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The Properties of Topological Manifolds of Simplicial Polynomials

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Abstract: The formulations of polynomials over a topological simplex combine the elements of topology and algebraic geometry. This paper proposes the formulation of simplicial polynomials and the properties of resulting topological manifolds in two classes, non-degenerate forms and degenerate forms, without imposing the conditions of affine topological spaces. The non-degenerate class maintains the degree preservation principle of the atoms of the polynomials of a topological simplex, which is relaxed in the degenerate class. The concept of hybrid decomposition of a simplicial polynomial in the non-degenerate class is introduced. The decompositions of simplicial polynomial for a large set of simplex vertices generate ideal components from the radical, and the components preserve the topologically isolated origin in all cases within the topological manifolds. Interestingly, the topological manifolds generated by a non-degenerate class of simplicial polynomials do not retain the homeomorphism property under polynomial extension by atom addition if the simplicial condition is violated. However, the topological manifolds generated by the degenerate class always preserve isomorphism with varying rotational orientations. The hybrid decompositions of the non-degenerate class of simplicial polynomials give rise to the formation of simplicial chains. The proposed formulations do not impose strict positivity on simplicial polynomials as a precondition.

Keywords: topology; polynomials; simplex; zero-sets; manifolds

MSC: 55Mxx; 57Nxx; 14Hxx; 14Jxx

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1. Introduction

The representations of homotopy paths as a set of polynomials are interesting in the domains of algebraic topology and algebraic geometry. In this context, the decompositions of polynomials in real algebraic sets or real semi-algebraic sets expose several interesting properties. It is known that if R denotes an algebraically closed real field and $g \in R[x]$ such that $g > 0$, then there exists $\{f_k : 0 < k \leq m\} \subset R[x]$, revealing the decomposition to be $g = \sum_k f_k^2$ considering the respective real algebraic sets [1]. However, a similar result is not guaranteed if we consider $g \in R[x_1, x_2, \ldots, x_n]$, and Hilbert showed that, in this case, rational functions are required [1,2]. The generalized decomposability of $f \in R[x_1, x_2, \ldots, x_n]$ in terms of a set of rational functions $\{h_k : 0 < k \leq m\} \subset R[x_1, x_2, \ldots, x_n]$ in the form $f = \sum_k h_k^2$ in a subset $A \subset R^n$ is an interesting area to study, and it invites the notion of semi-algebraic sets in Euclidean spaces [1]. Moreover, topological concepts are often required in algebraic geometry, including the *Zariski algebraic sets*. For example, it has been shown that the decomposability of $f \in R[x_1, x_2, \ldots, x_n]$ and topological separations of two convex cones (i.e., algebraically ordered cones) by linear functionals are interrelated [1,3]. Often, *monomials* play important roles in determining the decomposability of polynomials. Interestingly, special classes of polynomials, for example, Hermite polynomials, can reveal quadratic decomposition in bivariate forms under certain conditions [4]. First, we present the preliminary concepts, motivations and the contributions made in this paper in the following sections.

1.1. Preliminaries

Recall that if $S \subset F[x_1, x_2, \ldots, x_n]$ is given over an algebraically closed field F, then $Zr(S)$ is an algebraic set, which is topologically Zariski closed. On the other hand, the decompositions of multivariable (positive real) polynomials often involve real semi-algebraic sets. The definition of a *real semi-algebraic set* is presented as follows [3].

Definition 1. *Let $A \subset R$ be a real subring and $B_f(R) = \{\langle a_i \rangle_{i=1}^n \in R^n : f(\langle a_i \rangle_{i=1}^n) > 0\}$ be a set such that $f \in A[x_1, x_2, \ldots, x_n]$. The $S \subset R^n$ is a semi-algebraic set over $A \subset R$ if $S \subset R^n$ is a Boolean set of algebraic combinations of $B_f(R)$.*

Interestingly, the concept of the semi-algebraic set has close similarity to the concept of the quasi-algebraic set. Note that the positivity of $f \in R[x_1, x_2, \ldots, x_n]$ depends on the lower bound of the minima of the respective polynomial on a bounded algebraic set or semi-algebraic set in R^n, and it can be computed by using Lojasiewicz inequalities involving an $n - simplex$ in R^n [2,5]. The concept of *Lojasiewicz inequality* is presented in the following definition [6].

Definition 2. *Let us consider $f \in R[x]$ with $f(0) = 0$ and $A^n(R) \subset R^n$ such that $Zr(f) \subset A^n(F)$. If a topological subspace $B^n(R) \subset R^n$ is compact, then there exists the constants $b > 0$ and $c > 0$ such that $\left|f\right| \geq cd(x, Zr(f))^b$ for every $x \in B^n(R)$.*

Note that the Lojasiewicz inequality considers a real $n -$ space, which is metrizable. It is possible to decompose a geometric topological object such as a *polyhedral space* by employing the polynomial maps and Minkowski summation considering that the polynomials are in the non-degeneracy class [6]. Let us consider a set of non-degenerate polynomials $P = \{p_1, p_2, \ldots, p_k\}$ and a corresponding polynomial map $P : R^n \backslash \{0, 0, \ldots, 0\} \to R^k$ such that $1 \leq k \leq n$. Suppose we denote the Minkowski summation as $\sum_{j=1}^{k} M(p_j)$ such that $\sum_{j=1}^{k} M(p_j) \subset R^k \backslash \{0\}$. Thus, a polyhedral face $\Delta \in \sum_{j=1}^{k} M(p_j)$ under the polynomial map can be presented in the form $\Delta = \sum_{j=1}^{k} \Delta_j$, where Δ_j is a face of the respective $M(p_j)$ of the polyhedral space. Note that the positivity of a polynomial $f \in R[x_1, x_2, \ldots, x_n]$ and the $n - simplex$ are interrelated through the applications of *Pólya's theorem*. The interrelationship is presented in the following theorem [5].

Theorem 1. *Let $f \in R[x_1, x_2, \ldots, x_n]$ be positive as well as homogeneous with $deg(f) = d$ and the absolute values of coefficients be bounded below 2. If λ is the minimum of $f \in R[x_1, x_2, \ldots, x_n]$ over the $n - simplex$ $\sigma_{(n)} = \left\{\sum_{j=1}^{n} x_j = 1 : \forall x_j \geq 0\right\}$, then λ^{-1} is bounded above, and all coefficients of $f\left(\sum_{j=1}^{n} x_j\right)^p$ are strictly positive, where p is a constant.*

The proof of the aforesaid theorem is detailed in [5]. Often the polynomials can be formulated in combinatorial forms considering a smooth function. The Bernstein polynomials are in such a class, which can be extended over the $n - simplex$. The definition of Bernstein polynomials and the extension over an $n - simplex$ are presented in the following definition [7].

Definition 3. *Let $f([0, 1])$ be a smooth function. The Bernstein polynomial $B(n > 1, f, x)$ is defined as $B(n > 1, x, f) = \sum_{j=0}^{n} \binom{n}{j} . f\left(\frac{j}{n}\right) . x^j . (1 - x)^{n-j}$. Moreover, if $\sigma_{(n)} = \sum_{j=0}^{n} \lambda_j . x^j$*

is an $n - simplex$, *then the Bernstein polynomial extended over the* $n - simplex$ *is given as* $B(n > 1, x, f) = \sum_{|s|=n} f(x_s).\binom{n}{s}.\lambda^s$, *where* $x_s = (1/n)\sum_{j=0}^{n} s_j.x^j$ *and* $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n)$.

Note that the Bernstein polynomials can be asymptotically expanded, and it includes an error term [7]. A similar type of formulation of a polynomial over a simplicial complex involving combinatorial terms is the Sterling polynomial [8]. If we consider the complex field C and $f \in C[z]$, then it is proved that the polynomial $f(z) = 1 + \sum_{j=1}^{n} a_j.z^j$ has real zeros, and it represents a simplicial complex [9]. This leads to the following lemma [9].

Lemma 1. *If* $f \in C[z]$ *is a polynomial of* $deg(f) = n$ *with all real zeros in* $[-1, 0]$, *then* $f \in C[z]$ *represents a simplicial complex.*

The proof of the lemma and the associated properties are detailed in [9].

1.2. Motivations

The generation of algebraic curves over the topological $n - simplex$ within an affine topological space $B^n \subset R^n$ is an interesting domain of study. It involves the elements of geometric topology and algebraic geometry. The polynomial forms and their decomposability have applications in analyzing dynamics of complex systems, non-linear systems, biological data analysis and graph theory based on polynomials and simplicial complexes [10]. In general, the formation of positive polynomials over a simplex considers that the coefficients are in R [3]. On the other hand, the formulation of $f \in R[x_1, x_2, \ldots, x_n]$ over a topological $n - simplex$ $\sigma_{(n)}$ in $B^n \subset R^n$ involving the respective polynomial ring considers that $f \in Z[x_1, x_2, \ldots, x_n]$ and $f > 0$, indicating that it is strictly positive when coefficients are in the field of integers [2]. It was shown that the lower bound of $f > 0$ largely depends on deg(f) and the dimension of the space. Interestingly, the complex analytic function over a Newton polyhedral space can be given in the form $h : (C^0, 0) \rightarrow (C^p, 0)$ considering the non-degeneracy class [11]. It is important to note that it is often assumed that the polynomials are in the non-degeneracy class irrespective of the algebraic fields [2,11]. On the other hand, it was noted earlier that the polynomials can be formulated in combinatorial forms over an $n - simplex$. An example is the univariate Bernstein polynomial extended over an $n - simplex$ [7]. Moreover, it was shown earlier that the finite simplicial complexes can be determined by employing the algebra of polynomials involving the baricentric coordinates of the simplexes with coefficients of the polynomials in the integral domains [12]. Furthermore, the homotopy theory of algebraic topology can be employed to find the isolated solutions of polynomials within the zero-sets [13]. However, in this case, the complex space C^n is considered, which is in the Euclidean class. On the other hand, the polyhedral homotopies are employed to solve the polynomial systems [14]. It was shown that an $f - polynomial$ can be generated over a simplicial complex $\sigma_{(n-1)}$, which is given in the form $f(t) = \sum_{j=0}^{n} f_{j-1}t^j$, where f_j is a $j - simplex$ in the $\sigma_{(n-1)}$ complex [15]. This formulation has applications in analyzing *convex polytopes* as topological objects. It is known that the topological properties have close interrelationships with the elements of algebraic geometry [16–19]. It was shown that the simplicial triangulations of an affine topological space can be established by employing $h - polynomials$, and the topology of an algebraic curve changes when the changes in its coefficients force it to pass through singularities [17,18,20]. Evidently, the formations of polynomials over simplexes offer new perspectives in the views of geometric topology and algebraic geometry. This serves as the motivation to ask the following topological as well as algebraic questions: (1) *What are the roles of monomials (called atoms in this paper considering the degrees in the real algebraic field maintaining atomic degree conditions, as explained in Remark 1) in the formations of simplicial polynomials in topological spaces without restriction to affine spaces?* (2) *Can it reveal the degener-*

acy class in some forms? (3) *Is it possible to relax the requirement of positivity of polynomials in the formation of simplicial polynomials as a generalization*? And, finally, (4) *what are the properties of the resulting topological manifolds generated by them*? This article addresses these questions in relative detail by combining the elements of geometric topology and algebraic geometry.

1.3. Contributions

The contributions made in this paper can be summarized as follows. This paper proposes the formulation of polynomials over the n-simplex (which are called simplicial polynomials) in a topological space (without imposing the condition of topologically affine spaces), preserving the degree preservation principle of atoms of the polynomials representing a simplex and the formation of topological manifolds as a result. The simplicial polynomials preserve the properties of the Noetherian class of ideals so that the respective zero-set is not empty and Zariski closed. We introduce the concept of hybrid decomposition of a simplicial polynomial and the inclusion of the degeneracy class in the formulations. It is shown that the topological manifolds generated by the simplicial polynomials in a non-degenerate class do not retain the homeomorphism property if we increase the number of atoms of the simplicial polynomials in additive forms including the multiplicative scaling. The generations of the degeneracy class of simplicial polynomials relax the atomic degree preservation principle. Interestingly, the resulting manifolds show homeomorphisms with varying orientations (i.e., manifold rotation). The properties of decomposition of simplicial polynomials in a non-degenerate class are analyzed in detail. It is illustrated that the decomposition forms ideal components for sufficiently large degrees, which preserves the topologically isolated origin in topological manifolds in all cases. Moreover, the hybrid decomposition of a non-degenerate class of simplicial polynomial is admissible, and it invites the formation of a simplex chain within the topological spaces in lower dimensions. The two distinctive properties of the proposed formulations are: (1) the formulations do not consider that the simplicial polynomials are strictly positive as a precondition, and (2) they consider the formations of simplicial polynomials involving the non-degeneracy class as well as the degeneracy class, generating different varieties of topological manifolds. The relaxation of the strict positivity of a simplicial polynomial allows the formation of topological manifolds over the positive as well as negative regions, exposing various axes of symmetries.

The rest of the paper is organized as follows. The concept of simplicial polynomials in different classes and the related decompositions are presented in Section 2. Section 3 presents a set of topological as well as algebraic properties of the proposed formulations. The comparative analysis of the proposed formulations with respect to the related works is presented in Section 4. Finally, Section 5 concludes the paper.

2. Concept of Simplicial Polynomials

In this section, we present the concept of simplicial polynomials in a topological $n - space$ and the related degenerates in lower dimensional spaces by combining the elements of algebraic topology and algebraic geometry. We consider that the underlying real algebraic field F is always closed in topological $n - space\ A^n(F)$, and the set of integers is denoted as Z. Let $\sigma_{(n)}$ be an $n - simplex$ representing a topological space $\left|\sigma_{(n)}\right|$ generated over the vertex set $V(\sigma_{(n)}) = \langle x_1, x_2, \ldots, x_n \rangle$ such that $\sigma_{(n)} = \sum_{j=1}^{n} t_j x_j$, where $t_j \in [0,1]$ and $\sum_{j=1}^{n} t_j = 1$. We consider that $\sigma^o_{(n)}$ is a simply connected open topological space with dimension $n < +\infty$. First, we present the formulation of simplicial polynomials and the concept of hybrid decomposition.

2.1. Simplicial Polynomials and Hybrid Decomposition

Let F be a closed real algebraic field. If $F[x_1, x_2, \ldots, x_n]$ is a polynomial ring and $f \in F[x_1, x_2, \ldots, x_n]$, then every $a_{1\ldots j}x_1^{b(1)}x_2^{b(2)}\ldots x_n^{b(n)}$ is called an *atom* of $f \in F[x_1, x_2, \ldots, x_n]$, where $a_{1\ldots j} \in F$ and $b(j) \in Z$ (the details about the concept of the atom are given in Remark 1). Let us consider the algebraic curve $f \in F[x]$ in the form given by $f = \sum_{j=1}^{n} a_j x^j$, maintaining generality. The corresponding simplicial polynomial with respect to the pair $\left\langle f, \sigma_{(n)} \right\rangle$ is defined as follows.

Definition 4. *If $f \in F[x]$ is a polynomial, then the simplicial polynomial $f_{\sigma(n)} \in F[x, y]$ generated over $\left\langle f, \sigma_{(n)} \right\rangle$ is defined as $f_{\sigma(n)} = \sum_{j=1}^{n} (t_j a_j)x^{-j}y^{j+1}$ such that $\sum_{j=1}^{n} (t_j a_j) = 1$.*

Remark 1. *Note that every monomial $\beta_j = t_j x_j$ in $\sigma_{(n)}$ maintains $deg(\beta_j) = 1$. On the other hand, the simplicial polynomial $f_{\sigma(n)} \in F[x, y]$ preserves the condition that, for every monomial $\beta_{fj} = (t_j a_j)x^{-j}y^{j+1}$ in $f_{\sigma(n)}$, the degrees are not altered such that $deg(\beta_{fj}) = deg(\beta_j)$. We refer to it as the atomic degree condition (ADC), and the monomials preserving ADC are called atoms. Moreover, if the $\sum_{j=1}^{n} (t_j a_j) = 1$ condition is maintained, then it is said to be a proper simplicial polynomial preserving the simplicial condition. Note that we are not requiring that $(t_j a_j) \in [0, 1]$ in all cases (which is a strict simplicial condition) by generalizing the concept of proper simplicial polynomials.*

Example 1. *We present the manifold structure (MS) generated by the simplicial polynomials considering three different cases. First, we consider the polynomial in $R[x, y]$ with two atoms, which is given by $-2x^{-1}y^2 + 3x^{-2}y^3$. The resulting MS is illustrated in Figure 1.*

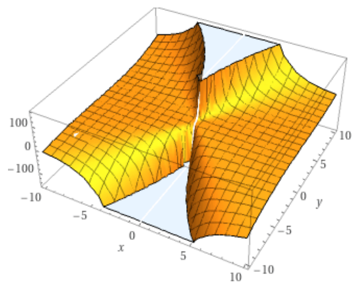

Figure 1. Manifold of simplicial polynomial with two atoms.

Next, we increase the number of atoms in additive forms such that $f_{\sigma(n)} = -2x^{-1}y^2 + 3x^{-2}y^3 - 4x^{-3}y^4$, and the resulting MS is illustratedin Figure 2. Note that it maintains the ADC, but it violates the proper simplicial condition. The generated MS for $f_{\sigma(n)} = 2(-2x^{-1}y^2 + 3x^{-2}y^3 - 4x^{-3}y^4) + 7x^{-4}y^5$, preserving the ADC and proper simplicial condition, is illustrated in Figure 3.

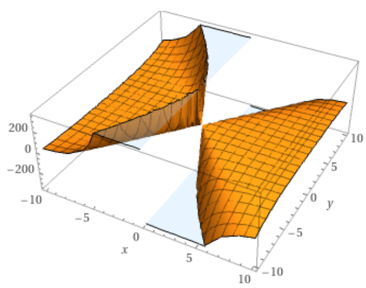

Figure 2. Manifold of simplicial polynomial with three atoms (addition).

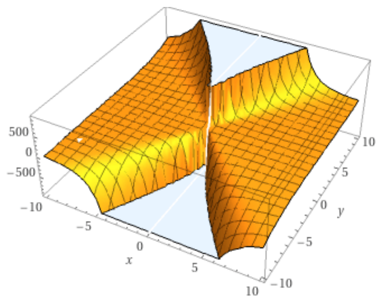

Figure 3. Manifold of simplicial polynomial with four atoms (with multiplicative scaling and addition).

Note that the topological homeomorphism property is not always preserved by the manifolds due to the increase in the numbers of atoms in the simplicial polynomials in additive forms if the simplicial condition is violated while maintaining the ADC. Let us consider polynomial $f_{\sigma(n)} \in F[x,y]$, which is algebraically decomposable. The definition of the hybrid decomposition is given as follows.

Definition 5. *A decomposable simplicial polynomial $f_{\sigma(n)} \in F[x,y]$ is in the class of hybrid decomposition if it can be expressed in the form given by $f_{\sigma(n)} = u.v.q(w)$, where $u \in F[y]$ and $v, w \in F[x,y]$.*

Note that the hybrid decomposition reveals the condition on the corresponding algebraic set that $Zr(f_{\sigma(n)}) = Zr(u) \cup Zr(v) \cup Zr(q(w))$. Moreover, the dimensions of the irreducible components in the hybrid decomposed class vary, and the compositions of polynomials are involved.

2.2. Degenerated Simplicial Polynomials

We can formulate the degenerated forms of simplicial polynomial $f_{\sigma(n)} \in F[x,y]$ in topological spaces if we relax the ADC while allowing topological decomposition of the degenerated $f_{\sigma(n)} \in F[x,y]$, where at least one topological component is an irreducible variety (i.e., irreducible algebraic zero-set). Note that the degenerated simplicial polynomials retain the simplicial condition on coefficients to form the corresponding topological space. The degenerate form of $f_{\sigma(n)} \in F[x,y]$ is defined as follows.

Definition 6. *The degenerated form of $f_{\sigma(n)} \in F[x,y]$ over $\left\langle f, \sigma_{(n)} \right\rangle$ in a topological space is defined as $f_{\sigma(n)} = p(x,y) = qh$, where $q \in F[x]$ and $h \in F[y]$ without retaining the ADC.*

Note that a degenerated $f_{\sigma(n)} \in F[x,y]$ is algebraically decomposed, and, as a result, it is topologically separable in $A^3(F)$.

Example 2. *Let us consider the degenerated form of $f_{\sigma(n)} \in F[x,y]$ in $A^3(F)$, which is given as $q(x) = x$ and $h(y) = t_1 + t_2y^2 + t_3y^3$ such that it preserves the simplicial condition $\sum_{j=1}^{3} t_j = 1$. Suppose we choose the coefficients in a combinatorial selection as $\{t_1, t_2 = t_1^2, t_3 = (1 - t_1 - t_2)\}$ such that $\sum_{j=1}^{3} t_j = 1$. The resulting topological manifold generated by the degenerated $f_{\sigma(n)}$ is illustrated in Figure 4 considering $t_1 = 0.1$ (i.e., keeping the first coefficient within a small neighborhood of zero). The z-axis represents the MS generated by the polynomials in degenerate forms.*

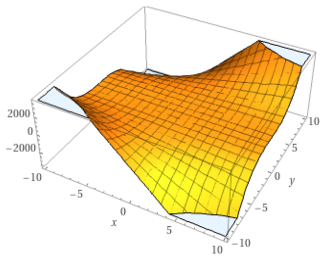

Figure 4. Topological manifold of degenerated $f_{\sigma(n)}$ for $t_1 = 0.1$.

If we change the choice of first coefficient to higher values (i.e., $t_1 \to 1$ within an expanded neighborhood of zero), then the resulting degenerated polynomial forms a topologically isomorphic manifold with altered orientation through rotation. This is illustrated in Figure 5, where $t_1 = 0.99$.

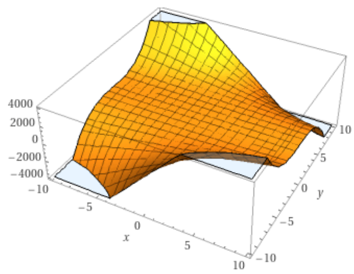

Figure 5. Isomorphic topological manifold of degenerated $f_{\sigma(n)}$ for $t_1 = 0.99$.

Note that the generated isomorphic manifolds are rotationally symmetric depending on the values of coefficients in $(0, 1)$. However, if we shift the coefficients in the negative real field (by relaxing the simplicial condition about coefficients being always positive), then the generated manifolds do not show any altered rotational orientations, and the topological isomorphism is preserved, as illustrated in Figures 6 and 7.

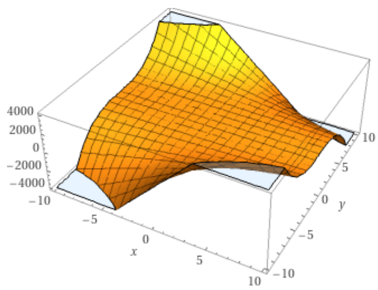

Figure 6. Topological manifold of degenerated $f_{\sigma(n)}$ for $t_1 = -0.99$.

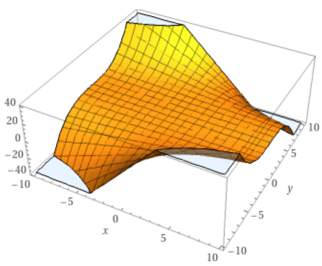

Figure 7. Topological manifold of degenerated $f_{\sigma(n)}$ for $t_1 = -0.1$.

Note that, in all cases, the degenerated simplicial polynomials generate isomorphic topological manifolds with varying orientations.

3. Properties of Simplicial Polynomials

In this section, we present a set of topological as well as algebraic properties of the non-degenerate class of simplicial polynomials, and the associated hybrid decompositions. First, we present the properties of the topological as well as algebraic (irreducible) decomposition of simplicial polynomials in multiplicative forms and the preservation of the additive extension property.

Theorem 2. *If $S \subset F[x,y]$ is in the non-degenerate class such that $S = \left\{ f_{\sigma(k)} : k \in [1,n] \right\}$, then the zero-set of every simplicial polynomial in S is decomposable in multiplicative 3-components with at least one irreducible component and an ideal component generated from a radical if $k > 1$.*

Proof. Let us consider a set of simplicial polynomials $S \subset F[x,y]$ in the non-degenerate class such that $S = \left\{ f_{\sigma(k)} : k \in [1,n] \right\}$, and let us select an $f_{\sigma(m)}$ such that $m > 1$. If we consider that $f_{\sigma(m)} = \sum_{j=1}^{m} \lambda_j x^{-j} y^{j+1}$ and $\lambda_j \in F$, then it results in the following derivation:

$$\begin{aligned} & f_{\sigma(m)} = \sum_{j=1}^{m} \lambda_j x^{-j} y^{j+1}, \\ & \Rightarrow f_{\sigma(m)} = \left(x^{-1}y\right)^m.(\lambda_m y + \lambda_{m-1} x + \lambda_{m-2} x^2 y^{-1} + \lambda_{m-3} x^3 y^{-2} + ...), \\ & \Rightarrow f_{\sigma(m)} = \left(x^{-1}y\right)^m.(h_m \in F[x,y]). \end{aligned}$$

Hence, the polynomial $g = \left(x^{-1}y\right)^m$ is an ideal for $m > 1$ generated from the radical for $m = 1$, and the multiplicative decomposition reveals the components given by $Zr(f_{\sigma(m)}) = Zr(g^m) \cup Zr(y) \cup Zr(h_m/y)$, where $Zr(y)$ is an irreducible component in the decomposition. $\square$

The following corollary illustrates that a set of finitely generated simplicial polynomials of the non-degenerate class preserves the Noetherian property of zero-sets when such polynomials are extended in additive forms.

Corollary 1. *In non-degenerate class $S \subset F[x,y]$, the algebraic set $Zr(S) = \bigcap_k Zr(f_{\sigma(k)})$ is not empty for a sufficiently large $k \in [m, \infty) \subset Z^+$.*

Proof. Let us consider simplicial polynomials $S \subset F[x,y]$ such that every $f_{\sigma(k)} \in S$ is finitely generated in additive forms for $k \in [m, \infty) \subset Z^+$. Note that F represents the real algebraic field. Thus, the field is Noetherian, and we can conclude that $I(Zr(S)) = \bigcup_k I_k$, where $\{I_k\}$ is a Noetherian class. Hence, the algebraic zero-set $Zr(S)$ cannot be empty if $k \in [m, \infty) \subset Z^+$. $\square$

Example 3. *The set $A \subset \{(x,y) : x \in R\backslash\{0\}; y = 0\}$ is an algebraic zero-set of a set of finitely generated simplicial polynomials.*

Remark 2. *Evidently, the non-degenerate class of simplicial polynomials given as set $S \subset F[x,y]$ reveals the formation of a Noetherian coordinate ring $A_{CR}(Zr(S)) = F[x,y]/I(Zr(S))$. Moreover, if $g = (x^{-1}y)$ and $I(M \subset A^2(F)) = \left\{ g^{-k}.f_{\sigma(k)} \right\}$, then $I(M) \subset F[x,y]$ is in a finitely generated Noetherian class.*

The concept of *isolated zero* is a topological property, and it is widely employed in algebraic geometry. The following theorem illustrates that any non-degenerate class of simplicial polynomials always preserves the respective topologically isolated origin.

Theorem 3. *If $S \subset F[x,y]$ is in a non-degenerate class of simplicial polynomials, then every $f_{\sigma(m)} \in S$ preserves the topologically isolated origin for all $m > 0$.*

Proof. Let $S \subset F[x,y]$ be in a non-degenerate class of simplicial polynomials, and let $N((0,0), \varepsilon > 0)$ be a neighborhood of $(0,0)$ in the topological space $A^2(F)$. Thus, we can find a unique decomposition of every $f_{\sigma(m)} \in S$ given in the form $f_{\sigma(m)} = (g^m \in F[x,y]).(h_m \in F[x,y])$, where $g = x^{-1}y$ and $m > 0$. Note that the zero-set of the component $Zr(g^m)$ is reducible in a topological subspace $A^2(F) \backslash N((0,0), \varepsilon > 0)$ for sufficiently large m. Hence, every $f_{\sigma(m)} \in S$ in a non-degenerate class of simplicial polynomials preserves the respective topologically isolated origin. □

Example 4. *The preservations of the topologically isolated origin by the decomposable and separable radical as well as ideal components of $f_{\sigma(m)}$ are presented in the following figures (Figures 8–11).*

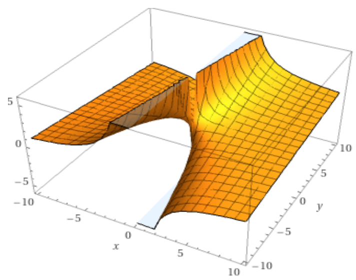

Figure 8. Formation of topologically isolated origin in manifold for m = 1.

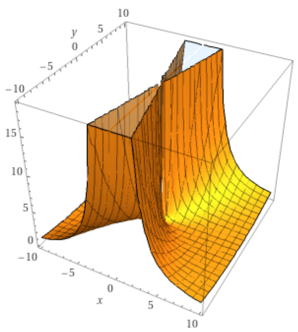

Figure 9. Formation of topologically isolated origin in manifold for m = 2.

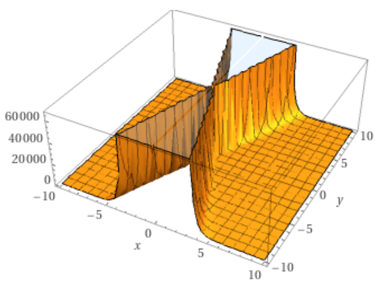

Figure 10. Formation of topologically isolated origin in manifold for m = 10.

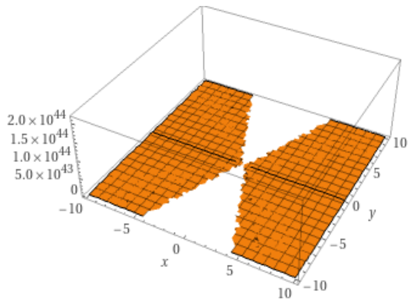

Figure 11. Formation of topologically isolated origin in manifold for m = 100.

Interestingly, the distinct behaviors of the respective decomposable and separable radicals and ideals are observable in all cases outside of $N((0,0), \varepsilon > 0)$. Furthermore, the simplicial polynomials in a non-degenerate class can reveal hybrid decomposition, as presented in the following theorem.

Theorem 4. *Every $f_{\sigma(k)} \in S \subset F[x,y]$ admits hybrid decomposition if $k > 1$.*

Proof. Let $f_{\sigma(m)} \in S \subset F[x,y]$ be a simplicial polynomial in a non-degenerative class such that $m > 1$. The simplicial polynomial can be decomposed as $f_{\sigma(m)} = (g^m \in F[x,y]).(h_m \in F[x,y])$, where a component is given by $h_m(x,y) = (\lambda_m y + \lambda_{m-1}x + \lambda_{m-2}x^2y^{-1} + \lambda_{m-3}x^3y^{-2} + ...)$. Suppose we denote $q(w) = [\lambda_m + \lambda_{m-1}w + \lambda_{m-2}w^2 + ...]$, where $w \in F[x,y]$. We can derive it further as follows:

$$\begin{aligned} &h_m = u.q(w), \\ &(u = y) \in F[y], \\ &w = xy^{-1}. \end{aligned} \tag{1}$$

Note that $q \in F[x,y]$ is a composite irreducible. Hence, we conclude that $Zr(f_{\sigma(m)}) = Zr(g^m) \cup Zr(u) \cup Zr(q)$, and it reveals hybrid decomposition. □

The hybrid decomposition leads to the following lemma, providing an interesting insight.

Lemma 2. *If $l : F \to E^2$ is a function in topological 2-space, then $(l \circ w^m)$ generates a simplex chain by the component $q(w)$ in hybrid decomposed $h_m \in F[x,y]$ for all m.*

Proof. We consider the irreducible component $Zr(q) \subset A^2(F)$ of an $h_m \in F[x,y]$ under the hybrid decomposition. Note that $\lambda_{m-j} \in F$, $w^m(x,y) \in F$ for the values of x, y, where $j = 0, 1, 2.......$ If we consider that $l : F \to E^2$ is a function within the topological 2-space, then $(l \circ w^m)(x,y) \in E^2$ is finite, and it generates $\sigma_{q(w)} = \sum_j \lambda_{m-j}.(l \circ w^j)$, which is a simplex chain in E^2. □

4. Topological and Algebraic Comparisons

In this section, we present the comparative analysis of the proposed formulations with respect to the related works in the domain. The simplicial complexes are the fundamental objects in geometric topology, and the polynomial algebras are fundamental to the algebraic geometry. Earlier, several attempts were made to bridge between the simplexes and polynomial algebras [9,21–23]. In general, the formulations of simplicial polynomials in a multicomplex consider the algebraic field of natural numbers N involving the complex variables [9]. For example, the simplicial polynomial with real zeros can be given as $p(z) \in N[z]$. Note that the formulations of simplicial polynomials proposed in this paper consider the closed real algebraic field R, and the variables are also in the set of reals. The computational approaches to form simplicial polynomials consider that $\{K_n : n \in N\}$ is a graded simplicial set indexed over the set of natural numbers, which is equipped with two functions given as $\partial_i^n : K_n \to K_{n-1}$ and $\mu_i^n : K_n \to K_{n+1}$, representing the face map and degeneracy map, respectively [21,22]. As a result, the simplicial polynomials are formulated as a list of monomials such as $p = 3\mu_4\mu_1\partial_3\partial_6\partial_7 - 2\mu_1\partial_3\partial_4$ involving the respective face maps and degeneracy maps [21]. Note that the indexes of face maps should employ strictly increasing indexes in the computational formulations, where the coefficients cannot be zero and the degrees cannot be negative for computations. On the other hand, as a distinction, the formulations proposed in this paper employ polynomial algebraic forms to construct the simplicial polynomials in the settings of algebraic geometry, and the formulations are generalized in nature because they involve coefficients as well as degrees of polynomials in the real algebraic field R. As a result, the proposed formulations in this paper reveal formations of ideals from the radical component for sufficiently long simplicial chains. The formulations of the $k - polynomial$ over a simplicial complex k are represented as $p_k(x) = \sum_{i \geq 1} f_i x^i$, which is in an algebraic polynomial form [23]. The $k - polynomial$ is constructed by employing the counting of $i - faces$. However, the formulations of simplicial polynomials proposed in this paper do not follow a similar approach, and the indexed face counting is not employed in the proposed constructions.

Finally, we present the distinctive topological properties of the proposed simplicial polynomial and the Laurent polynomial [24,25]. Let us consider an example of the Laurent polynomial over the real algebraic field R with three monomials as $f_L(x, y) = 0.2 + 0.7x^{-1}y + 0.1xy^{-1}$, maintaining the strict simplicial conditions in standard form. On the other hand, consider the corresponding simplicial polynomial $f_\sigma(x, y) = 0.2x^{-1}y^2 + 0.7x^{-2}y^3 + 0.1x^{-3}y^4$ by following the proposed formulations. The topological manifolds generated by the Laurent polynomial and the simplicial polynomial are illustrated in Figures 12 and 13, respectively.

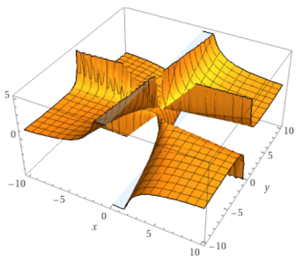

Figure 12. Topological manifold of Laurent polynomial.

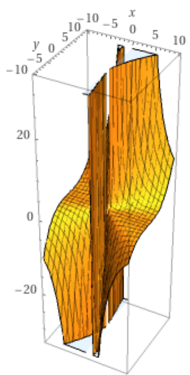

Figure 13. Topological manifold of corresponding simplicial polynomial.

Note that the topological homeomorphism property is not preserved by the manifolds although the coefficients maintain strict simplicial conditions in standard form in both the cases. The axes of symmetries are different in the respective manifolds. Note that the decompositions of multivariate polynomials are formulated by employing the composition of functions to form degenerate classes [26]. However, as a distinction, the degenerate class of simplicial polynomials proposed in this paper does not employ any function compositions or generator functions. However, the proposed simplicial polynomials admit hybrid decomposition.

Simplicial Polynomial in Alternate Form

In this section, we compare the structure of topological manifolds given in an alternative form $f_{\sigma(n)} = \sum_{j=1}^{n} (t_j a_j) x^j y^{-j+1}$ such that $\sum_{j=1}^{n} (t_j a_j) = 1$. Note that it preserves the ADC. Let us consider the simplicial polynomials in alternate form as $f_{\sigma(n)} = -2x + 3x^2 y^{-1}$ and $f_{\sigma(n)} = -2x + 3x^2 y^{-1} - 4x^3 y^{-2}$ such that the coefficients are equal to the respective simplicial polynomials given in Example 1. The respective topological manifolds generated by the simplicial polynomials in alternate form are presented in Figures 14 and 15.

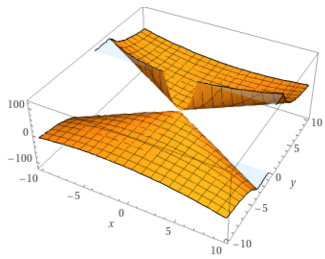

Figure 14. Topological manifold of simplicial polynomial $f_{\sigma(n)} = -2x + 3x^2 y^{-1}$.

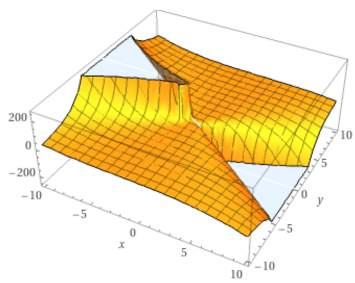

Figure 15. Topological manifold of simplicial polynomial $f_{\sigma(n)} = -2x + 3x^2 y^{-1} - 4x^3 y^{-2}$.

Note that the topological manifold generated from two atoms in the alternate form of the simplicial polynomial is topologically homeomorphic to the topological manifold generated from three atoms of the original form of the simplicial polynomial given in Example 1 (i.e., by comparing Figures 2 and 14). The topological homeomorphism is observable if we compare Figures 1 and 15. Moreover, the manifolds have attained rotational symmetry due to the *changes in the composition of atoms* in the simplicial polynomials considering the original form and the alternative form. It indicates that the changes in atomic structures of simplicial polynomials, while maintaining ADC, induce combinatorial formations of homeomorphic topological manifolds with varying rotational orientations. Finally, we compare the decomposed components and degeneration of other classes of polynomials with respect to the simplicial polynomials proposed in this paper. Interestingly, the special class of polynomials, called Hermite polynomials, can be decomposed in the forms of monomials under certain conditions [27]. However, the formulations of simplicial polynomials proposed in this paper are not fully decomposable into a set of monomials or atoms. The decompositions of the proposed simplicial polynomials generate topologically irreducible components and a radical ideal. Note that, in general, the degenerate classes of polynomials are formulated by employing the generator functions [28]. However, as a distinction, the degenerate class of simplicial polynomials proposed in this paper does not employ any specific generator functions.

5. Conclusions

The formulations of simplicial polynomials consider the elements of topology and algebraic geometry in a combination, exposing a set of interesting algebraic as well as topological properties. The simplicial polynomials can be formulated in a non-degenerate class and in a degenerate class depending upon the preservation of atomic degrees. The topological manifolds generated by different classes of simplicial polynomials exhibit various topological properties. The topological manifolds generated by the non-degenerate class of simplicial polynomials do not retain homeomorphism, whereas the topological manifolds generated by the degenerate class of simplicial polynomials retain isomorphism with varying rotational orientations. The simplicial polynomials in the non-degenerate class are decomposable, and it gives rise to radical components for the small number of vertices of simplexes preserving the topologically isolated origin in the topological manifolds. The hybrid decomposition of a simplicial polynomial leads to the formation of simplex chains in topological spaces in lower dimensions. In future, it would be interesting to investigate the interrelationships between the decompositions of simplicial polynomials and the resulting deformations of topological manifolds.

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