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Functional and Operatorial Equations Defined Implicitly and Moment Problems

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Abstract: The properties of the unique nontrivial analytic solution, defined implicitly by a functional equation, are pointed out. This work provides local estimations and global inequalities for the involved solution. The corresponding operatorial equation is studied as well. The second part of the paper is devoted to the full classical moment problem, which is an inverse problem. Two constraints are imposed on the solution. One of them requires the solution to be dominated by a concrete convex operator defined on the positive cone of the domain space. A one-dimensional operator is valued, and a multidimensional scalar moment problem is solved. In both cases, the existence and the uniqueness of the solution are proved. The general idea of the paper is to provide detailed information on solutions which are not expressible in terms of elementary functions.

Keywords: holomorphic function; implicitly defined function; self-adjoint operators; quadratic forms; operator coefficients; polynomial approximation; moment problem

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1. Introduction

The implicit function theorem is a very important tool in obtaining the properties and information on an implicitly defined function, which cannot be expressed in terms of elementary functions. However, the equations or problems that implicitly define the solution can be stated in terms of elementary functions. Only the exact solution is unknown. This is the purpose of the first part of the present work. On the other hand, finding the solutions for moment problems, only in terms of the given moments and one or two constraints, is an inverse problem. In other words, an unknown measure or function or linear operator should be determined in terms of the given moments and other given data. This is the aim of the second part of this work. From this viewpoint, one can say that a relationship and common point exists between the first and the second part of this paper. From the point of view of the methodology, classical analysis and functional analysis are applied. To use these methods for solving our problems in Section 3, we partially use results in real, complex, and functional analysis, which can be found in the books/monographs [1–7]. The books and respective articles [6,8–15] refer to the moment problem. In many of these references, Hahn-Banach-type theorems and/or their consequences play a significant role (for example, see [9]). For the study of the Maximum Entropy solution for the reduced moment problem on unbounded intervals and its approximation, see [14,15]. In [16], stability questions in the truncated trigonometric moment problem are under attention. For the results on sequences of important special polynomials, not necessarily related to the moment problem, see [17,18]. These two articles could provide new ideas for other authors and for the readers to continue and/or complete the present work. As is well-known, the Hahn-Banach typer results are not only used in moment and related problems (for example, see [7]). In the paper [19], we complete our study of the functional equation:

$$g(x) = g(f(x)),\ x \in (a,b) \subseteq \mathbb{R},$$

where g is given with natural properties, and $f \not\equiv id$ is the unique nontrivial solution. Namely, in [19], one proves that the analyticity of g appearing in the above-stated equation in a complex open neighborhood V of the interval (a,b), implies the analyticity of the solution f on the entire V. We assume that $g(x) \in \mathbb{R}$ and $f(x) \in \mathbb{R}$ for all real.

$x \in V \cap \mathbb{R}$. In our previous proofs, one assumes that there is a unique point α in the open interval (a,b) at which the derivative $g'(\alpha)$ is null. It results that α is the unique fixed point of the solution f. The analyticity of f at α was also proved. Finally, the papers [20,21] are devoted especially to Hahn-Banach and polynomial approximation-type results on compact and on unbounded subsets, applied to the moment problem. In the first part of the present work, we provide detailed information on the properties and estimates of the nontrivial solution f of the equation $ze^{-z} = f(z)e^{-f(z)}$, in a complex neighborhood of the interval $(0, +\infty)$. The behavior around $\alpha = 1$ is pointed out. The related operatorial equation in a commutative algebra of self-adjoint operators is also studied. For this first subject, see Theorems 1 and 2 and Corollary 1. The second part of the paper is devoted to a one dimensional and a multidimensional moment problem, respectively, with two constraints on the linear operator (respectively, linear functional) satisfying the moment interpolation conditions. Unlike our previous paper on this topic, here, we prove results that are valid on concrete spaces and concrete constraints defined by convex and null operators, respectively, on the positive cone of the domain space. In the end, a multidimensional moment problem on $[0, +\infty)^d$, $d \in \mathbb{N}, d \geq 2$ is solved and a related example is emphasized. For Hahn-Banach-type results on the extension of linear operators applied to the moment problem and to other subjects, see [21]. The rest of the paper is organized as follows. Section 2 resumes the methods that are going to be applied. In Section 3, the results are stated and proved. Section 4 (Discussion) concludes the paper.

2. Methods

The basic methods used in this work are:

1. Real and complex functions. Basic inequalities and expansions in Taylor series [1,2]. Applications to functional equations.
2. Properties of holomorphic functions. Implicit function theorem for real and complex differentiable functions, respectively [19]. Properties of unknown functions implicitly defined, expressed in terms of elementary functions. Local approximation; local and global inequalities.
3. Equalities and inequalities for self-adjoint operators deduced via functional calculus [3,4,7], from the new inequalities proved in the present work. For recent results in operator theory referring to the moment problem, see [11,16].
4. Using an order complete Banach lattice of self-adjoint operators, which is also a commutative algebra [5].
5. Applying Hahn-Banach-type results to the existence of a positive linear solution dominated by a convex operator for the full classical moment problem; application of polynomial approximation on unbounded subsets to the multidimensional full moment problem [20,21]. To this aim, we use the notion of a moment determinate measure and a sufficient condition for determinacy [12].
6. In Corollary 3, a necessary passing to the limit condition related to Theorem 3 is expressed in terms of quadratic forms with operator coefficients.

3. Results

3.1. Implicitly Defined Solutions of Functional and Operatorial Equations

In what follows, we apply results from [1–7,12,19–21].

Theorem 1. *Let us consider the functional equation:*

$$ze^{-z} = f(z)e^{-f(z)}, \quad Re(z) > 0. \tag{1}$$

There exists a unique nontrivial solution f of the Equation (1), with $f(z) \neq z$ for $z \neq 1$, f is holomorphic in an open complex neighborhood Ω of $(0, +\infty)$, $f(x) \in (0, +\infty)$ for all $x \in (0, +\infty)$, and f satisfies the following conditions:

(i) *The restriction of f to the interval $(0, +\infty)$ is decreasing, $f(0+) = +\infty$, $f(\infty-) = 0+$.*
(ii) $f(1) = 1$, $f'(1) = -1$.
(iii) $f(f(z)) = z$ *for all* $z \in \Omega$.
(iv) *The function f is strictly convex in an interval $(1-\varepsilon, 1+\varepsilon)$, with $\varepsilon > 0$ being sufficiently small, and the following inequalities hold:*

$$f(x) \geq 2 - x \geq 3x - 2x^2.$$

In each of these inequalities, equality occurs if and only if $x = 1$.
(v) *In a disc $D(1;\varepsilon) = \{z;\ |z-1| < \varepsilon\}$, of small radius, the following two-degree polynomial approximation of f holds:*

$$f(z) - z \approx \frac{1}{4}\left(3z - \left(9z^2 - 48z^2(1-z)\right)^{1/2}\right).$$

(vi) *If $\varepsilon > 0$ is sufficiently small and $x \in [1, 1+\varepsilon)$, then the following inequalities hold:*

$$\frac{1}{4}x[3 - (9 - 48(1-x))^{1/2}] \leq f(x) - x \leq 0.$$

If $x \in (1-\varepsilon, 1]$, and $0 < \varepsilon \leq 3/16$, then:

$$0 \leq f(x) - x \leq \frac{1}{4}x[3 - (9 - 48(1-x))^{1/2}\Big] \leq 4\varepsilon,$$

Proof. Let $g : \mathbb{C} \longrightarrow \mathbb{C}$ be the function defined by:

$$g(z) = ze^{-z}, \ \ z \in \mathbb{C}. \tag{2}$$

For $x \in [0, +\infty)$, $g(x) = xe^{-x}$ has the properties easily deduced from the calculus in one real variable: $g \in C^{(\infty)}((0, +\infty), \mathbb{R})$, g is increasing on the interval $[0, 1]$ and decreases on the interval $[1, +\infty)$. The global maximum point is $\alpha = 1$, with $g_{max} = g(\alpha) = g(1) = e^{-1}$. On the other hand, elementary computations also using L'Hopital's Rule yield:

$$\lim_{x \searrow 0} g(x) = \lim_{x \searrow 0}(xe^{-x}) = 0, \ \ \lim_{x \nearrow +\infty} g(x) = \lim_{x \nearrow +\infty}(xe^{-x}) = \lim_{x \nearrow +\infty}(x/e^x) = \lim_{x \nearrow +\infty}(1/e^x) = 0.$$

Thus, all of the requirements in the hypothesis of Theorem 3.1 from [19] (for detailed proofs, see also the reference there) are accomplished. Applying the conclusion of the theorem, the assertions (i) and (ii) follow; the equality

$$f(f(x)) = x, \ \ x \in (0, +\infty)$$

follows as well. On the other hand, since $g(z) = ze^{-z}$ is holomorphic on the entire complex filed, the application of Theorem 3.4 from [19] and the uniqueness of analytic continuation [2] lead to the conclusion that f is holomorphic on a complex open connected neighborhood Ω of $(0, +\infty)$, and the equality stated at point (iii) holds for all $z \in \Omega$. To prove (iv), from (1) written for $z = x \in (0, +\infty)$, also using (i) and (ii), we infer that $\varepsilon \in (0, 1)$ is sufficiently small, such that:

$$x \in (1-\varepsilon, 1) \Longrightarrow 0 < \frac{f(x) - x}{x} < 1 \Longrightarrow -\frac{f(x) - x}{x} > -1, \tag{3}$$

one obtains:

$$(1) \Longrightarrow e^{f(x)-x} = f(x)/x.$$

All of these further yields:

$$\begin{gathered} f(x) - x = log(f(x)/x) = log\left(1 + \frac{f(x)-x}{x}\right) = \\ \int_0^{(f(x)-x)/x} \frac{1}{1+t} dt = \int_0^{(f(x)-x)/x} \left(1 - t + t^2 + \cdots + (-1)^n t^n + \cdots\right) dt = \\ \frac{f(x)-x}{x} - \frac{1}{2}\left(\frac{f(x)-x}{x}\right)^2 + \frac{1}{3}\left(\frac{f(x)-x}{x}\right)^3 - \cdots = \\ \frac{f(x)-x}{x} - \frac{1}{2}\left(\frac{f(x)-x}{x}\right)^2 + \sum_{n \geq 1}\left(\frac{f(x)-x}{x}\right)^{2n+1}\left(\frac{1}{2n+1} - \frac{1}{2n+2} \cdot \frac{f(x)-x}{x}\right). \end{gathered} \tag{4}$$

Writing (4) for $n = 1$, the following first conclusion holds:

$$f(x) - x = \frac{f(x) - x}{x} - \frac{1}{2} \cdot \left(\frac{f(x) - x}{x}\right)^2 + \left(\frac{f(x) - x}{x}\right)^3 \left(\frac{1}{3} - \frac{1}{4} \cdot \frac{f(x) - x}{x}\right). \tag{5}$$

For $x \neq 1$, we know that $f(x) - x \neq 0$, so that (5) may be written as:

$$1 = \frac{1}{x} - \frac{1}{2} \cdot \frac{f(x) - x}{x^2} + \frac{(f(x) - x)^2 \cdot}{x^3} \cdot \frac{1}{3} + \omega(x), \ \lim_{x \to 1} \frac{\omega(x)}{(f(x) - x)^2} = 0. \tag{6}$$

For small $\varepsilon > 0$, $|x - 1| < \varepsilon$, Equation (6) further yields:

$$\begin{gathered} \lim_{x \to 1} \frac{1 - 1/x + (f(x) - x)/\left(2x^2\right)}{\frac{(f(x)-x)^2}{x^3}} = \\ \lim_{x \to 1} \frac{x^2 - x + (f(x) - x)/2}{(f(x) - x)^2} = \frac{1}{3} > 0. \end{gathered} \tag{7}$$

From this last equality, it follows that for $x \in (1 - \varepsilon, 1 + \varepsilon)$ with small $\varepsilon > 0$, the nominator of the ration on the left-hand side must be positive; that is:

$$\begin{gathered} x^2 - x + (f(x) - x)/2 > 0, \\ f(x) > 3x - 2x^2, \ \ x \in (1 - \varepsilon, 1 + \varepsilon), \ \ x \neq 1. \end{gathered} \tag{8}$$

Using l'Hopital's rule twice in (7) and the equalities $f(1) = 1,\ f'(1) = -1$, it results in:

$$\frac{1}{3} = -\frac{1}{4} \lim_{x \to 1} \frac{2 + f''(x)/2}{f'(x) - 1} = \frac{1}{8}(2 + f''(1)/2) = \frac{1}{4} + \frac{f''(1)}{16},$$

$$f''(1) = \frac{16}{12} = \frac{4}{3}.$$

Since $f''(x)$ is continuous and positive at $x = 1$, we have $f''(x) > 0$ for all x in the small interval $(1 - \varepsilon, 1 + \varepsilon)$. Thus, f is strictly convex on $(1 - \varepsilon, 1 + \varepsilon)$. Consequently, through the subgradient inequality for strictly convex differentiable functions, the following conclusion holds:

$$f(x) - f(1) = f(x) - 1 \geq f'(1)(x - 1) = -(x - 1),$$

$$f(x) \geq 2 - x, \qquad x \in (1 - \varepsilon, 1 + \varepsilon).$$

In this inequality, equality occurs if and only if $x = 1$. Hence, the first inequality stated at point (iv) is proved. On the other hand, the function

$$\varphi(x) = 3x - 2x^2$$

is strictly concave on the entire real axes. Therefore, its graph is dominated by the line representing the graph of the tangent to the graph of φ at the point $(1,1) = (1, \varphi(1)) = (1, f(1))$. As

$$\varphi(1) = f(1) = 1, \qquad \varphi'(1) = f'(1) = -1,$$

the equation of this tangent is given by $y - 1 = (-1)(x-1)$; that is $y = 2 - x$. Now the desired inequalities from (iv) are proved. The assertion (v) follows quite easily too, as all the involved functions are analytic in a neighborhood centered at 1. Hence, the following remarks and equalities hold. The function: $h(z) := \frac{f(z)-z}{z}$ is holomorphic (hence, is continuous) on the disc:

$D(1;\varepsilon) = \{z;\ |z-1| < \varepsilon\}$. Thus, according to the assertion (ii), $h(1) = f(1) - 1 = 0$, from (4) written for complex variable z instead of x, by means of analytic continuation, we derive:

$$f(z) - z = \frac{f(z)-z}{z} - \frac{1}{2}\left(\frac{f(z)-z}{z}\right)^2 + \varphi(z), \tag{9}$$

with

$$\varphi(z) := \sum_{n\geq 1}\left(\frac{f(z)-z}{z}\right)^{2n+1}\left(\frac{1}{2n+1} - \frac{1}{2n+2}\cdot\frac{f(z)-z}{z}\right) = \left(\frac{f(z)-z}{z}\right)^3 w(z). \tag{10}$$

Here, w is holomorphic in $D(1;\varepsilon)$, with

$$\lim_{z\to 1} w(z) = \frac{1}{3}. \tag{11}$$

From all of these, as $f(z)$ differs from z for all $z \neq 1$, also using (6), it results:

$$1 = \frac{1}{z} - \frac{1}{2}\cdot\frac{f(z)-z}{z^2} + (f(z)-z)^2\frac{w(z)}{z^3}. \tag{12}$$

Hence,

$$\begin{gathered} z - 1 \approx 0 \Longrightarrow 1 - \left(\tfrac{1}{z} - \tfrac{1}{2}\cdot\tfrac{f(z)-z}{z^2}\right) = (f(z)-z)^2\tfrac{w(z)}{z^3} \Longrightarrow \\ 2z^2 - 2z + f(z) - z = 2(f(z)-z)^2\tfrac{w(z)}{z}. \end{gathered}$$

This can be written as:

$$z(f(z)-z) - 2z^2(1-z) = 2(f(z)-z)^2 w(z),\ \ \lim_{z\to 1} w(z) = \frac{1}{3}.$$

$$\begin{gathered} 2w(z)(f(z)-z)^2 - z(f(z)-z) + 2z^2(1-z) = 0, \\ 2(f(z)-z)^2 - 3z(f(z)-z) + 6z^2(1-z) \approx 0. \end{gathered} \tag{13}$$

As $z \approx 1$, $f(z) - z \approx 0$, $w(z) \approx \frac{1}{3}$, these approximate equalities imply:

$$\begin{gathered} f(z) - z \approx \tfrac{1}{4}\left(3z - \left(9z^2 - 48z^2(1-z)\right)^{1/2}\right) \approx \\ \tfrac{1}{4}\left(3 - (9 - 48(1-z))^{1/2}\right). \end{gathered} \tag{14}$$

Thus, the assertion stated at point (vi) is proved. To prove (vii), we write (10), where first z is replaced by $x \in (1, 1+\varepsilon)$. For such x, we have: $0 < x - f(x) \to 0$ as $x \searrow 1$,

$$3x - \left(9x^2 - 48x^2(1-x)\right)^{1/2} = x\left(3 - (9 - 48(1-x))^{1/2}\right) < 0.$$

The equation

$$2y^2 - 3xy + 6x^2(1-x) = 0,$$

where $y(x)$ stands for $f(x) - x < 0$, has two roots $y_1(x),\ y_2(x)$ with the product

$$y_1(x) \cdot y_2(x) = 3x^2(1-x) < 0.$$

On the other hand, Equations (8) and (4) yield:

$$1 = \frac{1}{x} - \frac{1}{2} \cdot \frac{f(x) - x}{x^2} + \frac{(f(x) - x)^2}{x^3}\left(\frac{1}{3} - \frac{1}{4} \cdot \frac{f(x) - x}{x}\right) + 0\left((f(x) - x)^2\right). \tag{15}$$

As $-\frac{1}{4} \cdot \frac{f(x)-x}{x} > 0$, for sufficiently small $\varepsilon > 0$, $x \in (1, 1+\varepsilon]$ implies:

$$1 \geq \frac{1}{x} - \frac{1}{2} \cdot \frac{f(x) - x}{x^2} + \frac{1}{3} \cdot \frac{(f(x) - x)^2}{x^3}. \tag{16}$$

Multiplying by $6x^3 > 0$, the following inequality follows:

$$2(f(x) - x)^2 - 3x(f(x) - x) + 6x^2(1-x) \leq 0.$$

This implies:

$$y_1(x) = \frac{1}{4}[3x - \left(9x^2 - 48x^2(1-x)\right)^{1/2}] \leq f(x) - x \leq 0.$$

We next prove the second assertion form point (vii). For $x \in [1-\varepsilon, 1)$, we have $f(x) - x > 0$. Since $x < 1$, according to Equation (4), $n = 1$, it follows that:

$$1 \leq \frac{1}{x} - \frac{1}{2} \cdot \frac{f(x) - x}{x^2} + \frac{1}{3} \cdot \frac{(f(x) - x)^2}{x^3}. \tag{17}$$

In Equations (12) and (13), equality occurs if and only if $x = 1$. Multiplying by $6x^3 > 0$, from (12), after short calculation, with the reversed sense of the inequality (12), we infer that:

$$\begin{gathered} 2(f(x) - x)^2 - 3x(f(x) - x) + 6x^2(1-x) \geq 0, \\ 0 \leq f(x) - x \leq \tfrac{1}{4}[3x - \left(9x^2 - 48x^2(1-x)\right)^{1/2}],\ \ x \in [1-\varepsilon, 1]. \end{gathered} \tag{18}$$

To prove the last inequalities from the assertion (vii), we estimate the right-hand side member of (18). Namely, as x $\in (1-\varepsilon, 1]$ and $0 < \varepsilon \leq 3/16$, we have $0 \leq 1 - x < \varepsilon$,

$$\begin{gathered} 9 - 48(1-x) > 9 - 48\varepsilon = 3(3 - 16\varepsilon) \geq 0, \\ 0 \leq f(x) - x \leq \tfrac{1}{4}x[3 - (9 - 48(1-x))^{1/2}] = \\ \tfrac{12x(1-x)}{3 + (9 - 48(1-x))^{1/2}} \leq \tfrac{12(1-x)}{3} < 4\varepsilon. \end{gathered}$$

This ends the proof. □

Next, we apply the results proved in Theorem 1 to the spaces of bounded linear operators acting on a real or complex Hilbert space H. Most of the results and notations are those from [3,5,7]. Namely, $\mathcal{A}$ will be the real vector space of all (bounded) self-adjoint operators $A : H \longrightarrow H$, endowed with its natural order relation: $A, B \in \mathcal{A}$, $A \leq B$ if and only if $(Ah, h) \leq (Bh, h)$ for all $h \in H$, and with the operatorial norm: $\|A\| := \sup\limits_{\|h\| \leq 1} \|Ah\| = \sup\limits_{\|h\| \leq 1} |(Ah; h)|$. (We have denoted by (;) the inner (scalar) product on $H \times H$). Then, $\mathcal{A}$ is a real ordered Banach space—that is, $\mathcal{A}$ is topologically complete (see below)—and the norm is monotone, increasing on the positive cone $\mathcal{A}_+$ of $\mathcal{A}$:

$$A, B \in \mathcal{A},\ \ 0 \leq A \leq B \Longrightarrow \|A\| \leq \|B\|.$$

With this linear order relation, $\mathcal{A}$ is not a lattice. The convex cone of all positive self-adjoint operators will be denoted by $\mathcal{A}_+$. It is easy to see that $\mathcal{A}$ is a closed subspace of the Banach space $\mathcal{B}(H)$ of all bounded linear operators from H into H. If $T \in \mathcal{B}(H)$ and $\sigma(T)$ are the spectrum of T, for a holomorphic function $f = f(z)$ defined on an open neighborhood of $\sigma(T)$, by $f(T)$ we mean the operator valued function of operator variable T, corresponding to $f(z)$, having the same coefficients of the Taylor expansion. If the expansion of $f(z)$ is around a point $z_0 \in \mathbb{C}$, the expansion of $f(T)$ will be around $z_0 I$, where I is the identity operator on H. If $A \in \mathcal{A}$, then $f(A)$ makes sense for any $f \in C(\sigma(A))$. Moreover, if $f \in C(\sigma(A))$ takes non-negative (real) values at all points in $\sigma(A)$, then $f(A) \in \mathcal{A}_+$. In Theorems 1 and 2 (see below), we consider only holomorphic functions f which verify $x \in \mathbb{R} \Longrightarrow f(x) \in \mathbb{R}$. We will take

$$z_0 = \alpha \in \mathbb{R},$$

and, consequently, all the coefficients of the Taylor expansions centered at α will be real numbers. If the operator T is self-adjoint, it will be denoted by A. In this case, as is well-known, $\sigma(A) \subseteq \mathbb{R}$, $(Ah;h) \in \mathbb{R}$ for all $h \in H$. Moreover, $A \in \mathcal{A}_+$ (i.e., $(Ah;h) \geq 0\ \forall h \in H$), if and only if $\sigma(A) \subseteq [0, +\infty)$.

Theorem 2. *Let f be the function from Theorem 1. For sufficiently small $\varepsilon \in (0,1)$, there exists a nontrivial decreasing solution f defined on the order interval $((1-\varepsilon)I, (1+\varepsilon)I)$, with values in $\mathcal{A}_+$, for the equation:*

$$Ae^{-A} = f(A)e^{-f(A)}, \ \ A \in ((1-\varepsilon)I, (1+\varepsilon)I). \tag{19}$$

This solution f satisfies the following conditions:

(a) $f(I) = I$.

(b) *f is differentiable in a neighborhood $D \subseteq ((1-\varepsilon)I, (1+\varepsilon)I)$ of I, with $\varepsilon > 0$ as above, and $f'(I) = -I$.*

(c) *$f(f(A)) = A$ for all $A \in D$.*

(d) *For small $\varepsilon > 0$ and $\sigma(A) \subseteq ((1-\varepsilon), (1+\varepsilon))$, the function f satisfies the following inequalities:*

$$f(A) \geq 2I - A \geq 3A - 2A^2,$$

$$\sigma(A) \subseteq [1, (1+\varepsilon)) \Longrightarrow \frac{1}{4}A[3I - (9I - 48(I-A))^{1/2}] \leq f(A) - A \leq 0.$$

If $\sigma(A) \subseteq ((1-\varepsilon), 1], 0 < \varepsilon \leq 3/16$, then the following inequalities hold:

$$0 \leq f(A) - A \leq \frac{1}{4}A[3I - (9I - 48(I-A))^{1/2}] \leq 4\varepsilon I, \ \ \|f(A) - A\| \leq 4\varepsilon.$$

If $0 < \delta < \varepsilon < 1$, then $f(A)$ and $f(A) - A$ are invertible and

$$\left\|(f(A))^{-1}\right\| \leq 1, \ \ \left\|(f(A) - A)^{-1}\right\| \leq 1/\delta.$$

(e) *If $T \in \mathcal{B}(H)$ is a normal operator with the spectrum $\sigma(T)$ contained in a disc of small radius equal to $\varepsilon > 0$, centered at 1, then:*

$$\varepsilon \longrightarrow 0 \Longrightarrow \left\|f(T) - T - \frac{1}{4}\left(3T - \left(9T^2 - 48T^2(I-T)\right)^{1/2}\right)\right\| \longrightarrow 0.$$

Proof. The general idea of the proof is to apply continuous functional calculus for self-adjoint operators [3,5,7] and holomorphic functional calculus for normal operators [3], respectively, to the holomorphic function $f = f(z)$ defined implicitly by Equation (1) in an open disc $D(1, \varepsilon)$ of sufficiently small radius $\varepsilon \in (0, 1)$.

$$\begin{gathered} 0 < \delta < \varepsilon, \quad 1-\delta > 1-\varepsilon, \quad \sigma(A) \subset (1-\varepsilon, 1-\delta) \Longrightarrow \\ A \leq (1-\delta)I, \quad -A \geq -(1-\delta)I. \end{gathered} \tag{20}$$

$$\begin{gathered} f(x) \geq f(1-\delta) \geq f(1) = 1 \ \forall x \in \sigma(A) \Longrightarrow \\ inf(f(\sigma(A))) = inf\sigma(f(A)) \geq 1 \Longrightarrow f(A) \geq I. \end{gathered} \tag{21}$$

Hence, $f(A) \geq I$, $0 \notin \sigma(f(A))$, so that $f(A)$ is invertible and $\left\|(f(A))^{-1}\right\| \leq \|I\| = 1$. Moreover, (20) and (21) lead to

$$f(A) - A \geq I - (1-\delta)I = \delta I.$$

Thus, $\sigma(f(A) - A) \subseteq [\delta, +\infty) \not\ni 0$, so that $f(A) - A$ is invertible and

$$0 \leq (f(A) - A)^{-1} \leq (1/\delta)I,$$

$$\left\|(f(A) - A)^{-1}\right\| \leq 1/\delta.$$

The desired conclusions are proved. □

In what follows, we denote by $Sym(n, \mathbb{R})$ the ordered Banach space of all $n \times n,\ n \geq 2$, symmetric matrices with real entries. The order relation on $Sym(n, \mathbb{R})$ is defined like that on $\mathcal{A}(H)$, when $H = \mathbb{R}^n$ is the usual $n-$ dimensional Hilbert space over the real field. The positive cone of $Sym(n, \mathbb{R})$ is denoted by $Sym^+(n, \mathbb{R})$. For results on this ordered Banach space, see [4].

Corollary 1. *Let $f : D(\delta) \subset Sym^+(n, \mathbb{R}) \longrightarrow Sym^+(n, \mathbb{R})$ be the unique monotone decreasing solution of the matrix Equation (19), where $D(\delta) := \{A \in Sym^+(n, \mathbb{R});\ \|I - A\| < \delta\}$, and $\delta > 0$ is sufficiently small. Then, all of the assertions of Theorem 2 hold for symmetric positive matrices A which satisfy the properties of the points (a)–(e) from Theorem 2.*

3.2. On the Moment Problem

In what follows, we apply a polynomial approximation result and properties of Banach lattices to solve a Markov-type moment problem on the spectrum $\sigma(A)$ of a self-adjoint positive A acting on a real or complex Hilbert space H. Through $Y(A)$, we denote the commutative algebra over the real field of self-adjoint operators, constructed in [5], pp. 303–305. This algebra is a good example of subspace of the ordered Banach space $\mathcal{A} = \mathcal{A}(H)$, endowed with the natural order relation reviewed above. Unlike $\mathcal{A}$, $Y(A)$ is a (vector) lattice, which is also order complete. We briefly recall the definition of this Banach lattice and a few of its main properties. Firstly, one denotes:

$$Z(A) := \{B \in \mathcal{A}(H); BA = AB\}.$$

Then, one defines:

$$Y(A) := \{V \in Z(A); BV = VB,\ \forall B \in Z(A)\}.$$

By this definition, clearly any two operators from $Y(A)$ commute. One proves that for any operator $V \in Y(A)$, the self-adjoint (Hermitian) operator $|V| := \sqrt{V^2}$ (the positive square root of the positive operator V^2) has the following properties in the lattice $Y(A)$:

$$V^+ := \frac{1}{2}(|V| + V) = sup\{V, 0\},\ V^- := \frac{1}{2}(|V| - V) = sup\{-V, 0\},$$

$$|V| = sup\{V, -V\} = V \vee (-V) = V^+ + V^-, \ \ V = V^+ - V^-, \ \ V^+V^- = 0.$$

Through $\Psi = \Psi_A : C(\sigma(A)) \longrightarrow Y(A)$, one denotes the isometry defined as $\Psi(f) := f(A)$, where $f(A)$ is obtained via functional calculus for continuous functions, attached to the self-adjoint operator A. As usual, $(C(\sigma(A)))_+$ is the positive cone of the space $C(\sigma(A))$.

Theorem 3. *Let A be a positive self-adjoint operator, $B \in (Y(A))_+$. We denote:*

$$p_j(t) := t^j, \ \ t \in [0, +\infty), \ \ j \in \mathbb{N}, n \in \mathbb{N}, n \geq 2.$$

Let $\left(E_j\right)_{j\in\mathbb{N}}$ be a sequence in $Y(A)$, $\widetilde{P} : C(\sigma(A)) \longrightarrow Y(A)$,

$$\widetilde{P}(g) := B(\Psi(g))^n = B(g(A))^n, \ \ g \in C(\sigma(A)). \tag{22}$$

The following statements are equivalent:

(i) *There exists a unique positive linear operator $T : C(\sigma(A)) \longrightarrow Y(A)$ satisfying the moment conditions:*

$$T\left(p_j\right) = E_j, \ \ j \in \mathbb{N}, \tag{23}$$

with

$$T(\varphi) \leq \widetilde{P}(\varphi) \text{ for all} \varphi \in C(\sigma(A))_+, \ \|T\| \leq \|B\|. \tag{24}$$

(ii) *For any finite subset $J_0 \subset \mathbb{N}$, any set of scalars $\left\{\alpha_j\right\}_{j\in J_0}$, and $g \in (C(\sigma(A)))_+$, the following implication holds. If*

$$\sum_{j\in J_0} \alpha_j t^j \leq g(t)$$

for all $t \in \sigma(A)$, then

$$\sum_{j\in J_0} \alpha_j E_j \leq B(g(A))^n.$$

Proof. The idea of the proof is to apply Theorem 2.30 from [21]. For detailed proof, see the corresponding reference citation. Here, E stands for $C(\sigma(A))$, $F := Y(A)$, $M := Span\left\{p_j; j \in \mathbb{N}\right\} = \mathbb{R}[t]$, and P is the restriction of $\widetilde{P}$ defined by (22) to $(C(\sigma(A)))_+$. If $J_0 \subset \mathbb{N}$ is a finite subset and $\left\{\alpha_j\right\}_{j\in J_0}$ is an arbitrary finite set of real coefficients, one defines $T_1 : \mathbb{R}[t] \longrightarrow Y(A)$, by:

$$T_1\left(\sum_{j\in J_0} \alpha_j p_j\right) := \sum_{j\in J_0} \alpha_j E_j. \tag{25}$$

The conclusion follows if we prove the convexity of P on $(C(\sigma(A)))_+$. To do this, we first prove that $p_n(W) := W^n$, $W \in (Y(A))_+$ is convex on $(Y(A))_+$. Assuming this is achieved, the difficult implication (ii) implies that (i) follows, as $g \in (C(\sigma(A)))_+$ leads to $\Psi(g) \in (Y(A))_+$. Then, $g \longrightarrow (\Psi(g))^n$ is convex on $(C(\sigma(A)))_+$ as a composition of the convex monotone increasing function p_n with linear mapping Ψ, whose restriction to $(C(\sigma(A)))_+$ is affine. Finally, multiplication with the positive operator $B \in (Y(A))_+$ preserves inequalities as the product of two self-adjoint positive commuting operators is self-adjoint and positive. If $V \in int(Y(A))_+$, then the order interval $[V - \varepsilon I, V + \varepsilon I]$ is contained in $(Y(A))_+$ for some $\varepsilon > 0$. It results: $V \geq \varepsilon I$, which means: $inf\sigma(V) = \inf_{\|h\|=1} (Vh; h) \geq \varepsilon \cdot \inf_{\|h\|=1} (h; h) = \varepsilon$. Thus, $0 \notin \sigma(V)$, so V is invertible. By the same reasoning, W is invertible. The inequality that should be proved, namely,

$$((1-\omega)V + \omega W)^n = (V + \omega(W - V))^n \leq (1-\omega)V^n + \omega W^n, \ \ \omega \in [0,1],$$

is equivalent to

$$\left((1-\omega)I+\omega WV^{-1}\right)^n \leq (1-\omega)I+\omega\left(WV^{-1}\right)^n. \quad (26)$$

As $WV^{-1} \in (Y(A))_+$, the inequality is true due to the functional calculus for the operator WV^{-1} and the continuous real function $p_n(t) = t^n$, on the spectrum of $\sigma\left(WV^{-1}\right) \subset (0,+\infty)$. Namely, one applies the elementary inequality $((1-\omega)1+\omega t)^n \leq (1-\omega)1+\omega t^n$, $\omega \in [0,1]$, which works for all $t \in [0,+\infty)$, for $t \in \sigma\left(WV^{-1}\right)$. If $id(t) := t$, this leads to: $((1-\omega)\mathbf{1}+\omega id)^n \leq (1-\omega)\mathbf{1}+\omega(id)^n$ in $C\left(\sigma\left(WV^{-1}\right)\right)$, and the conclusion follows via functional calculus for continuous functions. Assume now that at least one of the operators $V, W \in (Y(A))_+$, say V, is not in the interior of this cone. As the identity operator I is clearly in the interior of the same convex cone, consider the line segment of ends I, V. As it is well-known [7], all of the points on these line segments which differ from the end V, are in the interior of the cone $(Y(A))_+$. Now, we choose operators V_l on the line segment

$$[I,V) := \{I+t(V-I); t \in [0,1)\},\ V_l := I+t_l(V-I),\ t_l \nearrow 1.$$

Then, $(V_l)_l$ converges to V in $(Y(A))_+$ and $V_l \in int(Y(A))_+$ for all l. From what has already been proved above, V_l is invertible and (26) holds where V^{-1} stands for V_l^{-1} for each l. Hence, we conclude that:

$$((1-\omega)V_l+\omega W)^n \leq (1-\omega)V_l^n+\omega W^n,\ \ \omega \in [0,1],\ \ l \in \mathbb{N}.$$

Passing to the limit as $l \longrightarrow \infty$, we find that $((1-\omega)V+\omega W)^n \leq (1-\omega)V^n+\omega W^n$, $\omega \in [0,1]$. As $Y(A)$ is a Banach lattice, its positive cone $(Y(A))_+$ is topologically closed. Hence, passing to the limit in inequalities is allowed. Now all of the conditions from the hypothesis of Theorem 2.30 point (b) [21] are satisfied. According to the implication (b) implying (a) of the invoked theorem, the conclusion (a) of that theorem holds. To prove the assertions claimed at point (i) of the present theorem, the interpolation moment conditions (23) follow as, according to (25), $T\left(p_j\right) = T_1\left(p_j\right) := E_j,\ j \in \mathbb{N}$. To prove (24), for any $g \in C(\sigma(A))$, with $\|g\|_{C(\sigma(A))} \leq 1$, also using the positivity of the linear operator T on the positive cone, the following conclusion holds: $\|g\|_{C(\sigma(A))} \leq 1$ is equivalent to $|g| \leq \mathbf{1}$ on $\sigma(A)$, which implies:

$$|T(g)| \leq T(|g|) \leq T(\mathbf{1}) \leq P(\mathbf{1}) = B(\Psi(\mathbf{1}))^n = BI^n = B.$$

These further yields:

$$\|T(g)\| = \||T(g)|\| \leq \|B\|.$$

Thus, $\|T\| \leq \|B\|$. The uniqueness of any bounded linear operator satisfying the moment conditions (23) follows from the Weierstrass polynomial approximation theorem for continuous real functions on compact subsets of $\mathbb{R}^m$. To prove that the converse implication (i) implies (ii), assume that (i) holds. Then, from $\sum_{j \in J_0} \alpha_j p_j \leq g \in (C(\sigma(A)))_+$ on $\sigma(A)$, the positivity of the linear operator T, Equations (23) and (24) yield:

$$\sum_{j \in J_0} \alpha_j E_j = T\left(\sum_{j \in J_0} \alpha_j p_j\right) \leq T(g) \leq P(g) = B(\Psi(g))^n = B(g(A))^n.$$

This ends the proof. □

Corollary 2. *If the sequence $\left(E_j\right)_{j \in \mathbb{N}}$ satisfies the conditions (ii) of Theorem 3, then:*

$$0 \leq E_j \leq BA^{nj},\ \ \left\|E_j\right\| \leq \|B\| \cdot \|A\|^{nj}.$$

Proof. Under the hypothesis of the present corollary, according to Theorem 3, the assertion (i) of Theorem 3 holds as well. Consequently, we have:

$$E_j = T(p_j) \le \widetilde{P}(p_j) = B\cdot(\Psi(p_j))^n = B\cdot A^{nj},\ \ j \in \mathbb{N}.$$

On the other hand, as A is positive, its spectrum $\sigma(A)$ is contained in $\{0, +\infty)$, so that:

$$p_j(t) = t^j \ge 0 \text{ for all } t \in \sigma(A), \text{ and all } j \in \mathbb{N}.$$

From the positivity of the operator T, it results that $E_j = T(p_j) \ge 0$ in $Y(A)$. Hence, the desired conclusion follows. □

Corollary 3. *With the notations from Theorem 3, for each and $g \in (C(\sigma(A)))_+$, we have:*

$$\sum_{i,j\in J_1} \alpha_i(m)\alpha_j(m)E_{i,m+j,m} + \sum_{k,l\in J_2} \beta_k(m)\beta_l(m)E_{k,m+l,m+1} \searrow T(g),\ \ m \longrightarrow \infty,$$

for some finite subsets J_1, J_2 of $\mathbb{N}$ and real scalars $\alpha_i(m),\ \beta_k(m)$.

Proof. We apply Lemma 2 of [20] of the uniform approximation of g on $\sigma(A)$ through the restrictions to $\sigma(A)$ of polynomials $(p_m)_m$, which are non-negative on $[0, +\infty)$. Any such polynomial is a sum of squares and the polynomial $p_1(t) = t$ multiplied by another sum of squares of polynomials from $\mathbb{R}[t]$ (see [6]). In other words, p_m is a sum of special polynomials $\sum_{i,j\in J_1} \alpha_i(m)\alpha_j(m)p_{i,m+j_m}$ with a sum of polynomials of the form $\sum_{k,l\in J_2} \beta_k(m)\beta_l(m)p_{k,m+l,m+1},\ p_m \searrow g$ uniformly on $\sigma(A)$. Using the linearity, positivity and continuity of the solution T, and the moment conditions (23) as well, we infer that $T(p_m) \searrow T(g),\ m \longrightarrow \infty$ in $Y(A)$. This ends the proof. □

The next result can be quite easily deduced from the proof of Theorem 3. In a way, it is a scalar variant of the Theorem 2.30 from [21].invoked in the proof of Theorem 3 proved above. However, unlike the operator valued or vector valued problems, in the present scalar valued case, the positive linear solution of the moment problem will be represented by a means of a non-negative function $f \in L^\infty_\nu\left([0, +\infty)^d\right)$. For $d \ge 2$, the following notations are used:

$$j := (j_1, \ldots, j_d) \in \mathbb{N}^d,\ t := (t_1, \ldots, t_d) \in [0, +\infty)^d.$$

Theorem 4. *Let $d \in \mathbb{N}, d \ge 1$, $(m_j)_{j\in\mathbb{N}^d}$ be a sequence of real numbers, $p > 1$ a real number, $b > 0$ a real number and ν a probability moment determinate measure on $[0, +\infty)^d$, with finite moments of all orders. The following statements are equivalent:*

(a) *There exists a non-negative real valued function $f \in L^\infty_\nu\left([0, +\infty)^d\right)$, such that:*

$$\int_{[0,+\infty)^d} t^j f(t) d\nu = m_j,\ j \in \mathbb{N}^d,$$

$$\int_{[0,+\infty)^d} \varphi(t)f(t)d\nu \le b\cdot\left(\int_{[0,+\infty)^d} \varphi(t)d\nu\right)^p,\ \forall\varphi \in \left(L^1_\nu\left([0, +\infty)^d\right)\right)_+,\ \|f\|_{L^\infty_\nu([0,+\infty)^d)} \le b.$$

(b) *For any finite subset $J_0 \subset \mathbb{N}^d$, any set of real scalars $\{\alpha_j\}_{j\in J_0}$, and $g \in \left(L^1_\nu\left([0, +\infty)^d\right)\right)_+$, the following implication holds. If*

$$\sum_{j\in J_0} \alpha_j t^j \le g(t)$$

for all $t \in [0, +\infty)^d$, *then*

$$\sum_{j \in J_0} \alpha_j m_j \leq b \cdot \left(\int_{[0,+\infty)^d} g(t)dt \right)^p.$$

Proof. The proof that the main implication (b) implies (a) follows from Theorem 2.30 of [21], in the same way as Theorem 3 of the present paper did. We apply the former theorem to $E := C(S)$, and

$$P : \left(L_\nu^1 \left([0, +\infty)^d \right) \right)_+ \longrightarrow \mathbb{R}_+, \ P(\varphi) := b \cdot \left(\int_{[0,+\infty)^d} \varphi(t)dt \right)^p, \ \varphi \in \left(L_\nu^1 \left([0, +\infty)^d \right) \right)_+, M := \mathbb{R}[t],$$

$$T_1 : \mathbb{R}[t] \longrightarrow \mathbb{R}, \ T_1 \left(\sum_{j \in J_0} \alpha_j p_j \right) := \sum_{j \in J_0} \alpha_j m_j, \ \ p_j(t) = t^j, \ j \in \mathbb{N}^d.$$

The functional P is convex, as the composition of strictly convex function $t \longmapsto t^p, \ p > 1$, on $[0, +\infty)^d$, with the linear functional $\varphi \longmapsto \int_{[0,+\infty)^d} \varphi(t)dt$, then multiplying with $b > 0$. The notations are almost the same as those from Theorem 3, proved above. Firstly, we prove that (b) implies (a). As in the proof of Theorem 3, the conditions for the theorem invoked in the present Theorem 3 are satisfied. Consequently, there exists a linear positive extension T of T_1 to the entire Banach lattice $L_\nu^1 \left([0, +\infty)^d \right)$, which verifies:

$$T(\varphi) \leq P(\varphi) = b \cdot \left(\int_{[0,+\infty)^d} \varphi(t)d\nu \right)^p, \ \ \forall \varphi \in \left(L_\nu^1 \left([0, +\infty)^d \right) \right)_+.$$

$\|\varphi\|_{L_\nu^1([0,+\infty)^d)} \leq 1$ implies $|T(\varphi)| \leq T(|\varphi|) \leq b \cdot \left(\int_{[0,+\infty)^d} |\varphi| d\nu \right)^p = b \cdot \left(\|\varphi\|_{L_\nu^1([0,+\infty)^d)} \right)^p$ $\leq b$. Hence, $\|T\| \leq b$. The positive linear functional T is also continuous [7], so that it is represented [2] by a non-negative element

$$f \in L_\nu^\infty \left([0, +\infty)^d \right) : T(g) = \int_{[0,+\infty)^d} g f d\nu, \ \ g \in L_\nu^1 \left([0, +\infty)^d \right).$$

The conclusion (a) follows, with $\|f\|_\infty = \|T\| \leq b$. The uniqueness of the solution f is a consequence of the density of $\mathbb{R}[t]$ in $L_\nu^1 \left([0, +\infty)^d \right)$. The implication that (a) implies (b) is almost obvious. Now, we use the hypothesis that ν is a probability measure. Due to this assumption, the constant function $\mathbf{1}$, as well as any other $\varphi \in L_\nu^\infty \left([0, +\infty)^d \right)$, is an element of the Banach lattice $L_\nu^1 \left([0, +\infty)^d \right)$, and we can write $\|\varphi\|_{L_\nu^1([0,+\infty)^d)} \leq \|\varphi\|_{L_\nu^\infty([0,+\infty)^d)}$, $1 = \|\mathbf{1}\|_{L_\nu^1([0,+\infty)^d)} = \|\mathbf{1}\|_{L_\nu^\infty([0,+\infty)^d)}$. This ends the proof. □

Corollary 4. *If the sequence of moments* $(m_j)_{j \in \mathbb{N}^d}$ *satisfies the requirements stated at point (b) of Theorem 4, then it also satisfies the following properties:*

$$0 \leq m_j \leq b \cdot \left(\int_{[0,+\infty)^d} t^j d\nu \right)^p, \ \ j \in \mathbb{N}^d, \ 0 \leq m_0 \leq b.$$

Example 1. *According to [12], each of the following measures* ν *is a Borel regular moment determinate probability measure on* $[0, +\infty)$:

$$e^{-t}dt; \ te^{-t}; \ \ \alpha e^{-\alpha t}, \ \alpha > 0; \ te^{-t^2/2}; \ \frac{2}{\sqrt{\pi}} e^{-t^2}.$$

The application of the polynomial approximation results recently recalled in [21] leads to the density of non-negative polynomials in $\left(L_\nu^1([0, +\infty)) \right)_+$. Consequently, $\mathbb{R}[t]$

is dense in $L^1_\nu([0,+\infty)$. For $d \geq 2$, if ν_l is a probability moment determinate measure on $[0,+\infty)$, $l = 1,..,d$, then

$$\nu := \nu_1 \times \cdots \times \nu_d$$

is a probability measure on $[0,+\infty)^d$, and the non-negative polynomials on $[0,+\infty)^d$ are dense in the convex cone $\left(L^1_\nu\left([0,+\infty)^d\right)\right)_+$. Consequently, $\mathbb{R}[t_1,\ldots,t_d]$ is dense in $L^1_\nu\left([0,+\infty)^d\right)$.

4. Discussion

We have proved two theorems studying the properties of the unique non-trivial solution of a functional equation and of the corresponding operatorial equation. The solution cannot be expressed in terms of elementary functions, although the equation is stated using elementary analytic functions. In the second part, two aspects of a moment-type problem are solved. The existence and uniqueness of the solution is characterized in terms of the given moments. However, the explicit form of the solution is not known. This proves the relationship between the two parts of this work. In Corollary 3, an approximation of the solution T in terms of the given moments E_j is sketched. Known results are only invoked and applied, not repeated. A possible subject for future work could be that of studying moment problems where the codomain of the solution is a $L^\infty_\nu(S)$ space, S being a special closed subset of $\mathbb{R}^d$, $d \in \mathbb{N}, d \geq 1$. Here, ν might be a moment determinate positive Borel regular measure on S.

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