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Article

Bilateral Connexive Logic

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Abstract: This paper proposes a bilateral analysis of connexivity, presenting a bilateral natural deduction system for a weak connexive logic. The proposed logic deviates from other connexive logics and other bilateral logics in the following respects: (1) The logic induces a difference in meaning between inner and outer occurrences of negation in the connexive axioms. (2) The logic allows incoherence—assertion and denial of the same formula—while still being non-trivial.

Keywords: connexive logic; bilateral logic; negation

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1. Introduction

The traditional characteristics of connexive logic (see [1] or [2] for a general survey) are the following (formal) theorems, which are *not* theorems of classical logic. Let '$\rightarrow$' denote a generic conditional and '$\neg$' a generic negation. In addition, let, φ, ψ range over arbitrary object language formulas.

$$\begin{array}{l} A_1 : \vdash \ \neg(\varphi \rightarrow \neg\varphi) \\ A_2 : \vdash \ \neg(\neg\varphi \rightarrow \varphi) \end{array} \tag{1}$$

Both are jointly known as *Aristotle's theses*.

$$\begin{array}{l} B_1 : \vdash \ (\varphi \rightarrow \psi) \rightarrow \neg(\varphi \rightarrow \neg\psi) \\ B_2 : \vdash \ (\varphi \rightarrow \neg\psi) \rightarrow \neg(\varphi \rightarrow \psi) \end{array} \tag{2}$$

Both are jointly known as *Boethius' theses*.

There is also a negative connexivity requirement[1] prohibiting interpreting the conditional as a bi-conditional:

$$\nvdash (\varphi \rightarrow \psi) \rightarrow (\psi \rightarrow \varphi) \tag{3}$$

For the history of connexive logics see [4,5].

The implicit assumptions in much[2] of the literature on connexive logics are as follows:

- The *outer negation* in the A_i axioms and the *inner negation* in those axioms *have the same meaning*. Note the difference from [7], in the context of dialectic logics. There, he considers two different negations, in contrast to my distinction between two meanings, depending on where the negation occurs.
- The *outer negation* in the consequent of the B_1 axiom and the *inner negation* in the consequent of this axioms *have the same meaning*.
- The *(outer) negation* in the consequent of the B_2 axiom and the *(inner) negation* in the antecedent of this axiom *have the same meaning*.

In this paper, I challenge this assumption, proposing an analysis of connexivity in the framework of *bilateral logic* (see Section 2.1).

The main idea is the following:

- Interpret the external negation in the A_i axioms as a *denial* (of $\varphi \rightarrow \neg\varphi$ and $\neg\varphi \rightarrow \varphi$, respectively), and interpret the inner negation as 'plain' negation.
- Interpret similarly the two occurrences of negation in the B_i axioms.

As a result, the following bilateral connexive system emerges, expressed in the bilateral *signed formulas* notation of Rumfitt [8] (explained below):

$$\begin{array}{l} A_1^b : \vdash\ -(\varphi \to \neg\varphi) \\ A_2^b : \vdash\ -(\neg\varphi \to \varphi) \end{array} \tag{4}$$

I refer to these axioms as Aristotle's b-theses.

$$\begin{array}{l} B_1^b :\ +(\varphi \to \psi)\ \vdash\ -(\varphi \to \neg\psi) \\ B_2^b :\ +(\varphi \to \neg\psi)\ \vdash\ -(\varphi \to \psi) \end{array} \tag{5}$$

I refer to these theses as Boethius' b-theses.

Because Boethius' b-theses appear in *rule form* (the reason for which is explained below; see Remark 1), such a logic is known[3] as *weakly connexive*.

A precursor to the distinction in the meaning of the two occurrences of negation in the connexive axioms can be found already in Boethius himself, who uses the inner occurrences as a *classifier* of conditionals into affirmative or negative, as shown by the following quotation of Boethius in [5]:

> Some hypothetical propositions, however, are affirmative and others are negative [...] Affirmatives are when we say "If it is A, it is B", or "If it is not A, it is B"; negative "If it is A, it is not B", "If it is not A, it is not B". For it depends on the consequent whether the proposition is judged to be affirmative or negative.

While not explicitly distinguishing between inner and outer negations, this quotation shows that the inner negation has a different role depending on whether it occurs in the antecedent or in the consequent. No such distinction applies to the outer negation.

The bilateral connexive logic defined below differs in an essential way from most other bilateral logics, such as bilateral classical logic [8] and bilateral intuitionistic logic [13,14]: it allows for *incoherence*, the assertion and denial of *the same* formula, without being trivial[4]. Thus, such a logic should turn out to be *bilaterally paraconsistent*. For other bilaterally paraconsistent logic see [15] and further references therein.

There are two main innovations in this paper:

1. The notion of a bilateral connexive logic.
2. The distinction in meaning between inner and outer occurrences of negation in the connexive axioms.

Note that I *am not* aiming here for a thorough investigation of the proposed bilateral connexive logic, only for its presentation for further consideration by the community.

2. A Bilateral Connexive Natural Deduction Proof System

In this section, I present a bilateral connexive natural deduction proof system $\mathcal{N}^b$.

Note that I view such a system as a *definitional tool*, i.e., a meaning-conferring proof system, specifying meaning in accordance with *proof-theoretic semantics* [16].

2.1. Bilateral Logics

Bilateralism is an approach to logic that takes both *assertion* (or *acceptance*) and *denial* (or *rejection*) as being primitive speech acts treated on a par, neither reducible to the other, but not independent either. This is in contrast to more standard *unilateral* approaches to logic, based on assertion only, and employing Frege's reduction, whereby the denial of a proposition is just the assertion of its negation.

2.2. Bilateral Natural Deduction Proof Systems

I use φ, ψ as metavariables over object-language formulas and Γ, Δ as finite sets of formulas.

Following Rumfitt [8], the bilateral ND-systems use *signed* formulas *both for assumptions and for conclusions* of I/E-rules: $+\varphi$ for asserting φ and $-\varphi$ for denying φ. In addition, $*\varphi$

means either $+\varphi$ or $-\varphi$, and $*^{-1}$, the *conjugate* of $*$, reverses the force (i.e., from '+' to '$-$' and from '$-$' to '+'). Collections of assertions and collections of denials (which are possibly empty) are denoted, respectively, by Γ^+ and Δ^-.

Note that '+'/'$-$' , called the *(formal) force markers*, are not in the object language of the bilateral logic defined by $\mathcal{N}^b$; they belong to the meta-language in which the proof-system $\mathcal{N}^b$ is formulated. In particular, the force markers cannot be *embedded* or *iterated*. Thus, the following are ill-formed:

$$++\varphi,\ \ +-\varphi,\ \ +\varphi \to -\psi$$

Remark 1.

1. *The prohibition of embedding the denial formal force marker makes bilateralism a suitable framework for my wish to distinguish the meaning of outer and inner occurrences of negation: the former can indicate denial, while the latter serves as 'plain' negation.*
2. *The prohibition of iteration and embedding of the formal force markers is the source of the weak connexivity of the logic defined below. If not this prohibition, the 'natural' bilateral reading of B_1 would have been*

$$\vdash +(\varphi \to \psi)\ \to\ -(\varphi \to \neg\psi)$$

which, however, is ill-formed.

2.3. Bilateral Connexive Natural-Deduction Proof Systems

Below are the I/E rules for the bilateral connexive $\mathcal{N}^b$ proof system.

$$\frac{[+\varphi]}{+\varphi}\ (Ass^+) \qquad \frac{[-\varphi]}{-\varphi}\ (Ass^-) \tag{6}$$

$$\frac{\begin{array}{c}[+\varphi]_i\\ \vdots\\ +\psi\end{array}}{+(\varphi \to \psi)}\ (\to^+ I^i) \qquad \frac{+(\varphi \to \psi) \quad +\varphi}{+\psi}\ (\to^+ E) \tag{7}$$

$$\frac{\begin{array}{c}[+\varphi]_i\\ \vdots\\ -\psi\end{array}}{-(\varphi \to \psi)}\ (\to^- I^i) \qquad \frac{-(\varphi \to \psi) \quad +\varphi}{-\psi}\ (\to^- E) \tag{8}$$

$$\frac{-\varphi}{+\neg\varphi}\ (\neg^+ I) \qquad \frac{+\neg\varphi}{-\varphi}\ (\neg^+ E) \tag{9}$$

$$\frac{+\varphi}{-\neg\varphi}\ (\neg^- I) \qquad \frac{-\neg\varphi}{+\varphi}\ (\neg^- E) \tag{10}$$

Remark 2.

1. *As usual, discharged assumptions are enclosed between square brackets, indexed by a* discharge index, *linking the assumption to an instance of a rule actually discharging it. Note that both $[+\varphi]$ and $[-\varphi]$ can serve as discharged assumptions.*
2. *A similar way of formulating the introduction of an assumption is attributed by von Plato [17] to Gentzen. The rule was intended to make the derivation of $\varphi \supset \varphi$ less awkward and was later abandoned by Gentzen.*

I revive this formulation here in order to make it explicit that φ, assumed in the premise, is also an explicit conclusion, carrying the same formal force indicator*! Thus, when this rule is applied, it uses the assumption in the premise, allowing later discharge of the latter. We thus obtain*

$$\dfrac{\dfrac{[+\varphi]_1}{+\varphi_1}\,(Ass^+)}{+(\varphi \to \varphi)}\,(\to^+ I^1) \tag{11}$$

However, to avoid notational clutter in displaying derivations, I occasionally omit the application of this rule, using the usual Prawitz-style presentation.

3. *The positive I/E-rules for the conditional are the usual intuitionistic I/E-rules.*
4. *The negative I/E-rules for the conditional are the source of connexivity, reflecting the modification of the falsity conditions of the conditional according to what has become known as the* Bochum plan *[18].*
5. *The negation rules are those of Rumfitt [8].*
6. *Note how the distinction in meaning of inner and outer negation is reflected in the form of the rules for the denied conditional: inner negation occurs only in the premise, while outer negation occurs in the conclusion.*
   Note that according to a common approach in proof-theoretic semantics (Dummett, Schroeder-Heister, Prawitz, and others), the meanings are captured by canonical derivations*. For a detailed account, see [16]. The indicated difference in rules induces a difference in canonical derivations.*

Proposition 1 (derivability of (dni))**.** *The following rules are derivable in $\mathcal{N}^b$:*

$$\frac{+\varphi}{+\neg\neg\varphi}\,(dni^+) \qquad \frac{-\varphi}{-\neg\neg\varphi}\,(dni^-)$$

Proof. The derivation of (dni^+) is shown below. The derivation of (dni^-) is similar and has been omitted.

$$\dfrac{\dfrac{+\varphi}{-\neg\varphi}\,(\neg^- I)}{+\neg\neg\varphi}\,(\neg^+ I)$$

□

Establishing b-Connexivity

I now show the b-connexivity of the logic defined by $\mathcal{N}^d$ by showing the derivations of the b-axioms.

Aristotle's b-theses:

$$\dfrac{\dfrac{\dfrac{[+\varphi]_1}{+\varphi}\,(Ass^+)}{-\neg\varphi}\,(\neg^- I)}{-(\varphi \to \neg\varphi)}\,(\to^- I^1_1) \qquad \dfrac{\dfrac{\dfrac{[+\neg\varphi]_1}{+\neg\varphi}\,(Ass^+)}{-\varphi}\,(\neg^+ E)}{-(\neg\varphi \to \varphi)}\,(\to^- I) \tag{12}$$

Boethius' b-theses:

$$\dfrac{\dfrac{\dfrac{+(\varphi \to \psi) \quad \dfrac{[+\varphi]_1}{+\varphi}\,(Ass^+)}{+\psi}\,(\to^+ E)}{-\neg\psi}\,(\neg^- I)}{-(\varphi \to \neg\psi)}\,(\to^- I^1) \qquad \dfrac{\dfrac{\dfrac{+(\varphi \to \neg\psi) \quad \dfrac{[+\varphi]_1}{+\varphi}\,(Ass^+)}{+\neg\psi}\,(\to^+ E)}{-\psi}\,(\neg^+ E)}{-(\varphi \to \psi)}\,(\to^- I_1) \tag{13}$$

Because I do not present here any means for establishing non-derivability, I will leave out the establishment of (3).

2.4. Bilateral (in)coherence

A central feature of bilateral classical logic [8] and bilateral intuitionistic logic [13,14] is the presence of *bilateral structural rules*, coordinating assertion and denial. A central role of such structural rules is the prevention of a simultaneous assertion and the denial of the same formula, i.e., *incoherence*.

Thus, both in bilateral classical logic and in bilateral intuitionistic logic, the following holds:

$$+\varphi, -\varphi \vdash \alpha$$

That is, every formula α, whether an assertion or a denial, is derivable from incoherence. This is the bilateral analog of *explosion* in the unilateral counterpart logics.

On the other hand, many connexive logics are known to be *negation-inconsistent* but not trivial—i.e., explosion fails. By analogy, incoherence is not a problem of the bilateral connexive logic defined by $\mathcal{N}^b$. There are instances of formulas both the assertion of which and the denial of which are derivable, but the logic is still not trivial:[5] not every assertion and every denial are derivable.

Example 1 (incoherence)**.**

$$\vdash_{\mathcal{N}^b} +(((\neg\varphi \to \varphi) \to \neg\varphi) \to \neg((\neg\varphi \to \varphi) \to \neg\varphi))$$

incoherent[6] *together with* A_1^b.

The derivation[7] *is:*

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[+((\neg\varphi \to \varphi) \to \neg\varphi)]_1 \quad [+(\neg\varphi \to \varphi)]_2}{+\neg\varphi}\,(\to^+ E) \quad [+(\neg\varphi \to \varphi)]_2}{+\varphi}\,(\to^+ E)}{+\neg\neg\varphi}\,(dni^+)}{-\neg\varphi}\,(\neg^+ E)}{-((\neg\varphi \to \varphi) \to \neg\varphi))}\,(\to^- I^2)}{+\neg((\neg\varphi \to \varphi) \to \neg\varphi))}\,(\neg^+ I)}{+(((\neg\varphi \to \varphi) \to \neg\varphi) \to \neg((\neg\varphi \to \varphi) \to \neg\varphi))}\,(\to^+ I^1)
$$

3. Conclusions

In this paper, I have proposed a bilateral analysis of connexivity, presenting a bilateral natural deduction system for a weak connexive logic. The proposed logic deviates from other connexive logics and other bilateral logics in the following respects:

- The logic induces a difference in meaning between inner and outer occurrences of negation in the connexive axioms.
- The logic allows incoherence—assertion and denial of the same formula while still being non-trivial.

I leave it for further research to investigate in more detail the bilateral connexive logic presented here only as a proof of concept.

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Notes

[1] While Aristotle's and Boethius axioms originate in antiquity, the negative requirement was added by McCall [3] (p.417).

[2] An exception is Ferguson's paper [6].

[3] This notion of weak connexivity is different from the one, with the same name, found in Pizzi [9], where the inner conditional is taken as material: $(\varphi \to \psi) \supset \neg(\varphi \to \neg\psi)$. It also differs from Kapsner's weak connexivity in [10,11]. It might be related to the weak connexivity in [12], where Boethius' axiom also holds only in rule form.

[4] A *proof* of non-triviality requires a definition of model theory in order to show non-derivability. I do not present here such a model theory, similarly to the general situation in bilateral logics over signed formulas, where model theories are not provided. Therefore, the claim of non-triviality, while seemingly correct, remains unproved.

[5] See Note 4 above.

[6] The same formula was used by McCall [3] to establish negation inconsistency in his connexive logic.

[7] I omit here the explicit use of the (*Ass*)-rules.

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