \in \mathbb{N}, x \in \mathbb{R}$ by

$$
\overline{E}_n(x) = (-1)^{[x]} E_n(\{x\}). \qquad (47)
$$

It is worth mentioning that Formula (46) was also discovered by Bayad ([27] Equation (1.2.13)), and has been used by Kim and Son [34] and Hu, Kim, and Kim [35] to establish some reciprocity formulas for the generalized Dedekind sums involving quasi-periodic Euler functions.

Corollary 6. *Let $a, b, n \in \mathbb{N}$ with $a \equiv b \pmod 2$. Then, for $x \in \mathbb{R}$,*

$$
a^n \sum_{j=0}^{a-1} (-1)^j \overline{E}_n \left(bx + \frac{bj}{a} \right) = b^n \sum_{j=0}^{b-1} (-1)^j \overline{E}_n \left(ax + \frac{aj}{b} \right). \qquad (48)
$$

Proof. Taking $\omega = 1$ in Theorem 2 gives the desired result. □

It is clear that the case $b = 1$ in Corollary 6 gives Bayad's multiplication formula (46). In fact, we can use Corollary 6 to yield the following result.

Corollary 7. *Let $a, b \in \mathbb{N}$ with $a \equiv b \pmod 2$. Then, for $x \in \mathbb{R}$,*

$$
\begin{aligned}
& a \sum_{j=0}^{a-1} (-1)^{j + [bx + \frac{bj}{a}]} \left(bx + \frac{bj}{a} - \left[bx + \frac{bj}{a} \right] - \frac{1}{2} \right) \\
&= b \sum_{j=0}^{b-1} (-1)^{j + [ax + \frac{aj}{b}]} \left(ax + \frac{aj}{b} - \left[ax + \frac{aj}{b} \right] - \frac{1}{2} \right). \qquad (49)
\end{aligned}
$$

Proof. Since $E_1(x) = x - \frac{1}{2}$, by taking $n = 1$ in Corollary 6, in view of (16) and (47), we obtain the desired result. □

Obviously, the case $b = 1$ in Corollary 7 gives for $a \in \mathbb{N}, x \in \mathbb{R}$ with $2 \nmid a$ that

$$
a \sum_{j=0}^{a-1} (-1)^{j + [x + \frac{j}{a}]} \left(x + \frac{j}{a} - \left[x + \frac{j}{a} \right] - \frac{1}{2} \right) = (-1)^{[ax]} \left(ax - [ax] - \frac{1}{2} \right), \qquad (50)
$$

which can be thought of as the complement of Hermite's identity (33). It is interesting to point out that Corollary 7 can be used to establish a symmetric identity for the sums (52) considered by Cetin et al. [29], as follows.

Corollary 8. *Let $a, b \in \mathbb{N}$ with $a \equiv b$ (mod 2) and $(a, b) = 1$. Then*

$$aC_1(b,a) = bC_1(a,b) + \frac{a-b}{2}, \tag{51}$$

where $C_1(b,a)$ is defined for $a, b \in \mathbb{N}$ by

$$C_1(b,a) = \sum_{j=0}^{a-1} (-1)^{j+[\frac{bj}{a}]} \overline{B}_1\left(\frac{bj}{a}\right). \tag{52}$$

Proof. Taking $x = 0$ in Corollary 7, it then follows from (16) and (32) that

$$\begin{aligned} & a\sum_{j=0}^{a-1} (-1)^{j+[\frac{bj}{a}]} \left(\overline{B}_1\left(\frac{bj}{a}\right) - \frac{1}{2}\delta_{\mathbb{Z}}\left(\frac{bj}{a}\right)\right) \\ &= b\sum_{j=0}^{b-1} (-1)^{j+[\frac{aj}{b}]} \left(\overline{B}_1\left(\frac{aj}{b}\right) - \frac{1}{2}\delta_{\mathbb{Z}}\left(\frac{aj}{b}\right)\right). \end{aligned} \tag{53}$$

Since a and b are relatively prime, we know from ([30] Theorem 3.8) that two lattice points (a, b) and $(0, 0)$ (two lattice points (b, a) and $(0, 0)$) are mutually visible. Hence, from (53) we have

$$aC_1(b,a) - \frac{a}{2} = bC_1(a,b) - \frac{b}{2},$$

as desired. This concludes the proof of Corollary 8. □

We here mention that Cetin [36] has shown that the sum in (52) is closely related to the Hardy sums. For some reciprocity formulas of the Hardy sums, one is referred to [37–41].

4. Conclusions

In this paper, we have used the Fourier expansions of the Apostol–Bernoulli polynomials and the Apostol–Euler polynomials to establish some symmetric identities for the Apostol–Bernoulli functions and the Apostol–Euler functions. The results presented here are the corresponding generalizations of Raabe's [8], Bayad's [27], and Hermite's [28] results. We also point out that a relation for the number of lattice points in the case of triangles, a symmetric identity for the sums considered by Cetin et al. [29], can be easily deduced. As shown in the third section, the topics explored in this paper are closely related to the generalized Dedekind sums and the Hardy sums. We will study some properties for the products of the Apostol–Bernoulli functions and the Apostol–Euler functions, and give some new reciprocity formulas for the generalized Dedekind sums and the Hardy sums in another papers.

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