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Green Measures for a Class of Non-Markov Processes

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Abstract: In this paper, we investigate the Green measure for a class of non-Gaussian processes in $\mathbb{R}^d$. These measures are associated with the family of generalized grey Brownian motions $B_{\beta,\alpha}$, $0 < \beta \leq 1$, $0 < \alpha \leq 2$. This family includes both fractional Brownian motion, Brownian motion, and other non-Gaussian processes. We show that the perpetual integral exists with probability 1 for $d\alpha > 2$ and $1 < \alpha \leq 2$. The Green measure then generalizes those measures of all these classes.

Keywords: fractional Brownian motion; generalized grey Brownian motion; green measure; subordination

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1. Introduction

In recent years, there has been a significant amount of research devoted to fractional dynamics related to fractional Brownian motion and related processes. These processes lack both the Markov and semimartingale properties from a mathematical standpoint. As a result, many traditional approaches in stochastic analysis do not apply, making their analysis more challenging. These processes are capable of modeling systems that exhibit long-range self-interaction and memory effects.

In 1992, Schneider introduced the grey Brownian motion [1], a class of non-Gaussian processes, to solve the time-fractional diffusion equation with a Caputo–Djrbashian derivative of fractional order. During the 1990s, Mainardi and their co-authors conducted a systematic investigation into fractional differential equations; see [2] and the references therein. They introduced the notion of generalized grey Brownian motion (ggBm for short), and the corresponding time-fractional differential equations governing its densities. This family of processes is denoted by $B_{\beta,\alpha}$ with parameters $0 < \beta \leq 1$ and $0 < \alpha \leq 2$. If $\beta \neq 1$, the process $B_{\beta,\alpha}$ is non-Gaussian with stationary increments and $\alpha/2$-self-similar; see Section 2 for details. The process $B_{\beta,\alpha}$ admits different representations (cf. (12) and (13) below) in terms of other known processes, which are useful for simulation and to derive other properties. In a recent work, Grothaus et al. [3] elaborated an infinite dimensional analysis for (non-Gaussian) measures of the Mittag-Leffler type. They used ggBm to solve the time-fractional heat equation, extending the fractional Feynman–Kac formula of Schneider [1].

The goal of this paper (see Theorem 1 and Corollary 1 below) is to prove the existence of the Green measure for the class of non-Gaussian processes ggBm in $\mathbb{R}^d$. This result will extend the results of Kondratiev et al. [4]. More precisely, for a Borel function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$, the potential of f (see [5,6] for details) is defined as

$$V_{\beta,\alpha}(f,x) = \int_0^\infty \mathbb{E}\big[f(x + B_{\beta,\alpha}(t))\big]\,\mathrm{d}t, \quad x \in \mathbb{R}^d. \tag{1}$$

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We would like to investigate the class of functions f for which the potential of f has the representation

$$V_{\beta,\alpha}(f,x) = \int_{\mathbb{R}^d} f(y)\mathcal{G}_{\beta,\alpha}(x,\mathrm{d}y), \tag{2}$$

where $\mathcal{G}(x,\cdot) := \mathcal{G}_{\beta,\alpha}(x,\cdot)$ is a Radon measure on $\mathbb{R}^d$ called the Green measure corresponding to the ggBm $B_{\beta,\alpha}$; see Definition 2 below. If $B_{\beta,\alpha}$ admits a generator $L_{\beta,\alpha}$, then the potential $V(x,f)$ can be obtained from the equation

$$-LV = f.$$

The Green measure can be seen as the fundamental solution for the generator $L_{\beta,\alpha}$ of the process $B_{\beta,\alpha}$. First, we establish the existence of the perpetual integral (cf. Theorem 1):

$$\int_0^\infty f(x + B_{\beta,\alpha}(t))\,\mathrm{d}t \tag{3}$$

with probability one. This leads to an explicit representation of the Green measure for ggBm, namely (cf. Corollary 1)

$$\mathcal{G}_{\beta,\alpha}(x,\mathrm{d}y) = \frac{D}{|x-y|^{d-2/\alpha}}\,\mathrm{d}y, \quad d\alpha > 2, \quad 1 < \alpha \leq 2,$$

where D is a constant that depends on β, α, and the dimension d; see (17) for the explicit expression. Note that as $d\alpha > 2$ and $1 < \alpha \leq 2$, the Green measure $\mathcal{G}_{\beta,\alpha}(x,\cdot)$ exists for $d \geq 2$, since $d > 2/\alpha \in [1,2)$. The Brownian case ($\alpha = 1$) is covered only for $d \geq 3$. We emphasize that the existence of the Green measure for a given process X is not always guaranteed. In addition, finding a proper space of functions $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ that guarantees the existence of (1) is crucial. As an example, the d-dimensional Bm starting at $x \in \mathbb{R}^d$ has a density given by $p_t(x,y) = (2\pi t)^{-d/2} \exp\big(-|x-y|^2/(2t)\big)$, $y \in \mathbb{R}^d$. It is not difficult to see that $\int_0^\infty p_t(x,y)\,\mathrm{d}t$ does not exist for $d = 1,2$. Hence, the Green measure of Bm for $d = 1,2$ does not exist. On the other hand, for $d \geq 3$, the Green measure of Bm on $\mathbb{R}^d$ exists and is given by $\mathcal{G}(x,\mathrm{d}y) = C(d)|x-y|^{2-d}\,\mathrm{d}y$, where $C(d)$ is a constant depending on the dimension d; see [4] and the references therein for more details. In a two-dimensional space, the Green measure of ggBm is determined by the parameter α that is related to the roughness of the path. The Green measure of ggBm for $d = 1$ requires further analysis (for Bm, see [7], Ch. 4), which we will postpone for a future paper.

This paper is organized as follows. In Section 2, we recall the definition and main properties of ggBm that will be needed later. In Section 3, we show the existence of the perpetual integral with probability one, which leads to the explicit formula for the Green measure for ggBm. In Section 4, we discuss the obtained results, connect them with other topics, and draw conclusions.

2. Generalized Grey Brownian Motion

We recall the class of non-Gaussian processes, called generalized grey Brownian motion, which we study below. This class of processes was first introduced by Schneider [8,9], and was generalized by Mura et al. (see [10,11]) as a stochastic model for slow/fast anomalous diffusion described by the time-fractional diffusion equation.

2.1. Definition and Properties

For $0 < \beta \leq 1$, the (entire) Mittag-Leffler function E_β is defined by the Taylor series

$$E_\beta(z) := \sum_{n=0}^\infty \frac{z^n}{\Gamma(\beta n + 1)}, \quad z \in \mathbb{C}, \tag{4}$$

where

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt, \quad z \in \mathbb{C},\, \mathrm{Re}(z) \geq 0$$

is the Euler gamma function.

The M-Wright function is a special case of the class of Wright functions $W_{\lambda,\mu}$, $\lambda > -1$, $\mu \in \mathbb{C}$, via

$$M_\beta(z) := W_{-\beta,1-\beta}(-z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\beta n + 1 - \beta)}.$$

The special choice $\beta = 1/2$ yields the Gaussian density on $[0, \infty)$:

$$M_{1/2}(z) = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{z^2}{4}\right). \tag{5}$$

The Mittag-Leffler function E_β is the Laplace transform of the M-Wright function, that is,

$$E_\beta(-s) = \int_0^\infty e^{-s\tau} M_\beta(\tau)\, d\tau. \tag{6}$$

The generalized moments of the density M_β of order $\delta > -1$ are finite and are given (see [10]) by

$$\int_0^\infty \tau^\delta M_\beta(\tau)\, d\tau = \frac{\Gamma(\delta+1)}{\Gamma(\beta\delta+1)}. \tag{7}$$

Definition 1. *Let $0 < \beta \leq 1$ and $0 < \alpha \leq 2$ be given. A d-dimensional continuous stochastic process $B_{\beta,\alpha} = \{B_{\beta,\alpha}(t),\, t \geq 0\}$, starting at $0 \in \mathbb{R}^d$ and defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, is a ggBm in $\mathbb{R}^d$ (see [11] for $d = 1$) if the following is satisfied:*

1. *$\mathbb{P}(B_{\beta,\alpha}(0) = 0) = 1$, that is, $B_{\beta,\alpha}$ starts at zero $\mathbb{P}$-almost surely ($\mathbb{P}$-a.s.).*
2. *Any collection $\{B_{\beta,\alpha}(t_1), \dots, B_{\beta,\alpha}(t_n)\}$ with $0 \leq t_1 < t_2 < \cdots < t_n < \infty$ has a characteristic function given, for any $\theta = (\theta_1, \dots, \theta_n) \in (\mathbb{R}^d)^n$ with $\theta_k = (\theta_{k,1}, \dots \theta_{k,d})$, $k = 1, \dots, n$, by*

$$\mathbb{E}\left[\exp\left(\mathrm{i} \sum_{k=1}^{n} (\theta_k, B_{\beta,\alpha}(t_k))_{\mathbb{R}^d}\right)\right] = E_\beta\left[-\frac{1}{2} \sum_{j=1}^{d} (\theta_{\cdot,j}, \gamma_\alpha \theta_{\cdot,j})_{\mathbb{R}^n}\right], \tag{8}$$

   where $\mathbb{E}$ denotes the expectation with regard to $\mathbb{P}$ and

$$\gamma_\alpha := \gamma_{\alpha,n} := \left(t_k^\alpha + t_j^\alpha - |t_k - t_j|^\alpha\right)_{k,j=1}^n.$$

3. *The joint probability density function of $(B_{\beta,\alpha}(t_1), \dots, B_{\beta,\alpha}(t_n))$ is equal to*

$$\rho_\beta(\theta, \gamma_\alpha) = \frac{(2\pi)^{-\frac{nd}{2}}}{(\det\gamma_\alpha)^{d/2}} \int_0^\infty \tau^{-\frac{nd}{2}} \mathrm{e}^{-\frac{1}{2\tau} \sum_{j=1}^d (\theta_{\cdot,j}, \gamma_\alpha^{-1} \theta_{\cdot,j})_{\mathbb{R}^n}} M_\beta(\tau)\, d\tau. \tag{9}$$

The following are the most important key properties of ggBm:

(P1). For each $t \geq 0$, the moments of any order of $B_{\beta,\alpha}(t)$ are given by

$$\begin{cases} \mathbb{E}\left[|B_{\beta,\alpha}(t)|^{2n+1}\right] & = 0, \\ \mathbb{E}\left[|B_{\beta,\alpha}(t)|^{2n}\right] & = \frac{(2n)!}{2^n \Gamma(\beta n+1)} t^{\alpha n}. \end{cases}$$

(P2). The covariance function has the form

$$\mathbb{E}\left[(B_{\beta,\alpha}(t), B_{\beta,\alpha}(s))\right] = \frac{d}{2\Gamma(\beta+1)}\left(t^\alpha + s^\alpha - |t-s|^\alpha\right), \quad t, s \geq 0. \tag{10}$$

(P3). For each $t, s \geq 0$, the characteristic function of the increments is

$$\mathbb{E}\left[e^{i(k,B_{\beta,\alpha}(t)-B_{\beta,\alpha}(s))}\right] = E_\beta\left(-\frac{|k|^2}{2}|t-s|^\alpha\right), \quad k \in \mathbb{R}^d. \tag{11}$$

(P4). The process $B_{\beta,\alpha}$ is non-Gaussian and $\alpha/2$-self-similar with stationary increments.

(P5). The ggBm is not a semimartingale. Furthermore, $B_{\alpha,\beta}$ cannot be of finite variation in $[0,1]$ and, by the scaling and stationarity of the increment, on any interval in $\mathbb{R}^+$.

(P5). For $n = 1$, the density $\rho_\beta(x,t)$, $x \in \mathbb{R}^d$, $t > 0$ is the fundamental solution of the following fractional differential equation (see [12]):

$$\mathbb{D}_t^{2\beta}\rho_\beta(x,t) = \Delta_x\rho_\beta(x,t),$$

where Δ_x is the d-dimensional Laplacian in x and $\mathbb{D}_t^{2\beta}$ is the Caputo–Dzherbashian fractional derivative; see [13] for the definition and properties.

2.2. Representations of Generalized Grey Brownian Motion

The ggBm admits different representations in terms of well-known processes. It follows from (8) that ggBm has an elliptical distribution; see Section 3 in [3]. On the other hand, ggBm is also given as a product (see [10] for $d = 1$) of two processes, as follows:

$$\{B_{\beta,\alpha}(t),\, t \geq 0\} \stackrel{\mathcal{L}}{=} \{\sqrt{Y_\beta}B^{\alpha/2}(t),\, t \geq 0\}. \tag{12}$$

Here, $\stackrel{\mathcal{L}}{=}$ means equality in law, the non-negative random variable Y_β has density M_β, and $B^{\alpha/2}$ is a d-dimensional fBm with Hurst parameter $\alpha/2$ and is independent of Y_β.

We give another representation of ggBm $B_{\beta,\alpha}$ as a subordination of fBm (see Section 2.14 in [14] for $d = 1$) which is used below. For completeness, we give a short proof.

Proposition 1. *The ggBm has the following representation:*

$$\{B_{\beta,\alpha}(t),\, t \geq 0\} \stackrel{\mathcal{L}}{=} \{B^{\alpha/2}(tY_\beta^{1/\alpha}),\, t \geq 0\}. \tag{13}$$

Proof. We must show that both representations (12) and (13) have the same finite-dimensional distribution. For every $\theta = (\theta_1, \ldots, \theta_n) \in (\mathbb{R}^d)^n$, we have

$$\begin{aligned}
\mathbb{E}\left[\exp\left(\mathrm{i}\sum_{k=1}^n\left(\theta_k, B^{\alpha/2}(t_kY_\beta^{1/\alpha})\right)\right)\right] &= \int_0^\infty \mathbb{E}\left[\exp\left(\mathrm{i}\sum_{k=1}^n\left(\theta_k, B^{\alpha/2}(t_ky^{1/\alpha})\right)\right)\right]M_\beta(y)\,\mathrm{d}y \\
&= \int_0^\infty \mathbb{E}\left[\exp\left(\mathrm{i}\sum_{k=1}^n\left(\theta_k, y^{1/2}B^{\alpha/2}(t_k)\right)\right)\right]M_\beta(y)\,\mathrm{d}y \\
&= \mathbb{E}\left[\exp\left(\mathrm{i}\sum_{k=1}^n\left(\theta_k, Y_\beta^{1/2}B^{\alpha/2}(t_k)\right)\right)\right].
\end{aligned}$$

In the second equality, we use the $\alpha/2$-self-similarity of fBm. This completes the proof. □

3. The Green Measure for Generalized Grey Brownian Motion

In this section, we show the existence of the Green measure for ggBm; see (1) and (2). Let us begin by discussing the existence of the Green measure for a general stochastic process X.

Let $X = \{X(t),\ t \geq 0\}$ be a stochastic process in $\mathbb{R}^d$ starting from $x \in \mathbb{R}^d$. If $X(t)$, $t \geq 0$, has a probability distribution $\rho_{X(t)}(x,\cdot)$, then Equation (1) becomes

$$V_X(x,f) = \int_0^\infty \int_{\mathbb{R}^d} f(y)\,\rho_{X(t)}(x,\mathrm{d}y)\,\mathrm{d}t. \tag{14}$$

Then, applying the Fubini theorem, the Green measure $\mathcal{G}_X(x,\cdot)$ of X is given by

$$\mathcal{G}_X(x,\mathrm{d}y) = \int_0^\infty \rho_{X(t)}(x,\mathrm{d}y)\,\mathrm{d}t,$$

assuming the existence of $\mathcal{G}_X(x,\cdot)$ as a Radon measure on $\mathbb{R}^d$. That is, for every bounded Borel set $B \in \mathcal{B}_b(\mathbb{R}^d)$ we have

$$\mathcal{G}_X(x,B) = \int_0^\infty \rho_{X(t)}(x,B)\,\mathrm{d}t < \infty.$$

If the probability distribution $\rho_{X(t)}(x,\cdot)$ is also absolutely continuous with respect to the Lebesgue measure, say $\rho_{X(t)}(x,\mathrm{d}y) = \rho_t(x,y)\,\mathrm{d}y$, then the function

$$g_X(x,y) := \int_0^\infty \rho_t(x,y)\,\mathrm{d}t, \quad \forall y \in \mathbb{R}^d, \tag{15}$$

is called the Green function of the stochastic process X. Moreover, the Green measure in this case is given by $\mathcal{G}_X(x,\mathrm{d}y) = g_X(x,y)\,\mathrm{d}y$.

This leads us to the following definition of the Green measure of a stochastic process X.

Definition 2. *Let $X = \{X(t),\ t \geq 0\}$ be a stochastic process on $\mathbb{R}^d$ starting from $x \in \mathbb{R}^d$ and $\rho_{X(t)}(x,\cdot)$ be the probability distribution of $X(t)$, $t \geq 0$. The Green measure of X is defined as a Radon measure on $\mathbb{R}^d$ by*

$$\mathcal{G}_X(x,B) := \int_0^\infty \rho_{X(t)}(x,B)\,\mathrm{d}t, \ \ B \in \mathcal{B}_b(\mathbb{R}^d),$$

or

$$\int_{\mathbb{R}^d} f(y)\mathcal{G}_X(x,\mathrm{d}y) = \int_{\mathbb{R}^d} f(y) \int_0^\infty \rho_{X(t)}(x,dy)\,\mathrm{d}t, \ \ f \in C_0(\mathbb{R}^d)$$

whenever these integrals exist.

In other words, $\mathcal{G}_X(x,B)$ is the expected length of time the process remains in B.

To state the main theorem that establishes the existence of the Green measure for ggBm, first, we introduce a proper Banach space of functions $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ such that the perpetual integral (3) is finite $\mathbb{P}$-a.s. Without a loss of generality, we can assume that $f \geq 0$ above. We define the space $CL(\mathbb{R}^d)$, of continuous real valued, on $\mathbb{R}^d$ by

$$CL(\mathbb{R}^d) := \big\{f : \mathbb{R}^d \to \mathbb{R} \mid f \text{ is continuous, bounded and } f \in L^1(\mathbb{R}^d)\big\}.$$

The space $CL(\mathbb{R}^d)$ becomes a Banach space with the norm

$$\|f\|_{CL} := \|f\|_\infty + \|f\|_1, \quad \forall f \in CL(\mathbb{R}^d),$$

where $\|\cdot\|_\infty$ denotes the sup-norm and $\|\cdot\|_1$ is the norm in $L^1(\mathbb{R}^d)$. The choice of $CL(\mathbb{R}^d)$ allows us to show that the family of random variables (3) with $f \in CL(\mathbb{R}^d)$ have finite expectations $\mathbb{P}$-a.s.

Theorem 1. *Let $f \in CL(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ be given and consider ggBm $B_{\beta,\alpha}$ with $d\alpha > 2$ and $1 < \alpha \leq 2$. Then, the perpetual integral functional $\int_0^\infty f(x+B(t))\,\mathrm{d}t$ is finite $\mathbb{P}$-a.s. and its expectation equals*

$$\mathbb{E}\left[\int_0^\infty f(x+B_{\beta,\alpha}(t))\,\mathrm{d}t\right] = D\int_{\mathbb{R}^d} \frac{f(x+y)}{|y|^{d-2/\alpha}}\,\mathrm{d}y, \tag{16}$$

where

$$D = D(\beta,\alpha,d) = \frac{1}{\alpha}2^{-1/\alpha}\pi^{-\frac{d}{2}}\Gamma\left(\frac{d}{2}-\frac{1}{\alpha}\right)\frac{\Gamma(1-\frac{1}{\alpha})}{\Gamma(1-\frac{\beta}{\alpha})}. \tag{17}$$

Proof. Given that $x \in \mathbb{R}^d$ and $f \in CL(\mathbb{R}^d)$ are non-negative, let $\rho_\beta(\cdot, t^\alpha)$ denote the density of $B_{\beta,\alpha}(t)$, $t \geq 0$, which is given by (see (9) with $n = 1$)

$$\rho_\beta(y, t^\alpha) = \frac{1}{(2\pi t^\alpha)^{d/2}} \int_0^\infty \tau^{-d/2} \mathrm{e}^{-\frac{|y|^2}{2t^\alpha \tau}} M_\beta(\tau)\, \mathrm{d}\tau, \quad y \in \mathbb{R}^d.$$

First, we show equality (16). It follows from the above considerations that

$$\begin{aligned}\mathbb{E}\left[\int_0^\infty f(x + B_{\beta,\alpha}(t))\, \mathrm{d}t\right] &= \int_0^\infty \int_{\mathbb{R}^d} f(x+y)\rho_\beta(y, t^\alpha)\, \mathrm{d}y\, \mathrm{d}t. \\ &= \int_0^\infty \int_{\mathbb{R}^d} f(x+y) \frac{1}{(2\pi t^\alpha)^{d/2}} \int_0^\infty \tau^{-d/2} M_\beta(\tau) e^{-\frac{|y|^2}{2t^\alpha \tau}}\, \mathrm{d}\tau\, \mathrm{d}y\, \mathrm{d}t.\end{aligned}$$

Using Fubini's Theorem, we first compute the t-integral and use the assumption $d\alpha > 2$. We obtain

$$\int_0^\infty \frac{1}{(2\pi t^\alpha \tau)^{d/2}} e^{-\frac{|y|^2}{2t^\alpha \tau}}\, \mathrm{d}t = C(\alpha, d) \frac{\tau^{-\frac{1}{\alpha}}}{|y|^{d-2/\alpha}},$$

where

$$C(\alpha, d) := \frac{1}{\alpha} 2^{-1/\alpha} \pi^{-\frac{d}{2}} \Gamma\left(\frac{d}{2} - \frac{1}{\alpha}\right).$$

Next, we compute the τ-integral using (7) so that

$$\int_0^\infty \tau^{-1/\alpha} M_\beta(\tau)\, \mathrm{d}\tau = \frac{\Gamma(1 - \frac{1}{\alpha})}{\Gamma(1 - \frac{\beta}{\alpha})}, \quad \alpha > 1.$$

Combining them gives the equality (16) where $D = D(\beta, \alpha, d) = C(\alpha, d) \frac{\Gamma(1-\frac{1}{\alpha})}{\Gamma(1-\frac{\beta}{\alpha})}$.

Now, we show that the right-hand side of (16) is finite for every non-negative $f \in CL(\mathbb{R}^d)$. To see this, we can use the local integrability of $|y|^{d-2/\alpha}$ in y and obtain

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{f(x+y)}{|y|^{d-2/\alpha}}\, \mathrm{d}y &= \int_{\{|y| \leq 1\}} \frac{f(x+y)}{|y|^{d-2/\alpha}}\, \mathrm{d}y + \int_{\{|y|>1\}} \frac{f(x+y)}{|y|^{d-2/\alpha}}\, \mathrm{d}y \\ &\leq C_1 \|f\|_\infty + C_2 \|f\|_1 \leq C\|f\|_{CL}.\end{aligned}$$

Therefore, the integral in (16) is, in fact, well defined. In other words, the integral $\int_0^\infty f\big(x + B_{\beta,\alpha}(t)\big)\, \mathrm{d}t$ exists with probability one. This completes the proof. □

As a consequence of the above theorem, we immediately obtain the Green measure of ggBm $B_{\beta,\alpha}$, that is, comparing (2) and (16).

Corollary 1. *The Green measure of ggBm $B_{\beta,\alpha}$ for $d\alpha > 2$ is given by*

$$\mathcal{G}_{\beta,\alpha}(x, \mathrm{d}y) = \frac{D}{|x-y|^{d-2/\alpha}}\, \mathrm{d}y,$$

where D is given by (17).

Remark 1.

1. *It is possible to show that, given $f \neq 0$, the perpetual integral* (3) *is a non-constant random variable. As a consequence, for $f \geq 0$ the variance of the random variable* (3) *is strictly positive. The proof uses the notion of conditional full support of ggBm. We do not provide a detailed explanation of this result that closely follows the ideas of Theorem 2.2 in* [4]*, to which we address interested readers.*

2. *Note also that the functional in* (1),

$$V_{\beta,\alpha}(\cdot, x) : CL(\mathbb{R}^d) \longrightarrow \mathbb{R}$$

is continuous. In fact, from the proof of Theorem 1, any $f \in CL(\mathbb{R}^d)$ yields

$$|V_{\beta,\alpha}(f, x)| \leq K\|f\|_{CL},$$

where K is a constant depending on the parameters β, α, and d.

4. Discussion and Conclusions

We derived the Green measure for the class of stochastic processes called generalized grey Brownian motion in Euclidean space $\mathbb{R}^d$ for $d \geq 2$. This class includes, in particular, fractional Brownian motion and other non-Gaussian processes. To address the case where $d = 1$, a renormalization process is needed. However, this will be postponed to future work. For $\beta = \alpha = 1$ ggBm, $B_{1,1}$ is nothing but a Brownian motion. In this case, the Green measure exists for $d \geq 3$. Green measures and Green functions are well known to be intrinsically connected and applied to (stochastic partial) differential equations. In this context, the Green measures discussed in this paper play the same role for space-time-fractional derivatives. The presented method can be applied to other processes with sufficient information on the density and existence of the integrals. If we consider a Markov process X that admits a Green measure and T, a random time change given by an inverse subordinator, then the Green measure of the subordinated process $X(T(t))$, $t \geq 0$ exists only after renormalization. Mixing different types of processes, e.g., fBm and scaled Bm, as described in [15], or Markovian and non-Markovian, as in [16], may lead us to a renormalization procedure to guarantee the existence of the Green measure.

The relationship between the Green measure and the local time of the ggBm can be described as follows. For any $T > 0$ and a continuous function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$, the integral functional

$$\int_0^T f(B_{\beta,\alpha}(t))\,\mathrm{d}t \tag{18}$$

is well defined. For $d = 1$, the integral (18) with $f \in L^1(\mathbb{R})$ is represented as

$$\int_0^T f(B_{\beta,\alpha}(t))\,\mathrm{d}t = \int_{\mathbb{R}} f(x) L_{\beta,\alpha}(T, x)\,\mathrm{d}x,$$

where $L_{\beta,\alpha}(T, x)$ is the local time of ggBm up to time T at the point x (see [3]). The Green measure corresponds to the asymptotic behaviour in T of the expectation of local time $L_{\beta,\alpha}(T, x)$. The existence of this asymptotic depends on the dimension d and the transient or recurrent properties in the process.

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