physics

MDPI

Review

Axion Electrodynamics and the Casimir Effect

Iver Brevik [1,*], Subhojit Pal [2], Yang Li [3,4], Ayda Gholamhosseinian [5] and Mathias Boström [2,6]

1 Department of Energy and Process Engineering, Norwegian University of Science and Technology, NO-7491 Trondheim, Norway
2 Centre of Excellence ENSEMBLE3 Sp. z o. o., Wolczynska Str. 133, 01-919 Warsaw, Poland; subhojit.pal@ensemble3.edu (S.P.); mathias.bostrom@ensemble3.eu (M.B.)
3 Department of Physics, Nanchang University, Nanchang 330031, China; leon@ncu.edu.cn
4 School of Physics and Materials Science, Nanchang University, Nanchang 330031, China
5 Department of Physics, Ferdowsi University of Mashhad, Mashhad 9177948974, Iran; ayda.gholamhosseinian@mail.um.ac.ir
6 Chemical and Biological Systems Simulation Laboratory, Centre of New Technologies, University of Warsaw, Banacha 2C, 02-097 Warsaw, Poland
* Correspondence: iver.h.brevik@ntnu.no

Abstract: We present a concise review of selected parts of axion electrodynamics and their application to Casimir physics. We present the general formalism including the boundary conditions at a dielectric surface, derive the dispersion relation in the case where the axion parameter has a constant spatial derivative in the direction normal to the conducting plates, and calculate the Casimir energy for the simple case of scalar electrodynamics using dimensional regularization.

Keywords: axion electrodynamics; Casimir effect; topological insulators

check for updates

Citation: Brevik, I.; Pal, S.; Li, Y.; Gholamhosseinian, A.; Boström, M. Axion Electrodynamics and the Casimir Effect. *Physics* **2024**, *6*, 407–421. https://doi.org/10.3390/physics6010027

Received: 24 November 2023
Revised: 13 January 2024
Accepted: 5 February 2024
Published: 14 March 2024

1. Introduction

The axion concept has actually a long history. It was introduced by Roberto Peccei and Helen Quinn back in 1977 [1,2] in connection with the CP (charge conjugation and parity symmetry) problem in high-energy physics. However, later it was understood as related to a natural extra term in the electromagnetic Lagrangian with a direct formal connection to materials like topological insulators and thus of obvious practical interest. So, we first describe the basic properties of topological materials. Topological insulators (TIs) are the new phases of matter, which exhibit unique electronic properties due to their nontrivial topological characteristics, and were discovered in 2005 [3,4]. These materials have insulating states inside the bulk with a bulk energy gap separating the highest occupied electronic band from the lowest empty band, like an ordinary insulator, while conducting states exist on their surfaces in the case of three-dimensional TIs or on their edges in the case of two-dimensional (2D) TIs, which are topologically protected (robust to local defects, imperfections, and disorders) by time-reversal symmetry [5]. In 2006, the TI phase was theoretically predicted [6] and experimentally realized in a CdTe-HgTe-CdTe quantum well; this quantum well behaves in bulk as an insulator. However, the electric current was observed across the interface, i.e., it behaves like a conductor in the surface region [7]. Additionally, one knows from band theory that conductors do not have a gap between their valence and conduction bands. In contrast, insulators are defined as materials with a gap between them. The most notable aspect is that Maxwell's equations are unable to explain the experimental behaviors of topological insulators. Notably, this kind of behavior was previously suggested by Frank Wilczek [8] in 1987, along with the possibility that it could be described by the axion electrodynamics he [9] and Steven Weinberg [10] developed. Their initial aim was to explain the breaking of combined symmetries of charge conjugation and parity in strong interactions.

With a topological invariant called the $\mathbb{Z}_2$ invariant, one can distinguish trivial insulators from topological insulators. Topological materials have interesting features, and one of them is the magnetoelectric effect caused by a term called the θ term. Since this term, referred to as the magnetoelectric polarizability, has exactly the same form as the action describing the coupling between a photon and an axion, these magnetoelectric phenomena are often depicted with axion electrodynamics. In the presence of time-reversal symmetry, θ takes on a quantized value $\theta = \pi(\mathrm{mod}\ 2\pi)$ for topological insulators and $\theta = 0$ for ordinary insulators [11–14]. The value of θ, nevertheless, can be arbitrary in systems with broken time-reversal symmetry, even depending on space and time as $\theta(r, t)$, like in various semimetallic phases. Moreover, when the dynamics of the axion field is included, the existence of new quasiparticles, such as the axion polariton [15], is also proposed. For more details, see [14] and the references therein.

In a separate paper [16] we reviewed the semiclassical electrodynamics and its link to Casimir physics. Notably, the history of this remarkable effect dates back to 1948 when it was predicted by Hendrik Casimir [17,18]. A formidable, and highly effective, theory for the retarded dispersion force between a pair of planar surfaces interacting across an intervening medium was developed in the 1950s by Evgeny Lifshitz and collaborators [19,20]. From the late 1960s, groups from around the world explored if a theory based upon the classical Maxwell's equations combined with the Planck quantization of light could by itself lead to a simple and useful semiclassical theory for van der Waals, Lifshitz, and Casimir interactions [21,22]. As one important example, a semiclassical derivation of Casimir effects in magnetic media was presented by Peter Richmond and Barry Ninham [23] already in 1971. Many theoretical [24–36] and experimental studies [37–44] have followed in the last 50 years. More information related to Casimir effects in traditional systems can be found in the extensive literature [45–51]. A number of studies have been carried out with a focus on predictive theories, nanobiotechnological applications, and novel materials' growth and characterization. Notably, going beyond the standard applications, specific ion effects in both biological systems and colloid chemistry have been proposed to occur partly due to ionic dispersion potentials [52] acting on ions in salt solutions [53–59]. This leads to a nonlinear coupling of electrodynamical and electrostatic interactions with a proposed role behind the so-called (ion-specific) Hofmeister effect [60]. Most interestingly, Casimir and van der Waals interactions may also have an impact on the growth of ice clusters within mist [61] and clouds [62,63]. It has been, and still is, relevant to explore the limits of validity of the different theories for dispersion forces (e.g., between two layered surfaces). A most natural extension of this conventional semiclassical electrodynamics, as well as quantum electrodynamics, in media is to allow for an extra pseudoscalar field, called conventionally the axion field $a(x)$ (x means here spacetime), pervading in the whole volume. Some of the pioneering papers on axion electrodynamics are listed in Refs. [1,2,10,64–69]. More recent investigations can be found in Refs. [70–94]. Here, we present an easy-to-follow and concise review of the current understanding of axion electrodynamics from our point of view.

2. Axion Electrodynamics

Different methods have been suggested to investigate the electromagnetic characteristics of 3D magnetic topological insulators (see, for example, Refs. [10,64]), which are based on axion field theory [95,96]. This approach introduces an additional term to the conventional Maxwell electromagnetic action, expressed as follows:

$$S_A = \frac{e^2}{32\pi^2\hbar c}\int d^3\vec{r}\, dt\, \theta\, \varepsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}, \tag{1}$$

where e denotes the elementary charge, $\hbar$ is the reduced Planck constant, c denotes the speed of light. Here and elsewhere below, $\varepsilon^{\mu\nu\alpha\beta}$ represents the completely antisymmetric tensor, $F_{\mu\nu}$ is the Maxwell field strength tensor, the Greek letter indices take the 0 (time, t), 1, 2, and 3 (space) values, $\vec{r}$ is the space vector, and θ denotes the axion coupling strength. Originally proposed in the context of quantum chromodynamics (QCD) to address the strong CP problem [1,97], the axion is a hypothetical pseudoscalar particle and has also been considered a potential candidate for cosmological dark matter. Since Equation (1) bears mathematical similarities to the description of cosmological/QCD axions, the term "axion" is used in the context of topological insulators. However, the axial coupling in topological insulators is related to the presence of a surface quantum Hall effect [98].

The reason why $a(x)$ is a pseudoscalar quantity is that the mentioned extra term in the Lagrangian implies a two-photon interaction with the electromagnetic field, and the pseudoscalar property of $a(x)$ ensures that its product with the polar vector $\mathbf{E}$ together with the axial vector $\mathbf{B}$ becomes a scalar in the Lagrangian. We shall in the following highlight some essential properties of the axion formalism, assuming a dielectric environment with the permittivity ε and the permeability μ being constants. It means that the constitutive relations are simply $\mathbf{D} = \varepsilon\mathbf{E}$, $\mathbf{B} = \mu\mathbf{H}$. The presentation in the following is largely based upon our earlier papers [91–94]. When magnetoelectric effects occur in topological material, the magnetic induction $\mathbf{B}$ changes the electric displacement vector $\mathbf{D}$, and the magnetic field intensity $\mathbf{H}$ is in turn influenced by the electric field. Then, the relations for $\mathbf{D}$ and $\mathbf{B}$ should be changed, $\mathbf{D} = \varepsilon\mathbf{E} - \theta\alpha\mathbf{B}/\pi$, $\mathbf{H} = \mathbf{B}/\mu + \theta\alpha\mathbf{E}/\pi$, where α is the fine-structure constant. These constitutive relations were given in [99] and in references therein; see also the later Ref. [100]. As a result, these two polarizations are coupled, meaning that the electromagnetic boundary conditions are off-diagonal components of the Fresnel coefficients [101].

In relation to our previous comments in Section 1, it is quite important to note parallelism between two analogous yet distinct phenomena. In the following discussion, our focus will be on the axion approach rooted in the Peccei–Quinn formalism, while a formal analogy arises with polariton excitations in condensed media. Our coupling constant $g_{a\gamma\gamma}$, below, will thus refer to the axion case. One might be interested in the corresponding coupling constant in the polarization case also, but this is a complicated subject into which we will not enter here. Interested readers may consult, for instance, the paper [102], to obtain detailed information about a strong coupling between a topological insulator and a III-V heterostructure.

2.1. Basic Equations

We choose the metric convention with signature $g_{00} = -1$ and introduce two field tensors, $F_{\alpha\beta}$ and $H^{\alpha\beta}$. The components of the original field tensor $F_{\alpha\beta}$ are as in vacuum, $F_{0i} = -E_i, F_{jk} = \epsilon_{ijk}B_i$ (where the indices in Latin letters denote the space coordinates), while the components of the contravariant response tensor $H^{\alpha\beta}$ are $H^{0i} = D_i, H_{jk} = \epsilon_{ijk}H_i$.

Including the pseudoscalar axion field $a = a(x)$, we obtain for the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\alpha\beta}H^{\alpha\beta} + \mathbf{A}\cdot\mathbf{J} - \rho\Phi - \frac{1}{2}\partial_\mu a\partial^\mu a - \frac{1}{2}m_a^2a^2 - \frac{1}{4}\theta(x)F_{\alpha\beta}\tilde{F}^{\alpha\beta}, \tag{2}$$

in which $\mathbf{A}$ denotes the magnetic potential, $\mathbf{J}$ denotes the current, ρ denotes the energy density, Φ denotes the scalar potential, m_a is the axion mass, $\tilde{F}^{\alpha\beta} = \epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}/2$, and we have defined the nondimensional field amplitude as

$$\theta(x) = g_{a\gamma\gamma}a(x). \tag{3}$$

The constant for the coupling of the axion with two photons can be defined as follows,

$$g_{a\gamma\gamma} = g_\gamma\frac{\alpha}{\pi}\frac{1}{f_a}, \tag{4}$$

where g_γ is a constant depending on the specific model used and is usually taken to be 0.36 [75]. The f_a represents the axion decay constant and for f_a, it is commonly assumed that its value is on the order of 10^{12} GeV. The Lagrangian's last term (2), $\mathcal{L}_{a\gamma\gamma}$, can be written as $\mathcal{L}_{a\gamma\gamma} = \theta(x)\mathbf{E}\cdot\mathbf{B}$, explicitly showing the pseudoscalar property of $\theta(x)$.

The expression (2) can be used to obtain the generalized Maxwell equations

$$\nabla\cdot\mathbf{D} = \rho - \mathbf{B}\cdot\nabla\theta, \tag{5}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial}{\partial t}\mathbf{D} + \frac{\partial \theta}{\partial t}\mathbf{B} + \nabla\theta \times \mathbf{E}, \tag{6}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{7}$$

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}. \tag{8}$$

These equations are general, and there are no restrictions on the spacetime variation of $a(x)$. The equations are relativistically covariant. It is crucial that the constitutive relations stated previously maintain their quite simple form $\mathbf{D} = \varepsilon \mathbf{E}$, $\mathbf{B} = \mu \mathbf{H}$ only in the rest system, and the electromagnetic formalism's covariance is achieved by introducing *two* distinct field tensors, $F_{\mu\nu}$ and $H^{\mu\nu}$. The field equations describing the system are

$$\nabla^2\mathbf{E} - \varepsilon\mu\frac{\partial^2}{\partial t^2}\mathbf{E} = \nabla(\nabla\cdot\mathbf{E}) + \mu\frac{\partial}{\partial t}\mathbf{J} + \mu\frac{\partial}{\partial t}\left[\frac{\partial\theta}{\partial t}\mathbf{B} + \nabla\theta\times\mathbf{E}\right], \tag{9}$$

$$\nabla^2\mathbf{H} - \varepsilon\mu\frac{\partial^2}{\partial t^2}\mathbf{H} = -\nabla\times\mathbf{J} - \nabla\times[\frac{\partial\theta}{\partial t}\mathbf{B} + \nabla\theta\times\mathbf{E}]. \tag{10}$$

In practice, the influence from axions is typically small in our case here. We do not consider the field equations for the axions explicitly for simplicity and clarity.

The field equations above contain the second-order derivatives of θ. These may conveniently be removed if we assume there is a strong external magnetic field $\mathbf{B}_e = B_e\hat{\mathbf{z}}$ present , where $\hat{\mathbf{z}}$ is the normal vector. Then, assuming the axion field

$$a(t) = a_0\cos\omega_a t, \tag{11}$$

one can separate out the part $\mathbf{E}_a(t)$ related to the $\ddot{\theta}$ term, where the dot on the top denotes the time derivative. From the governing equation for $\mathbf{E}_a(t)$, ignoring the current $\mathbf{J}$ as the axion-related field,

$$\nabla^2\mathbf{E}_a - \varepsilon\mu\frac{\partial^2}{\partial t^2}\mathbf{E}_a = \mu\ddot{\theta}\mathbf{B}_e. \tag{12}$$

Then,

$$\mathbf{E}_a(t) = -\frac{1}{\varepsilon}E_0\cos\omega_a t\,\hat{\mathbf{z}}, \tag{13}$$

where

$$E_0 = \theta_0 B_e. \tag{14}$$

After this separation, the field equations (9) and (10) take the reduced forms

$$\nabla^2\mathbf{E} - \varepsilon\mu\frac{\partial^2}{\partial t^2}\mathbf{E} = \nabla(\nabla\cdot\mathbf{E}) + \mu\frac{\partial}{\partial t}\mathbf{J} + \mu[\dot{\theta}\frac{\partial}{\partial t}\mathbf{B} + \nabla\theta\times\frac{\partial}{\partial \mathbf{t}}\mathbf{E}], \tag{15}$$

$$\nabla^2\mathbf{H} - \varepsilon\mu\frac{\partial^2}{\partial t^2}\mathbf{H} = -\nabla\times\mathbf{J} - \left[\dot{\theta}\nabla\times\mathbf{B} + (\nabla\theta)\nabla\cdot\mathbf{E} - (\nabla\theta\cdot\nabla)\mathbf{E}\right]. \tag{16}$$

In the following, we allow θ to be spatially varying but neglect the second-order derivatives, i.e., $\partial_i\partial_j\theta \approx 0$. This means that the model excludes situations where spatial boundaries lead to δ- and δ'-type terms. Actually, as shown in Equation (25) below, we in the mathematical analysis take θ to vary linearly in space over the field region of interest. There will accordingly be no second-order terms in the field region, while the electromagnetic boundary conditions must be imposed at the boundaries.

It is to be noted that the equations contain the dynamical fields $\mathbf{E}$ and $\mathbf{B}$ only.

2.2. Hybrid Form of Maxwell's Equations. Boundary Conditions

It turns out that one can construct a hybrid form of Maxwell's equations from which the generalized boundary conditions at a dielectric surface follow in a very transparent way. We introduce new fields $\mathbf{D}_\gamma$ and $\mathbf{H}_\gamma$ via

$$\begin{pmatrix} \mathbf{D}_\gamma \\ \mathbf{H}_\gamma \end{pmatrix} = \begin{pmatrix} \varepsilon & \theta \\ -\theta & 1/\mu \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}, \tag{17}$$

which shows how $\mathbf{D}_\gamma, \mathbf{H}_\gamma$ relate to the response tensor $H^{\mu\nu}$. The hybrid Maxwell equations thus become similar to those in the usual electrodynamics,

$$\nabla \times \mathbf{H}_\gamma = \mathbf{J} + \frac{\partial}{\partial t}\mathbf{D}_\gamma, \quad \nabla \cdot \mathbf{D}_\gamma = \rho, \tag{18}$$

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}, \quad \nabla \cdot \mathbf{B} = 0. \tag{19}$$

The boundary conditions at a dielectric boundary are then immediate:

$$E_\perp = E_{\gamma,\perp} + E_{a,\perp} \quad \text{is continuous}, \tag{20}$$

$$E_\perp, H_{\gamma,\perp} \quad \text{are continuous}, \tag{21}$$

$$D_{\gamma,\|}, B_\| \quad \text{are continuous}, \tag{22}$$

(the symbols $\perp$ and $\|$ mean perpendicular and parallel to the normal, thus parallel and perpendicular to the surface). A key quantity is the property of Poynting's vector, $\mathbf{S} = \mathbf{E} \times \mathbf{H}$. Taking the z component S_z orthogonal to a dielectric surface, one sees that

$$S_{1z} = S_{2z}, \tag{23}$$

showing that the energy flux density is continuous across the surface, dividing media 1 and 2. This is as one would expect as the surface is *at rest*; the force acting on it is not able to perform any mechanical work.

2.3. Dispersion Relations

We use now the standard plane wave expansion

$$\mathbf{E} \propto e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}, \quad \mathbf{k} = (k_x, k_y, k_z) \tag{24}$$

with $\mathbf{k}$ the wave number and the frequency ω.

We start from the reduced Maxwell Equations (15) and (16) and restrict ourselves to the case where $a(x)$ will vary with space only,

$$\boldsymbol{\beta} = \boldsymbol{\nabla}\boldsymbol{\theta}, \tag{25}$$

with $\boldsymbol{\beta}$ assumed constant. This form is mathematically convenient and often used in the literature. The form is of the same kind as assumed for Weyl semimetals [103,104], where the gradient of the axion is related to the separation of Weyl nodes in the Brillouin zone. One may here note that the case of topological insulators is different, as in such a case the gradient of the axion is taken to be zero except at the interfaces between trivial and nontrivial phases. The situation is, however, complex, and it should be mentioned that the configuration given here is very close to the one reported in Ref. [105], where a topological insulator slab is placed between two perfect conducting plates. We also point out that instructive papers on the new material, for example, those showing exotic Hall effects, those on TRS-broken semimetals, those on Chern insulators, etc., can be found in the Refs. [101,106,107].

Starting from Equations (15) and (16), assuming $\rho = \mathbf{J} = 0$, we find the determinant equation determining the dispersion relations. There are two dispersive branches. The first is a "normal" one, satisfying

$$\varepsilon\mu\omega^2 = \mathbf{k}^2, \tag{26}$$

corresponding to waves independent of the axions and polarizing parallel to $\boldsymbol{\beta}$. The second branch, polarizing perpendicular to $\boldsymbol{\beta}$, should have

$$\varepsilon\mu\omega^2 = \mathbf{k}^2 \pm \mu\beta\omega. \tag{27}$$

showing the splitting of this branch into two modes, equally separated from the normal mode on both sides. This sort of splitting has been encountered before under various circumstances; see, for instance, Refs. [75,76,108].

The following property of this kind of material should be noticed: the dispersive property does not influence the electromagnetic energy density. In a complex representation, the energy density can be written as

$$W_{\text{disp}} = \frac{1}{4}\left[\frac{d(\varepsilon_{\text{eff}}\omega)}{d\omega}|\mathbf{E}|^2 + |\mathbf{H}|^2\right], \tag{28}$$

where we have assumed for a moment that the medium is nonmagnetic, and we have introduced an effective permittivity ε_{eff} such that

$$\mathbf{k}^2 = \varepsilon_{\text{eff}}(\omega)\omega^2. \tag{29}$$

It then follows that

$$\frac{d(\varepsilon_{\text{eff}}\omega)}{d\omega} = \varepsilon, \tag{30}$$

which shows that the dispersive correction disappears. The energy density is the same as if dispersion were absent. This property is not quite trivial.

2.4. Dispersion Relations, When θ Is Time-Dependent

We introduce the variable α as

$$\alpha = \dot{\theta} \tag{31}$$

and assume α constant. The dispersion takes the form

$$\varepsilon\mu\omega^2 = \mathbf{k}^2 \pm \mu\alpha|\mathbf{k}|. \tag{32}$$

We may here for a nonmagnetic medium introduce the refractive index, $n_{\text{eff}} = \sqrt{\varepsilon_{\text{eff}}}$, and so obtain

$$n_{\text{eff}}(\omega) = \sqrt{\varepsilon + \frac{\alpha^2}{4\omega^2}} \pm \frac{\alpha}{2\omega}. \tag{33}$$

Given the assumed smallness of the axion contributions, we restrict the parameter values to the region $(\beta + \alpha)/k_z \ll 1$.

If k_z is the wave number for photons in the z direction, corresponding to axions with mass $m_a \sim \omega = 10^{-5}$ eV, we have $k_z = 10^{-5}$ eV, so that the condition above can be rewritten as

$$\beta + \alpha \ll 10^{-5}\ \text{eV}\left(\frac{m_a}{10^{-5}\ \text{eV}}\right). \tag{34}$$

3. Energy–Momentum Considerations

It is interesting to consider the electromagnetic energy–momentum tensor in interaction with the axion field. As this is a nonclosed physical system, the four-force density will in general be different from zero. The system is in this way analogous to the electromagnetic field in a medium in ordinary electrodynamics, since also, in that case, the system is nonclosed because the influence from the mechanical system itself is not accounted for (this is the point giving rise to the classic Abraham–Minkowski problem). We assume in this section that ε and μ can vary in space and time but do not restrict the derivatives with respect to space or time to be constants.

We may start with the electromagnetic energy density,

$$W = \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B}). \tag{35}$$

Together with the Poynting vector,

$$\mathbf{S} = \mathbf{E}\times\mathbf{H}, \tag{36}$$

this leads to the energy conservation equation

$$\nabla\cdot\mathbf{S} + \dot{W} = -\mathbf{E}\cdot\mathbf{J} - \dot{\theta}(\mathbf{E}\cdot\mathbf{B}). \tag{37}$$

There is thus an exchange of electromagnetic energy with the axion "medium" if $\mathbf{E}$ and $\mathbf{B}$ are different from zero and $\theta(t)$ is time-varying, even if $\mathbf{J} = 0$.

A more delicate question is the expression for the momentum density $\mathbf{g}$. In accordance with Planck's principle of inertia of energy, $\mathbf{g} = \mathbf{S}/c^2$, one would expect the Abraham momentum density $\mathbf{g}^{\mathrm{A}}$ to be right,

$$\mathbf{g}^{\mathrm{A}} = \mathbf{E}\times\mathbf{H}. \tag{38}$$

We notice that the Maxwell stress tensor,

$$T_{ik} = E_i D_k + H_i B_k - \frac{1}{2}\delta_{ik}(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B}), \tag{39}$$

is common for the Abraham and Minkowski alternatives, $T_{ik}^{\mathrm{A}} = T_{ik}^{\mathrm{M}} \equiv T_{ik}$. Here δ_{ik} is the Kronecker delta. The momentum conservation equation can thus be expressed in the form

$$\partial_k T_{ik} - \dot{g}_i^{\mathrm{A}} = f_i^{\mathrm{A}}, \tag{40}$$

where f_i^{A} are the components of Abraham's force density

$$\mathbf{f}^{\mathrm{A}} = \rho\mathbf{E} + (\mathbf{J}\times\mathbf{B}) + (\varepsilon\mu - 1)\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{H}) - g_{a\gamma\gamma}(\mathbf{E}\cdot\mathbf{B})\nabla a. \tag{41}$$

The third term on the right-hand side of Equation (41), the Abraham term, has experimentally turned up only in a few experiments, mainly at low frequencies where the mechanical oscillations of a test body are directly detectable.

In optics, the Abraham force will fluctuate out. It is therefore mathematically simpler, and in accordance with all observational experience in optics, to include the Abraham momentum (physically, a mechanical accompanying momentum) in the effective field momentum. Therewith, the momentum density becomes just the Minkowski momentum $\mathbf{g}^{\mathrm{M}}$, given by

$$\mathbf{g}^{\mathrm{M}} = \mathbf{D}\times\mathbf{B}. \tag{42}$$

The momentum conservation equation in the Minkowski case yields

$$\partial_k T_{ik} - \frac{\partial}{\partial t} g_i^{\mathrm{M}} = f_i^{\mathrm{M}}, \tag{43}$$

where

$$\mathbf{f}^{\mathrm{M}} = \rho\mathbf{E} + (\mathbf{J}\times\mathbf{B}) - (\mathbf{E}\cdot\mathbf{B})\nabla\theta. \tag{44}$$

Consider the relativistically covariant form for the energy–momentum balance: Minkowski's energy–momentum tensor,

$$S_{\mu}^{\mathrm{M}\nu} = F_{\mu\alpha}H^{\nu\alpha} - \frac{1}{4}g_{\mu}^{\nu}F_{\alpha\beta}H^{\alpha\beta}, \tag{45}$$

has the same form in all inertial frames. The conservation equations for electromagnetic energy and momentum can be written as

$$-\partial_{\nu}S_{\mu}^{\mathrm{M}\nu} = f_{\mu}^{\mathrm{M}}, \tag{46}$$

where $f_{\mu}^{\mathrm{M}} = (f_0, \mathbf{f}^{\mathrm{M}})$ is the four-force density. In the rest system,

$$f_0 = \mathbf{E}\cdot\mathbf{J} + (\mathbf{E}\cdot\mathbf{B})\dot{\theta}, \tag{47}$$

where $f_0^{\mathrm{M}} = f_0^{\mathrm{A}} \equiv f_0$.

4. Casimir Effect between Two Plates

We here restrict ourselves to the simplest case, namely, scalar electrodynamics, meaning that the photons' vector property is included but not their spin. We assume $\alpha = 0$ and also zero temperature. Assume the usual setup where there are two large metallic plates, with a gap L, orthogonal to the z direction. In the intervening region, there is an axion field $a(z)$ whose amplitude increases linearly with z,

$$a(z) = \frac{a_0 z}{L} = \beta z, \quad 0 < z < L, \tag{48}$$

where a_0, the amplitude at $z = L$, is fixed. There is no external magnetic field. The magnitude of β is not restricted to be small. The smallness expansion in the present theory applies rather to the spatial variation of the axion field, as embodied in the restriction $\partial_i\partial_j\theta \approx 0$, as also mentioned above.

We mention that the reduced Maxwell equations in this case can be written as

$$\nabla^2\mathbf{E} - \varepsilon\mu\ddot{\mathbf{E}} = g_{a\gamma\gamma}\mu\beta\hat{\mathbf{z}}\times\dot{\mathbf{E}}, \tag{49}$$

$$\nabla^2\mathbf{H} - \varepsilon\mu\ddot{\mathbf{H}} = g_{a\gamma\gamma}\beta\partial_z\mathbf{E}. \tag{50}$$

We obtain two dispersive branches, as before. The first, corresponding to Equation (26), can now be written

$$|\mathbf{k}| = \sqrt{\varepsilon\mu}\,\omega, \quad k_z = \frac{\pi n}{L}, \quad n = 1, 2, 3, \cdots, \tag{51}$$

showing no dependence on the axions. The second branch can be written as

$$\varepsilon\mu\omega^2 = \mathbf{k}^2 \pm g_{a\gamma\gamma}\beta\omega, \tag{52}$$

which can be recast in the form

$$\omega = \frac{1}{\sqrt{\varepsilon\mu}}\left(|\mathbf{k}| \pm \frac{1}{2}g_{a\gamma\gamma}\beta\sqrt{\frac{\mu}{\varepsilon}}\right), \tag{53}$$

in view of the smallness of $g_{a\gamma\gamma}^2$. This branch, composed of two modes, lies very close to the first mode above. For a given ω, there are in all three different values of $|\mathbf{k}|$.

Consider now the zero-temperature zero-point energy $\mathcal{E}$ of the field, defined by

$$\mathcal{E} = \frac{1}{2}\sum\omega. \tag{54}$$

From the second branch (53) only, one obtains, per unit plate area,

$$\mathcal{E} = \frac{1}{2\sqrt{\varepsilon\mu}} \sum_{n=1}^{\infty} \left[\int \frac{d^2k_\perp}{(2\pi)^2} \sqrt{k_\perp^2 + \frac{\pi^2 n^2}{L^2}} \pm \frac{g_{a\gamma\gamma}}{2L^2} \sqrt{\frac{\mu}{\varepsilon}} \right], \tag{55}$$

where $\mathbf{k}_\perp$ is the component of $\mathbf{k}$ orthogonal to the surface normal (note that the present definition of β differs from that in Ref. [108]). This expression can be further treated using dimensional regularization (see, for instance, Ref. [46]), with the result [108]

$$\mathcal{E} = \frac{1}{\sqrt{\varepsilon\mu}} \left(-\frac{\pi^2}{1440} \frac{1}{L^3} \mp \frac{g_{a\gamma\gamma}\beta}{4L^2} \sqrt{\frac{\mu}{\varepsilon}} \right). \tag{56}$$

Of main interest is the contribution from the photons and axions moving in the z direction. We associate this with the Casimir energy $\mathcal{E}_C$. Mathematically, the derivation involves the Hurwitz zeta function. One obtains

$$\mathcal{E}_C = \frac{1}{4\sqrt{\varepsilon\mu}} \left(-\frac{\pi}{6L} \pm \frac{1}{2} g_{a\gamma\gamma}\beta \sqrt{\frac{\mu}{\varepsilon}} \right) \frac{1}{L^2}. \tag{57}$$

Here, the first term comes from scalar photons propagating in the z direction, while the second term is the axionic contribution. With respect to the inverse L dependence, the Casimir energies for the electrodynamic and the axion parts behave similarly, as one would expect. In the above, the upper and lower signs match. In Equation (57), the small axion-induced increase in the Casimir energy arises from the superluminal mode in the dispersion relation (53) (meaning that the group velocity is larger than $1/\sqrt{\varepsilon\mu}$). This mode corresponds to a weak repulsive Casimir force. The other mode corresponds to a weak attractive force.

5. An Axion Echo from Reflection in Outer Space

Assume that the axion field is of the form $a = a(t) = a_0 \sin \omega_a t$, as is often chosen for outer space, and assume $\varepsilon = \mu = 1$. In view of the smallness of the axion velocity, $v \sim 10^{-3}c$, one has approximately $\omega_a = m_a$. An estimated value of $m_a = 10$ µeV is common. If the axions are associated with dark matter, we have to make do with a very low energy density, $\rho_{\rm DM} = 0.45$ GeV/cm^3 [71]. We will adopt the simple form

$$\rho_a = \frac{1}{2} m_a^2 a_0^2. \tag{58}$$

Assume now that an electromagnetic beam is sent from the Earth to an axion cloud. We take the initial form of the beam to be Gaussian,

$$\mathbf{A}_0(x,t) = c\, e^{-x^2/2D^2} \cos k_0 x\, \hat{\mathbf{y}}, \tag{59}$$

with $\mathbf{A}_0$ as the vector potential and the constant D as the Gaussian width. Making a Fourier decomposition,

$$\mathbf{A}_0(x.t) = \frac{1}{2} \int_{-\infty}^{\infty} \left[\mathbf{A}_0(k) e^{i(kx-\omega t)} + \mathbf{A}_0^*(k) e^{-i(kx-\omega t)} \right] dk, \tag{60}$$

one has as inversion

$$\mathbf{A}_0(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \left[\mathbf{A}_0(x,0) + \frac{i}{\omega} \frac{\partial \mathbf{A}_0}{\partial t}(x,0) \right] dx. \tag{61}$$

Then, imposing the condition $\partial \mathbf{A}_0/\partial t(x,0) = 0$, we obtain

$$\mathbf{A}_0(k) = \mathbf{A}_0^+(k) + \mathbf{A}_0^-(k), \tag{62}$$

where

$$\mathbf{A}_0^+(k) = \frac{cD}{2\sqrt{2\pi}} \exp\left[-\frac{1}{2}D^2(k-k_0)^2\right]\hat{\mathbf{y}}. \tag{63}$$

The expression for $\mathbf{A}_0^-(k)$ implies that $k - k_0 \to k + k_0$. The right-moving wave, to be considered henceforth, is

$$\mathbf{A}_0^+(x,t) = \int_0^\infty \mathbf{A}_0^+(k)\cos(kx - \omega t)dk, \quad \omega = k > 0. \tag{64}$$

In what follows, we omit the superscript. The incident wave fields are $\mathbf{E}_0(x,t) = -\dot{\mathbf{A}}_0(x,t)$, $\mathbf{H}_0(x,t) = \nabla \times \mathbf{A}_0(x,t)$.

The incident wave will interact with the axion cloud. As $\rho = \mathbf{J} = 0$, we see from Equations (41) or (44) that $\mathbf{f} = 0$, in accordance with the assumed homogeneity of the cloud. Furthermore, the component f_0 in (47) is zero, because $\mathbf{E}_0$ and $\mathbf{H}_0$ are orthogonal. We therefore go back to the field equations, from which we obtain

$$\nabla^2\mathbf{A} - \ddot{\mathbf{A}} = -g_{a\gamma\gamma}\dot{a}\nabla \times \mathbf{A}. \tag{65}$$

We here let $\mathbf{A} \to \mathbf{A}_0$ on the right-hand side of Equation (65). Neglecting $\nabla^2\mathbf{A}$ on the left-hand side, and using Equation (64), we obtain

$$\ddot{\mathbf{A}}(x,t) = -g_{a\gamma\gamma}a_0\omega_a \int_0^\infty A_0(k)k\sin(kx - \omega t)\cos\omega_a t\, dk\,\hat{\mathbf{z}}, \tag{66}$$

still with $\omega = k$.

In complex notation, extracting the resonance term,

$$\ddot{\mathbf{A}}(x,t) = \frac{1}{2}g_{a\gamma\gamma}a_0\omega_a \mathrm{Im}\int_0^\infty A_0(k)ke^{i(kx-\omega t+\omega_a t)}\,\hat{\mathbf{z}}. \tag{67}$$

We construct a new quantity $\mathbf{A}(k,t)$ as

$$\mathbf{A}(x,t) = \mathrm{Im}\int_0^\infty e^{ikx}\mathbf{A}(k,t)dk, \tag{68}$$

and can thus write

$$\ddot{\mathbf{A}}(k,t) = -\frac{1}{2}g_{a\gamma\gamma}a_0\omega_a A_0(k)ke^{-i(\omega-\omega_a)t}\,\hat{\mathbf{z}}. \tag{69}$$

With yet another quantity

$$\mathcal{A}(k,t) = \mathbf{A}(k,t)e^{-i\omega t}, \tag{70}$$

we have

$$\ddot{\mathbf{A}}(k,t) = \ddot{\mathcal{A}}(k,t)e^{i\omega t} + 2i\omega\dot{\mathcal{A}}(k,t)e^{i\omega t} - \omega^2\mathcal{A}(k,t)e^{i\omega t}, \tag{71}$$

Now, keeping the resonance term only, we obtain

$$\dot{\mathcal{A}}(k,t) = \frac{i}{4}g_{a\gamma\gamma}a_0\omega_a A_0(k)e^{-i(2\omega-\omega_a)t}\,\hat{\mathbf{z}}. \tag{72}$$

Integrating over t,

$$\mathcal{A}(k,t) = -g_{a\gamma\gamma}a_0\omega_a\frac{cD}{8\sqrt{2\pi}}\exp\left[-\frac{1}{2}D^2(k-k_0)^2\right]\frac{e^{-i(2\omega-\omega_a)t}}{2\omega-\omega_a}\,\hat{\mathbf{z}}. \tag{73}$$

The resonance at $\omega = \omega/2$ is as expected: an incoming photon 2ω can split an axion into two components with the same mass, $m_a = \omega_a$. We focus on the imaginary part of the above expression and use the relation

$$\lim \frac{\sin \alpha x}{\pi x}\Big|_{\alpha\to\infty} = \delta(x), \tag{74}$$

(with $\delta(x)$ the Dirac delta function) to obtain

$$\mathrm{Im}\mathcal{A}(k,t) = g_{a\gamma\gamma} a_0 \omega_a \frac{cD}{16}\sqrt{\frac{\pi}{2}} \exp\left[-\frac{1}{2}D^2(k-k_0)^2\right] \delta(\omega - \frac{1}{2}\omega_a)\,\hat{\mathbf{z}}. \tag{75}$$

From this, the axion echo can be found. This sort of calculation was pioneered by Pierre Sikivie and collaborators [75,82]. The present generalized form, containing the Gaussian width D, was given by one of the authors of this paper and Masud Chaichian [91].

6. Discussions and Future Outlooks

We would like to conclude with the following brief points.

1. As already mentioned, the influence of axions, at least in cosmology, is expected to be very weak. The cosmological axion energy density is often expected to be about 0.40 GeV/cm^3, corresponding to an axion mass of about 10^{-5} eV and a relative velocity of about 10^{-3}. Various experiments and proposals of experiments have been launched:
    (a) The haloscope experiment, proposed by Sikivie [75], in which the aim is to detect resonances between the electromagnetic eigenfrequencies of a dielectric cylinder and the axions (see also Refs. [70,91]). To date, no such resonance has been detected.
    (b) The idea, also due to Sikivie [75], to observe the axions via their electromagnetic "echo" returned back to the Earth from an outer cloud (see also Ref. [91]).
    (c) The broadband solenoidal haloscope proposed in Ref. [71], which proposes to make use of the axion "antenna" effect to focus the electromagnetic radiation emitted from dielectric boundaries towards a detector.
2. The above treatment provides a general review of axion electrodynamics and is, in principle, not limited to the semiclassical case. This constraint applies similarly to ordinary electrodynamics, usually when distances are small or temperatures are high.
3. The axion formalism is useful as regards application to topological insulators. Thus, the constitutive relations (17) can formally be taken over to this kind of modern material science as they stand. The case of chiral materials, for instance, a Faraday material, is more complicated since the coupling parameter θ becomes imaginary ; see, for instance, Ref. [74].
4. To conclude, we have presented a concise summary of the basics of axion electrodynamics, linking it to the general field of Casimir physics. Notably, additional contributions to the Casimir interaction are observed that occur as direct consequences of the extra pseudoscalar axion field. Novel physics, based on improved materials characterization requiring new and improved physical models, is likely to be discovered in the years to come.

Author Contributions: Initialization, conceptualization, supervision, project administration, funding acquisition, I.B. Writing—original draft preparation, writing—review and editing, I.B., M.B., A.G., Y.L., and S.P. All authors have read and agreed to the published version of the manuscript.

Funding: This research is part of the project No. 2022/47/P/ST3/01236 cofunded by the National Science Centre and the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 945339. We also thank the "ENSEMBLE3—Centre of Excellence for nanophotonics, advanced materials and novel crystal growth-based technologies" project (GA No. MAB/2020/14) carried out within the International Research Agendas programme of the Foundation for Polish Science cofinanced by the European Union under the European Regional Development Fund, the European Union's Horizon 2020 research and innovation programme Teaming for Excellence (GA. No. 857543) for support of this work.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Peccei, R.D.; Quinn, H.R. CP Conservation in the presence of pseudoparticles. *Phys. Rev. Lett.* **1977**, *38*, 1440–1443. [CrossRef]
2. Peccei, R.D. The strong CP problem and axions. In *Axions: Theory, Cosmology, and Experimental Searches*; Kuster, M., Raffelt, G., Beltrán, B., Eds.; Springer: Berlin/Heidelberg, Germany, 2008. pp. 3–17. [CrossRef]
3. Kane, C.L.; Mele, E.J. Z_2 topological order and the quantum spin Hall effect. *Phys. Rev. Lett.* **2005**, *95*, 146802. [CrossRef]
4. Kane, C.L.; Mele, E.J. Quantum spin Hall effect in graphene. *Phys. Rev. Lett.* **2005**, *95*, 226801. [CrossRef]
5. Hasan, M.Z.; Kane, C.L. Colloquium: Topological insulators. *Rev. Mod. Phys.* **2010**, *82*, 3045–3067. [CrossRef]
6. Bernevig, B.A.; Hughes, T.L.; Zhang, S.C. Quantum spin Hall effect and topological phase transition in HgTe quantum wells. *Science* **2006**, *314*, 1757–1761. [CrossRef]
7. Qi, X.L.; Zhang, S.C. The quantum spin Hall effect and topological insulators. *Phys. Today* **2010**, *63*, 33–38. [CrossRef]
8. Wilczek, F. Two applications of axion electrodynamics. *Phys. Rev. Lett.* **1987**, *58*, 1799–1802. [CrossRef]
9. Wilczek, F. Problem of strong P and T invariance in the presence of instantons. *Phys. Rev. Lett.* **1978**, *40*, 279–282. [CrossRef]
10. Weinberg, S. A new light boson? *Phys. Rev. Lett.* **1978**, *40*, 223–226. [CrossRef]
11. Fu, L.; Kane, C.L. Topological insulators with inversion symmetry. *Phys. Rev. B* **2007**, *76*, 045302. [CrossRef]
12. Liu, C.C.; Feng, W.; Yao, Y. Quantum spin Hall effect in silicene and two-dimensional germanium. *Phys. Rev. Lett.* **2011**, *107*, 076802. [CrossRef]
13. Fukui, T.; Hatsugai, Y.; Suzuki, H. Chern numbers in discretized Brillouin zone: Efficient method of computing (spin) Hall conductances. *J. Phys. Soc. Jpn.* **2005**, *74*, 1674–1677. [CrossRef]
14. Sekine, A.; Nomura, K. Axion electrodynamics in topological materials. *J. Appl. Phys.* **2021**, *129*, 141101. [CrossRef]
15. Li, R.D.; Wang, J.; Qi, X.L.; Zhang, S.C. Dynamical axion field in topological magnetic insulators. *Nat. Phys.* **2010**, *6*, 284–288. [CrossRef]
16. Boström, M.; Gholamhosseinian, A.; Pal, S.; Li, Y.; Brevik, I. Semi-classical electrodynamics and the Casimir effect. *Physics* **2024**, *6*, 456–467. [CrossRef]
17. Casimir, H.B.G. On the attraction between two perfectly conducting plates. *Proc. Kon. Ned. Akad. Wetensch. B* **1948**, *51*, 793–795. Available online: https://dwc.knaw.nl/DL/publications/PU00018547.pdf (accessed on 1 February 2024).
18. Casimir, H.B.G.; Polder, D. The influence of retardation on the London–van der Waals forces. *Phys. Rev.* **1948**, *73*, 360–372. [CrossRef]
19. Lifshitz, E.M. The theory of molecular attractive forces between solids. *Sov. Phys. JETP* **1956**, *2*, 73–83. Available online: http://jetp.ras.ru/cgi-bin/e/index/e/2/1/p73?a=list (accessed on 1 February 2024).
20. Dzyaloshinskii, I.E.; Lifshitz, E.M.; Pitaevskii, L.P. The general theory of van der Waals forces. *Adv. Phys.* **1961**, *10*, 165–209. [CrossRef]
21. Parsegian, V.A.; Ninham, B.W. Application of the Lifshitz theory to the calculation of van der Waals forces across thin lipid films. *Nature* **1969**, *224*, 1197–1198. [CrossRef]
22. Ninham, B.W.; Parsegian, V.A.; Weiss, G.H. On the macroscopic theory of temperature-dependent van der Waals forces. *J. Stat. Phys.* **1970**, *2*, 323–328. [CrossRef]
23. Richmond, P.; Ninham, B.W. A note on the extension of the Lifshitz theory of van der Waals forces to magnetic media. *J. Phys. C Solid State Phys.* **1971**, *4*, 1988–1993. [CrossRef]
24. Richmond, P.; Ninham, B.W. Calculations, using Lifshitz theory, of the height vs. thickness for vertical liquid helium films. *Solid State Commun.* **1971**, *9*, 1045–1047. [CrossRef]
25. Richmond, P.; Ninham, B.W.; Ottewill, R.H. A theoretical study of hydrocarbon adsorption on water surfaces using Lifshitz theory. *J. Coll. Interf. Sci.* **1973**, *45*, 69–80. [CrossRef]
26. Boström, M.; Sernelius, B.E. Thermal effects on the Casimir force in the 0.1–5 μm range. *Phys. Rev. Lett.* **2000**, *84*, 4757–4760. [CrossRef]
27. Bordag, M.; Geyer, B.; Klimchitskaya, G.L.; Mostepanenko, V.M. Casimir force at both nonzero temperature and finite conductivity. *Phys. Rev. Lett.* **2000**, *85*, 503–506. [CrossRef]
28. Ninham, B.W.; Boström, M.; Persson, C.; Brevik, I.; Buhmann, S.Y.; Sernelius, B.E. Casimir forces in a plasma: Possible connections to Yukawa Potentials. *Eur. Phys. J. D* **2014**, *68*, 328. [CrossRef]
29. Dou, M.; Lou, F.; Boström, M.; Brevik, I.; Persson, C. Casimir quantum levitation tuned by means of material properties and geometries. *Phys. Rev. B* **2014**, *89*, 201407. [CrossRef]
30. Esteso, V.; Carretero-Palacios, S.; Miguez, H. Nanolevitation phenomena in real plane-parallel systems due to the balance between Casimir and gravity forces. *J. Phys. Chem. C* **2015**, *119*, 5663–5670. [CrossRef] [PubMed]

31. Boström, M.; Dou, M.; Malyi, O.I.; Parashar, P.; Parsons, D.F.; Brevik, I.; Persson, C. Fluid-sensitive nanoscale switching with quantum levitation controlled by α-Sn/β-Sn phase transition. *Phys. Rev. B* **2018**, *97*, 125421. [CrossRef]
32. Esteso, V.; Carretero-Palacios, S.; Miguez, H. Casimir–Lifshitz force based optical resonators. *J. Phys. Chem. Lett.* **2019**, *10*, 5856–5860. [CrossRef] [PubMed]
33. Esteso, V.; Carretero-Palacios, S.; MacDowell, L.G.; Fiedler, J.; Parsons, D.F.; Spallek, F.; Míguez, H.; Persson, C.; Buhmann, S.Y.; Brevik, I.; et al. Premelting of ice adsorbed on a rock surface. *Phys. Chem. Chem. Phys.* **2020**, *22*, 11362–11373. [CrossRef] [PubMed]
34. Li, Y.; Brevik, I.; Malyi, O.I.; Boström, M. Different pathways to anomalous stabilization of ice layers on methane hydrates. *Phys. Rev. E* **2023**, *108*, 034801. [CrossRef] [PubMed]
35. Boström, M.; Khan, M.R.; Gopidi, H.R.; Brevik, I.; Li, Y.; Persson, C.; Malyi, O.I. Tuning the Casimir-Lifshitz force with gapped metals. *Phys. Rev. B* **2023**, *108*, 165306. [CrossRef]
36. Klimchitskaya, G.L.; Mostepanenko, V.M. Casimir effect invalidates the Drude model for transverse electric evanescent waves. *Physics* **2023**, *5*, 952–967. [CrossRef]
37. Hauxwell, F.; Ottewill, R. A study of the surface of water by hydrocarbon adsorption. *J. Coll. Interf. Sci.* **1970**, *34*, 473–479. [CrossRef]
38. Anderson, C.H.; Sabisky, E.S. Phonon interference in thin films of liquid helium. *Phys. Rev. Lett.* **1970**, *24*, 1049–1052. [CrossRef]
39. Lamoreaux, S.K. Demonstration of the Casimir force in the 0.6 to 6 μm range. *Phys. Rev. Lett.* **1997**, *78*, 5–8. [CrossRef]
40. Harris, B.W.; Chen, F.; Mohideen, U. Precision measurement of the Casimir force using gold surfaces. *Phys. Rev. A* **2000**, *62*, 052109. [CrossRef]
41. Decca, R.S.; López, D.; Fischbach, E.; Krause, D.E. Measurement of the Casimir force between dissimilar metals. *Phys. Rev. Lett.* **2003**, *91*, 050402. [CrossRef]
42. Feiler, A.A.; Bergström, L.; Rutland, M.W. Superlubricity using repulsive van der Waals forces. *Langmuir* **2008**, *24*, 2274–2276. [CrossRef]
43. Munday, J.N.; Capasso, F.; Parsegian, V.A. Measured long-range repulsive Casimir–Lifshitz forces. *Nature* **2009**, *457*, 170–173. : [CrossRef] [PubMed]
44. Somers, D.; Garrett, J.; Palm, K.; Munday, J.N. Measurement of the Casimir torque. *Nature* **2018**, *564*, 386–389. [CrossRef]
45. Mahanty, J.; Ninham, B.W. *Dispersion Forces*; Academic Press Inc. Ltd.: London, UK, 1976.
46. Milton, K.A. *The Casimir Effect: Physical Manifestations of Zero-Point Energy*; World Scientific Co. Ltd.: Singapore, 2001. [CrossRef]
47. Parsegian, V.A. *Van der Waals Forces: A Handbook for Biologists, Chemists, Engineers, and Physicists*; Cambridge University Press: New York, NY, USA, 2006. [CrossRef]
48. Bordag, M.; Klimchitskaya, G.L.; Mohideen, U.; Mostepanenko, V.M. *Advances in the Casimir Effect*; Oxford Science Publications: New York, NY, USA, 2009. [CrossRef]
49. Sernelius, B.E. *Surface Modes in Physics*; Wiley-VCH Verlag Berlin GmbH: Berlin, Germany, 2005. [CrossRef]
50. Buhmann, S.Y. *Dispersion Forces I: Macroscopic Quantum Electrodynamics and Ground-State Casimir, Casimir–Polder and Van der Waals Forces*; Springer: Berlin/Heidelberg, Germany, 2012. [CrossRef]
51. Sernelius, B.E. *Fundamentals of van der Waals and Casimir Interactions*; Springer Nature Switzerland AG: Cham, Switzerland, 2018. [CrossRef]
52. Boström, M.; Williams, D.R.M.; Ninham, B.W. Specific ion effects: Why DLVO theory fails for biology and colloid systems. *Phys. Rev. Lett.* **2001**, *87*, 168103. [CrossRef]
53. Parsons, D.F.; Ninham, B.W. Ab initio molar volumes and Gaussian radii. *J. Phys. Chem. A* **2009**, *113*, 1141–1150. [CrossRef]
54. Parsons, D.F.; Ninham, B.W. Nonelectrostatic ionic forces between dissimilar surfaces: A mechanism for colloid separation. *J. Phys. Chem. C* **2012**, *116*, 7782–7792. [CrossRef]
55. Medda, L.; Barse, B.; Cugia, F.; Boström, M.; Parsons, D.F.; Ninham, B.W.; Monduzzi, M.; Salis, A. Hofmeister challenges: Ion binding and charge of the BSA protein as explicit examples. *Langmuir* **2012**, *28*, 16355–16363. [CrossRef] [PubMed]
56. Duignan, T.T.; Parsons, D.F.; Ninham, B.W. A continuum model of solvation energies including electrostatic, dispersion, and cavity contributions. *J. Phys. Chem. B* **2013**, *117*, 9421–9429. [CrossRef]
57. Parsons, D.F.; Salis, A. The impact of the competitive adsorption of ions at surface sites on surface free energies and surface forces. *J. Chem. Phys.* **2015**, *142*, 134707. [CrossRef]
58. Boström, M.; Malyi, O.I.; Thiyam, P.; Berland, K.; Brevik, I.; Persson, C.; Parsons, D.F. The influence of Lifshitz forces and gas on premelting of ice within porous materials. *EPL (Europhys. Lett.)* **2016**, *115*, 13001. [CrossRef]
59. Thiyam, P.; Fiedler, J.; Buhmann, S.Y.; Persson, C.; Brevik, I.; Boström, M.; Parsons, D.F. Ice particles sink below the water surface due to a balance of salt, van der Waals, and buoyancy forces. *J. Phys. Chem. C* **2018**, *122*, 15311–15317. [CrossRef]
60. Ninham, B.W.; Yaminsky, V. Ion binding and ion specificity: The Hofmeister effect and Onsager and Lifshitz theories. *Langmuir* **1997**, *13*, 2097–2108. [CrossRef]
61. Boström, M.; Li, Y.; Brevik, I.; Persson, C.; Carretero-Palacios, S.; Malyi, O.I. van der Waals induced ice growth on partially melted ice nuclei in mist and fog. *Phys. Chem. Chem. Phys.* **2023**, *25*, 32709–32714. [CrossRef]
62. Luengo-Márquez, J.; MacDowell, L.G. Lifshitz theory of wetting films at three phase coexistence: The case of ice nucleation on Silver Iodide (AgI). *J. Coll. Interf. Sci.* **2021**, *590*, 527–538. [CrossRef]

63. Luengo-Marquez, J.; Izquierdo-Ruiz, F.; MacDowell, L.G. Intermolecular forces at ice and water interfaces: Premelting, surface freezing, and regelation. *J. Chem. Phys.* **2022**, *157*, 044704. [CrossRef] [PubMed]
64. Sikivie, P. Experimental tests of the "invisible" axion. *Phys. Rev. Lett.* **1983**, *51*, 1415–1417. [CrossRef]
65. Preskill, J.; Wise, M.B.; Wilczek, F. Cosmology of the invisible axion. *Phys. Lett. B* **1983**, *120*, 127–132. [CrossRef]
66. Abbott, L.F.; Sikivie, P. A cosmological bound on the invisible axion. *Phys. Lett. B* **1983**, *120*, 133–136. [CrossRef]
67. Dine, M.; Fischler, W. The not-so-harmless axion. *Phys. Lett. B* **1983**, *120*, 137–141. [CrossRef]
68. Sikivie, P. Axion cosmology. In *Axions: Theory, Cosmology, and Experimental Searches*; Kuster, M., Raffelt, G., Beltrán, B., Eds.; Springer: Berlin/Heidelberg, Germany, 2008; pp. 19–50. [CrossRef]
69. Sikivie, P.; Sullivan, N.; Tanner, D.B. Proposal for axion dark matter detection using an LC circuit. *Phys. Rev. Lett.* **2014**, *112*, 131301. [CrossRef] [PubMed]
70. Millar, A.J.; Redondo, J.; Steffen, F.D. Dielectric haloscopes: Sensitivity to the axion dark matter velocity. *J. Cosmol. Astropart. Phys.* **2017**, *2017*, 006. . [CrossRef]
71. Liu, J.; et al. [BREAD Collaboration]. Broadband solenoidal haloscope for terahertz axion detection. *Phys. Rev. Lett.* **2022**, *128*, 131801. [CrossRef] [PubMed]
72. Li, X.; Shi, X.; Zhang, J. Generalized Riemann ζ-function regularization and Casimir energy for a piecewise uniform string. *Phys. Rev. D* **1991**, *44*, 560–562. [CrossRef] [PubMed]
73. Lawson, M.; Millar, A.J.; Pancaldi, M.; Vitagliano, E.; Wilczek, F. Tunable axion plasma haloscopes. *Phys. Rev. Lett.* **2019**, *123*, 141802. [CrossRef] [PubMed]
74. Jiang, Q.D.; Wilczek, F. Chiral Casimir forces: Repulsive, enhanced, tunable. *Phys. Rev. B* **2019**, *99*, 125403. [CrossRef]
75. Sikivie, P. Invisible axion search methods. *Rev. Mod. Phys.* **2021**, *93*, 015004. [CrossRef]
76. McDonald, J.I.; Ventura, L.B. Optical properties of dynamical axion backgrounds. *Phys. Rev. D* **2020**, *101*, 123503. [CrossRef]
77. Zyla, P.A.; et al. [Particle Data Group]. Review of Particle Physics. *Prog. Theor. Exp. Phys* **2020**, *2020*, 083C01. Ch. 79. [CrossRef]
78. Arza, A.; Schwetz, T.; Todarello, E. How to suppress exponential growth—On the parametric resonance of photons in an axion background. *J. Cosmol. Astropart. Phys.* **2020**, *2020*, 013. [CrossRef]
79. Carenza, P.; Mirizzi, A.; Sigl, G. Dynamical evolution of axion condensates under stimulated decays into photons. *Phys. Rev. D* **2020**, *101*, 103016. [CrossRef]
80. Leroy, M.; Chianese, M.; Edwards, T.D.; Weniger, C. Radio signal of axion-photon conversion in neutron stars: A ray tracing analysis. *Phys. Rev. D* **2020**, *101*, 123003. [CrossRef]
81. Ouellet, J.; Bogorad, Z. Solutions to axion electrodynamics in various geometries. *Phys. Rev. D* **2019**, *99*, 055010. [CrossRef]
82. Arza, A.; Sikivie, P. Production and detection of an axion dark matter echo. *Phys. Rev. Lett.* **2019**, *123*, 131804. [CrossRef]
83. Qiu, Z.; Cao, G.; Huang, X.G. Electrodynamics of chiral matter. *Phys. Rev. D* **2017**, *95*, 036002. [CrossRef]
84. Dror, J.A.; Murayama, H.; Rodd, N.L. Cosmic axion background. *Phys. Rev. D* **2021**, *103*, 115004. [CrossRef]
85. Fukushima, K.; Imaki, S.; Qiu, Z. Anomalous Casimir effect in axion electrodynamics. *Phys. Rev. D* **2019**, *100*, 045013. [CrossRef]
86. Tobar, M.E.; McAllister, B.T.; Goryachev, M. Modified axion electrodynamics as impressed electromagnetic sources through oscillating background polarization and magnetization. *Phys. Dark Univ.* **2019**, *26*, 100339. [CrossRef]
87. Bae, S.; Youn, S.; Jeong, J. Tunable photonic crystal haloscope for high-mass axion searches. *Phys. Rev. D* **2023**, *107*, 015012. [CrossRef]
88. Adshead, P.; Draper, P.; Lillard, B. Time-domain properties of electromagnetic signals in a dynamical axion background. *Phys. Rev. D* **2020**, *102*, 123011. [CrossRef]
89. Tobar, M.E.; McAllister, B.T.; Goryachev, M. Poynting vector controversy in axion modified electrodynamics. *Phys. Rev. D* **2022**, *105*, 045009. [CrossRef]
90. DeRocco, W.; Hook, A. Axion interferometry. *Phys. Rev. D* **2018**, *98*, 035021. [CrossRef]
91. Brevik, I.H.; Chaichian, M.M. Axion electrodynamics: Energy–momentum tensor and possibilities for experimental tests. *Int. J. Mod. Phys. A* **2022**, *37*, 2250151. [CrossRef]
92. Brevik, I.H.; Chaichian, M.M. Electric current and heat production by a neutral carrier: An effect of the axion. *Eur. Phys. J. C* **2022**, *82*, 202. [CrossRef]
93. Brevik, I.; Favitta, A.M.; Chaichian, M. Axionic and nonaxionic electrodynamics in plane and circular geometry. *Phys. Rev. D* **2023**, *107*, 043522. [CrossRef]
94. Brevik, I.; Chaichian, M.; Favitta, A.M. On the Axion Electrodynamics in a two-dimensional slab and the Casimir effect. *arXiv* **2023**, arXiv:2310.05575. [CrossRef]
95. Qi, X.L.; Zhang, S.C. Topological insulators and superconductors. *Rev. Mod. Phys.* **2011**, *83*, 1057–1110. [CrossRef]
96. Qi, X.L.; Hughes, T.L.; Zhang, S.C. Topological field theory of time-reversal invariant insulators. *Phys. Rev. B* **2008**, *78*, 195424. [CrossRef]
97. Peccei, R.D.; Quinn, H.R. Constraints imposed by CP conservation in the presence of pseudoparticles. *Phys. Rev. D* **1977**, *16*, 1791–1797. [CrossRef]
98. Lu, B.S. The Casimir effect in topological matter. *Universe* **2021**, *7*, 237. [CrossRef]
99. Martín-Ruiz, A.; Cambiaso, M.; Urrutia, L. The magnetoelectric coupling in electrodynamics. *Int. J. Mod. Phys. A* **2019**, *34*, 1941002. [CrossRef]

100. Nogueira, F.S.; van den Brink, J. Absence of induced magnetic monopoles in Maxwellian magnetoelectrics. *Phys. Rev. Res.* **2022**, *4*, 013074. [CrossRef]
101. Woods, L.M.; Krüger, M.; Dodonov, V.V. Perspective on some recent and future developments in Casimir interactions. *Appl. Sci.* **2020**, *11*, 293. [CrossRef]
102. To, D.Q.; Wang, Z.; Ho, D.Q.; Hu, R.; Acuna, W.; Liu, Y.; Bryant, G.W.; Janotti, A.; Zide, J.M.; Law, S.; et al. Strong coupling between a topological insulator and a III-V heterostructure at terahertz frequency. *Phys. Rev. Mater.* **2022**, *6*, 035201. [CrossRef]
103. Kargarian, M.; Randeria, M.; Trivedi, N. Theory of Kerr and Faraday rotations and linear dichroism in topological Weyl semimetals. *Sci. Rep.* **2015**, *5*, 12683. [CrossRef]
104. Wilson, J.H.; Allocca, A.A.; Galitski, V. Repulsive Casimir force between Weyl semimetals. *Phys. Rev. B* **2015**, *91*, 235115. [CrossRef]
105. Martín-Ruiz, A.; Cambiaso, M.; Urrutia, L. A Green's function approach to the Casimir effect on topological insulators with planar symmetry. *Europhys. Lett.* **2016**, *113*, 60005. [CrossRef]
106. Woods, L.; Dalvit, D.A.R.; Tkatchenko, A.; Rodriguez-Lopez, P.; Rodriguez, A.W.; Podgornik, R. Materials perspective on Casimir and van der Waals interactions. *Rev. Mod. Phys.* **2016**, *88*, 045003. [CrossRef]
107. Khusnutdinov, N.; Woods, L.M. Casimir effects in 2D Dirac materials (Scientific Summary). *JETP Lett.* **2019**, *110*, 183–192. [CrossRef]
108. Brevik, I. Axion electrodynamics and the axionic Casimir effect. *Universe* **2021**, *7*, 133. [CrossRef]