*fractal and fractional*

*Article*
# Supplemental Material

## 1. Relation between DIV theory and effective-dimension theory

In this supplement we will verify both the Effective-dimension theory[1] and the dangerous irrelevant variable(DIV) theory[2] have compeletly same dynamics scaling form and finite-size scaling(FSS)[3,4]. We set out to the langevin equation of $\phi^4$ model

$$\frac{\partial\phi(x,t)}{\partial t} = -\lambda\frac{\delta H}{\delta\phi(x,t)} + \zeta \tag{S1}$$

with

$$H = \int d^d x[\frac{1}{2}\tau\phi(x,t)^2 + \frac{1}{2}(\nabla^{\frac{\sigma}{2}}\phi(x,t))^2 - h\phi(x,t) + \frac{1}{4}u\phi(x,t)^4] \tag{S2}$$

Where $H$ is effective Hamitonian, kinetic coefficient $\lambda$ is positive, $u$ and $\tau$ represent coupling constant and reduced temperature corresponding to distance between $T$ and critical temperature $T_c$ respectively. $\nabla^{\sigma/2}$ represents spatial long-range interactions algebraically decaying with an exponent $d+\sigma$[5]. For $\sigma = 2$, short-range interaction is recovered. The Gaussian noise $\zeta$ in Eq.(S1) is satisfied with

$$\begin{aligned} <\zeta(x,t)> &= 0, \\ <\zeta(x,t)\zeta(x',t')> &= 2\lambda\delta(x-x')\delta(t-t'). \end{aligned} \tag{S3}$$

Correspondingly, integrating over the noise $\zeta$ and introduing the auxiliary field $\tilde{\phi}(x,t)$, the effective Lagrangian $\mathcal{L}$ including $t$ is well known as[6–9]

$$\mathcal{L} = \int d^d x dt\tilde{\phi}(x,t)[\frac{\partial\phi(x,t)}{\partial t} + \lambda\frac{\delta H}{\delta\phi} - \lambda\tilde{\phi}(x,t)\tilde{\phi}(x,t)], \tag{S4}$$

Given the dimensions of the coordinate or correlation length as $[x] = [\xi] = -1$ and assuming that $\mathcal{L}$ is dimensionless, we can determine the dimensions of other variables, [] denotes dimension, as

$$\begin{aligned} [\tau] = \sigma, [\phi] = \frac{d-\sigma}{2}, [\tilde{\phi}] = \frac{d+\sigma}{2}, \\ [u] = 4-2\sigma, [h] = \frac{d+\sigma}{2}, [t] = -\sigma. \end{aligned} \tag{S5}$$

due to the shadow relation

$$[\phi] + [\tilde{\phi}] = d, \tag{S6}$$

the dimensional analysis for $\mathcal{L}$ is equivalent to the dimensional analysis for $H$. The mean-field exponents is only recovered and equal to the Gaussian exponents at the upper critical dimension $d = d_c$, where $d_c = 2\sigma$. The upper critical dimension is derived from dimensionless of $u$ and solely depends on the $\sigma$ of spatial interaction. According to the definitions $\xi = \tau^{-\nu}$, for $d = d_c$ the relations between critical exponents and the dimensions of variables are given as:

$$[\tau] = \frac{1}{\nu}, [\phi] = \frac{\beta}{\nu}, [h] = \frac{\beta\delta}{\nu}, [t] = -z, \tag{S7}$$

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where $\nu$, $\beta$, and $\delta$ are critical exponents. The remaining exponents, such as $\alpha$, $\gamma$, and $\eta$, can also be obtained through scaling laws[10]. $z$ is the dynamic exponent. Above the upper critical dimension, due to the influence of the DIV $u$, the Gaussian exponents no longer align with the mean-field exponents, and hyperscaling law breaks down[11]. Neither the Gaussian exponents nor the mean-field exponents match the numerical results obtained through FSS[12]. DIV theory is a method to unveil the critical properties above the upper critical dimension by introducing a transformation $\phi' = u^{1/4}\phi$, the others also have the transformations[2]

$$\tau' = u^{-\frac{1}{2}}\tau, h' = hu^{-\frac{1}{4}}. \tag{S8}$$

After transformations, the dimension of variables can be expressed as

$$[\tau'] = \frac{d}{2}, [\phi'] = \frac{d}{4}, [h'] = \frac{3d}{4} \tag{S9}$$

according to Eq.(S9), the scaling function of free energy density and FSS can be expressed as:

$$f(\tau', h', L^{-1}) = b^{-d} f(\tau' b^{\frac{d}{2}}, h' b^{\frac{3d}{4}}, L^{-1}b), \tag{S10}$$

for $b = L$, we can obtain the FSS

$$f(\tau', h') = L^{-d} f(\tau' L^{\frac{d}{2}}, h' L^{\frac{3d}{4}}). \tag{S11}$$

Applying the transformations along the lines of DIV theory to the dynamic part yields a corresponding translation as $\tilde{\phi}' = u^{1/4}\tilde{\phi}$, thus, the transformation also derives $t' = tu^{1/2}$ corresponding to the form $u^{-1/2}\partial\phi'/\partial t$ in $\mathcal{L}$. The scaling function of free energy density including $t'$ instead ot $t$ for $L \to \infty$ can be expressed as

$$f(\tau', h', t') = b^{-d} f(\tau' b^{\frac{d}{2}}, h' b^{\frac{3d}{4}}, t' b^{-\frac{d}{2}}), \tag{S12}$$

where the scaling form of $t'$ is derived from $[t'] = -d/2$, by setting the scaling factor as $b = t'^{2/d}$, Eq.(S12) becomes

$$f(\tau', h', t') = t'^{-2} f(\tau' t', h' t'^{\frac{3}{2}}, 1). \tag{S13}$$

Eq.(S13) is dynamics scaling above the upper critical dimension, the scaling form is independent with $\sigma$ and spatial dimension $d$.

Another method to solve the argument about mean-field theory and FSS is Effective-dimension theory. In this theory, the effective spatial dimension are still fixed at upper critical dimension when $d > d_c$. It replaces the corrections of the scale field with corrections of spatial dimension by using the following transformations

$$\phi' = u^{\frac{1}{2}}\phi, \tilde{\phi}' = u^{\frac{1}{2}}\tilde{\phi}, h^* = hu^{\frac{1}{2}}. \tag{S14}$$

Correspongly, the Hamitonian $H$ and Lagrangian $\mathcal{L}$ become

$$\begin{aligned} H = u^{-1} \int d^d x [\frac{1}{2}\tau\phi'(x,t)^2 + \frac{1}{2}(\Delta^{\frac{\sigma}{2}}\phi'(x,t))^2 \\ -h^*\phi'(x,t) + \frac{1}{4}\phi'(x,t)^4], \\ \mathcal{L} = u^{-1} \int d^d x dt \tilde{\phi}'(x,t)[\frac{\partial\phi'(x,t)}{\partial t} + u\lambda\frac{\delta H}{\delta\phi'} \\ -\lambda\tilde{\phi}'(x,t)\tilde{\phi}'(x,t)], \end{aligned} \tag{S15}$$

As shown in Eq.(S15), the temporal integral $\int dt$ is not influenced by $u$, but the integrals for space change to $u^{-1}\int d^dx$. Since FSS is dominated by the zero wave-number mode, and finite wave-number modes are damped by gradients for $d > d_c$[13], this leads to an effective-dimension of $d + [u] = d_c$, corresponding to the dimension of $u^{-1}\int d^dx$, and the effective volume is $V' = Vu^{-1}$. Thus, $L' = Lu^{-1/d}$ and $[L'] = d_c/d$. Therefore, the scaling function for free energy density and FSS are given by

$$\begin{aligned} f(\tau, h, t, L'^{-1}) = b^{-d_c} f(\tau b^{\sigma}, h^* b^{\frac{3\sigma}{2}}, tb^{-\sigma}, L'^{-1} b^{\frac{d_c}{d}}), \\ f(\tau, h^*) = L'^{-d} f(\tau L'^{\frac{d}{2}}, h^* L'^{\frac{3d}{4}}). \end{aligned} \tag{S16}$$

FSS derived from Effective-dimension theory is also agreement with QFSS, QFSS suggests that the correlation length is no longer directly proportional to the system size $L$, but instead, it is represented as $L \propto \xi^{d/d_c}$[14]. Compare Eq.(S16) and Eq.(S11), both DIV theory and Effective-dimension theory have indentical FSS form. On other hand due to $t$ is not influenced by $u^{-1}$, setting $b = t^{1/\sigma}$, the dynamic scaling form of free energy density is described as

$$f(\tau, h, t) = t^{-\frac{d_c}{2}} f(\tau t, h^* t^{\frac{3}{2}}, 1). \tag{S17}$$

Since $d_c = 2\sigma$, Eq.(S17) is the same as Eq.(S13). As results of scaling analysis about FSS and dynamics scaling form, the same FSS and dynamics scaling form have been obtained by Effective-dimension theory and DIV theory, this phenomena also result in the same values about $\beta$, $\gamma$, $\delta$ and $\alpha$.

## 2. Landau-Ginzburg model for the fractal time process

In order to verify the critical exponents above upper critical dimension, we simulate 2D Landau-Ginzburg model with temporal fractional derivatives as

$$ {}_{t_0}^{C}D_t^{\theta}\phi(x,t) = -\lambda \frac{\delta H}{\delta \phi(x,t)} + \zeta, \tag{S18}$$

where

$$H = \int d^dx[\frac{1}{2}\tau\phi(x,t)^2 + \frac{1}{2}(\nabla\phi(x,t))^2 - h\phi(x,t) + \frac{1}{4}u\phi(x,t)^4]. \tag{S19}$$

since $\lambda$ and $u$ are constant, their values will not affect the critical exponents, therefore we set $\lambda = u = 1$ for convenience, $\tau = \tau_0 + Rt$ where $R$ is heating rate. Moreover, the fractional order $\theta$ should be smaller than 0.5, we choose $\theta = 0.2$. The left of Eq.(S18) is Caputo fractional derivative and can be expressed as

$$ {}_{t_0}^{C}D_t^{\theta}\phi(t) = \frac{1}{\Gamma(1-\theta)} \int_{t_0}^{t} d\tilde{t} \frac{\dot{\phi}(\tilde{t})}{(t-\tilde{t})^{\theta}}. \tag{S20}$$

We discretize the integral of $\tilde{t}$ from $t_0$ to $t$ uniformly as $t_1$, $t_2$, $t_3 ... t_i ... t_N$, set $N\Delta t = t - t_0$ and $\Delta t = t_i - t_{i-1} = 0.01^{\theta}$, where $N = 40$ is cut-off, according to linear interpolation we can obtain

$$\frac{1}{\Gamma(1-\theta)} \int_{t_0}^{t} d\tilde{t} \frac{\dot{\phi}(\tilde{t})}{(t-\tilde{t})^{\theta}} = \frac{1}{\Gamma(1-\theta)} \sum_i \int_{t_{i-1}}^{t_i} \frac{\dot{\phi}(\tilde{t})}{(t-\tilde{t})^{\theta}} d\tilde{t}, \tag{S21}$$

where

$$\dot{\phi}(\tilde{t}) = \frac{\phi(t_i) - \phi(t_{i-1})}{\Delta t}. \tag{S22}$$

Insert Eq.(S22) into Eq.(S21) can obtain

$$\frac{1}{\Gamma(1-\theta)}\sum_i \int_{t_{i-1}}^{t_i} \frac{\dot{\phi}(\tilde{t})}{(t-\tilde{t})^\theta} d\tilde{t} = \frac{\phi(t_i)-\phi(t_{i-1})}{\Gamma(1-\theta)\Delta t}\sum_i \int_{t_{i-1}}^{t_i} \frac{1}{(t-\tilde{t})^\theta} d\tilde{t}, \tag{S23}$$

since the interal of $\tilde{t}$ can be solved in Eq.(S23), then the left of Eq.S20 becomes

$${}_{t_0}^{C}D_t^\theta \phi(t) = \frac{1}{\Gamma(2-\theta)}\sum_i^N \frac{\phi(t_i)-\phi(t_{i-1})}{\Delta t}[(t-t_{i-1})^{1-\theta}-(t-t_i)^{1-\theta}], \tag{S24}$$

due to $t_N = t$, we can obtain the finite difference scheme of Caputo fractional derivative as

$${}_{t_0}^{C}D_t^\theta \phi(t) = \frac{1}{\Gamma(2-\theta)}\frac{\phi(t)}{\Delta t^\theta} + \frac{1}{\Gamma(2-\theta)} f_1 + \frac{1}{\Gamma(2-\theta)} f_0, \tag{S25}$$

where

$$\begin{aligned} f_1 &= \sum_i^{N-1} \frac{\phi(t_i)}{\Delta t^\theta}[(N-1-i)^{1-\theta}-(N-i)^{1-\theta}] - [(N-i)^{1-\theta}-(N+1-i)^{1-\theta}], \\ f_0 &= \frac{\phi(t_i)}{\Delta t^\theta}[(N-1)^{1-\theta}-N^{1-\theta}]. \end{aligned} \tag{S26}$$

If $N$ is big enough, $f_0$ also can be expressed as

$$f_0 = 1 - \sum_i^{N-1}[(N-1-i)^{1-\theta}-(N-i)^{1-\theta}] - [(N-i)^{1-\theta}-(N+1-i)^{1-\theta}]. \tag{S27}$$

In addition to the form on the left of the Eq(.S18), the evolution of $\phi(t)$ also depends on the state on the right side of the previous moment, $\delta H/\delta t$ is expressed as

$$\left.\frac{\delta H}{\delta t}\right|_{t=t_{N-1}} = \tau\phi(t_{N-1}) + u\phi(t_{N-1})^3 + \nabla^2\phi(t_{N-1}), \tag{S28}$$

For 2D model, $\nabla^2\phi$ is expressed as

$$\nabla^2\phi(x,y) = \phi(x+1,y)+\phi(x-1,y)+\phi(x,y+1)+\phi(x,y-1)-4\phi(x,y), \tag{S29}$$

where $(x,y)$ is spatial coordinate, thus the $\phi(t)$ can be obtained by the past value of $\phi$ and expressed as

$$\phi(t) = \Gamma(2-\theta)\sqrt{\Delta t^\theta} PP - \Delta t^\theta f_1 - \Delta t^\theta f_0 - \Gamma(2-\theta)\Delta t^\theta \left.\frac{\delta H}{\delta t}\right|_{t=t_{N-1}}, \tag{S30}$$

where $PP$ is random number derived from Gaussian distribution.

## 3. Ising model with temporal long-range interactions

We also simulate Ising model with temporal long-range interactions as

$$H = -J\sum_{t_1<t}\sum_{<i,j>}\frac{s_i(t)s_j(t_1)}{(t-t_1)^{1+\theta}} \tag{S31}$$

where $J$ is a positive constant and fixed as 1, $s_i(t) = \pm 1$ denotes spin at $i$ site and $t$ time, the interactions not only include the nearest-neighbor coupling in space which is completely same as standard Ising model but also consider the temporal long-range interactions decaying as $(t-t_1)^{1+\theta}$ where $0 < \theta < 1$. In order to avoiding divergence we just account for the interactions until $t_1 < t$. Subsequently, we obtain the magnetization and susceptibility which are defined as $M =< m >$ and $\chi = (< m^2 > - < m >^2)L^d$, where $L$ is lattice size and $m = \sum_i s_i / L^d$, respectively.

Another model we simulate is Ising model with temporal long-range interactions decaying as

$$H = -J \sum_{t_1 < t} \sum_{<i,j>} s_i(t) s_j(t_1) \exp(-\frac{t - t_1}{T_0}) \tag{S32}$$

where $T_0$ is characteristic time, the stronger the connection with the past, the greater $T_0$. Although the latter's interaction takes a completely different form from the former, they both take into account temporal long-range impacts. Above the upper critical dimension, to demonstrate that the critical exponents caused by time merely take into account the effect of time as a background and are independent of the specific form, we suggest an interactions form for exponential decay. Metropolis Monte carlo is a method of simulating random processes through a random number and is widely used to describe equilibrium states and dynamic processes[15–18]. In model(S31) and model(S32) the flipping probability $P$ of spin $s_i$ is satisfied with

$$P = \exp(\frac{-\Delta H}{KT}) \tag{S33}$$

where $\Delta H$ is the energy difference between before and after spin flipping, $K$ is Boltzmann constant and $T$ is temperature. Although Metropolis Monte carlo has argument about simulating interactions including $t$, it is just a method to assist in verifying that the critical exponents is independent with specifics of temporal interactions. If the results show that both of model(S31) and (S32) are consistent with theoretical prediction and similar to the results of Landau-Ginzurg model with fractal time process, then our relatively casual simulations are also quite convincing.

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