axioms

Article

A Novel View of Closed Graph Function in Nano Topological Space

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Abstract: The objective of this research is to describe and investigate a novel class of separation axioms and discuss some of their fundamental characteristics using a nano weakly generalized closed set. In nano topological space, $\mathcal{N}wg$-closed graph and strongly $\mathcal{N}wg$-closed graph functions are introduced and explored. We also analyse some of the characterizations of closed graph functions with the separation axioms via a nano weakly generalized closed set.

Keywords: $\mathcal{N}wg$-Harsdorff space; $\mathcal{N}wg$-Urysohn space; $\mathcal{N}wg$-open maps; $\mathcal{N}wg$-irresolute; $\mathcal{N}wg$-closed graphs

MSC: 54B10; 54C10; 54C35; 54D10

1. Introduction

The concept "closed sets" in general topology is important, and many topologists are currently focusing their research in this area. Topologists have developed generalizations for this concept, leading to the discovery of interesting results. One of the most well-known concepts and sources of inspiration is the notion of the g-closed set, which was first proposed by Levine [1]. In 1969, Long [2] discussed properties induced by functions with closed graphs on their domain and range spaces. Subsequently, Long and Herrington [3] defined strongly closed graphs in 1975. Based on this, in 1978, Noiri [4] examined functions with strongly closed graphs. Then, in 2009, Noiri and Popa [5] investigated generalized closed graphs and strongly generalized closed graphs. Lellis Thivagar [6] introduced nano topological space using an equivalence relation on the boundary area of a universal set and its approximations. Weakly generalized closed sets and weakly generalized closed graphs were defined in topological space by Nagaveni et al. [7–9], who also expanded their research into nano topological spaces.

In 2016, Lellis Thivagar et al. [10] defined the concept of nano topological space using general graphs. Arafa et al. (2020) [11] introduced a new method for generating nano topological spaces using the vertices of a graph. A novel approach for creating a nano topological structure using the ideas of the graph's boundary, closure and interior was also introduced. Using nano continuity, Atik et al. [12,13] examined the isomorphisms between simple graphs. Khalifa et al. (2021) [14] introduced a nano topological space based on graph theory that depends on neighbourhood relationships between the vertices within an undirected graph, illustrated with examples. The idea of continuity has been generalized through graph theory to provide additional characterizations and is applicable to simple graphs.

In this article, we define and analyse the characterizations of $\mathcal{N}wg$-closed graph and strongly $\mathcal{N}wg$-closed graph functions using nano weakly generalized closed sets. We also examined the relationships of strongly $\mathcal{N}wg$-closed graphs with $\mathcal{N}wg$-irresolute, nano quasi wg-irresolute, nano θwg-irresolute, etc. We extensively explored this concept and ascertained that those functions with strongly $\mathcal{N}wg$-closed graphs are a stronger

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notion than $\mathcal{N}w\text{-}g$-closed graphs. We investigated some separation properties induced by closed graph functions on their domain, range, or both spaces. We discuss an example of $\mathcal{N}w\text{-}g$-closed graphs using graph theory, which depends on neighbourhood relationships between vertices in a simple graph.

In this article, $(\mathcal{U}, \tau_{\mathcal{R}1}(\mathcal{X}))$ and $(\mathcal{V}, \sigma_{\mathcal{R}2}(\mathcal{Y}))$ represent two nano topological spaces (NTSs) with respect to $\mathcal{X}$ and $\mathcal{Y}$, where $\mathcal{X} \subseteq \mathcal{U}$ and $\mathcal{Y} \subseteq \mathcal{V}$. Additionally, $\mathcal{R}1$ is an equivalence relation defined on the set $\mathcal{U}$, and $\mathcal{U}/\mathcal{R}1$ denotes the collection of equivalence classes of $\mathcal{U}$ by $\mathcal{R}1$. Similarly, $\mathcal{R}2$ is an equivalence relation on $\mathcal{V}$, and $\mathcal{V}/\mathcal{R}2$ denotes the collection of equivalence classes of $\mathcal{V}$ induced by $\mathcal{R}2$.

2. Preliminaries

Descriptions of some of the terminologies used in this sequel are provided in this section.

Definition 1. ([6]). *Let $\mathcal{U}$ be a non-empty finite set of objects called the universe and $\mathcal{R}$ be an equivalence relation on $\mathcal{U}$, referred to as the indiscernibility relation. The pair ($\mathcal{U}$, $\mathcal{R}$) is said to be the approximation space. Let $\mathcal{X} \subseteq \mathcal{U}$. The lower approximation, upper approximation and boundary of the region of $\mathcal{X}$ with respect to $\mathcal{R}$ is defined as $L_{\mathcal{R}}(\mathcal{X}) = \bigcup_{\mathcal{X} \in \mathcal{U}}\{\mathcal{R}(x) : \mathcal{R}(x) \subseteq \mathcal{X}\}$, $\mathcal{U}_{\mathcal{R}}(\mathcal{X}) = \bigcup_{\mathcal{X} \in \mathcal{U}}\{\mathcal{R}(x) : \mathcal{R}(x) \cap \mathcal{X} \neq \varnothing\}$ and $B_{\mathcal{R}}(\mathcal{X}) = \mathcal{U}_{\mathcal{R}}(\mathcal{X}) - L_{\mathcal{R}}(\mathcal{X})$, where $\mathcal{R}$(x) denotes the equivalence class determined by $\mathcal{X} \in \mathcal{U}$. Then, the nano topology (NT) $\tau_{\mathcal{R}}(\mathcal{X}) = \{\mathcal{U}, \varnothing, \mathrm{L}_{\mathcal{R}}(\mathcal{X}), \mathcal{U}_{\mathcal{R}}(\mathcal{X}), \mathrm{B}_{\mathcal{R}}(\mathcal{X})\}$ is defined on $\mathcal{U}$. The $\tau_{\mathcal{R}}(\mathcal{X})$ satisfies the following axioms:*

(i) *$\mathcal{U}$ and $\varnothing \in \tau_{\mathcal{R}}(\mathcal{X})$.*
(ii) *The union of the elements of any subcollection of $\tau_{\mathcal{R}}(\mathcal{X})$ is in $\tau_{\mathcal{R}}(\mathcal{X})$.*
(iii) *The intersection of the elements of any finite subcollection of $\tau_{\mathcal{R}}(\mathcal{X})$ is in $\tau_{\mathcal{R}}(\mathcal{X})$. We call ($\mathcal{U}$, $\tau_{\mathcal{R}}(\mathcal{X})$) a nano topological space (briefly NTS).*

Definition 2. ([7]). *Let $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{X}))$ be a nano topological space. Subset $\mathcal{A}$ of $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{X}))$ is referred to as a nano weakly generalized closed set (briefly $\mathcal{N}w\text{-}g$ CS) if $\mathcal{N}$Cl($\mathcal{N}$Int($\mathcal{A}$)) $\subseteq \mathcal{V}$, where $\mathcal{A} \subseteq \mathcal{V}$ and $\mathcal{V}$ is nano open. The complement of the $\mathcal{N}w\text{-}g$-closed set is an $\mathcal{N}w\text{-}g$-open set (briefly $\mathcal{N}w\text{-}g$ -OS). The family of all nano weakly generalized open sets is denoted by $\mathcal{NWGO}(\mathcal{U})$. We set $\mathcal{NWGO}(\mathcal{U}, x) = \{\mathcal{M} \in \mathcal{NWGO}(\mathcal{U})$ such that $x \in \mathcal{M}\}$. Similarly, the family of all nano weakly generalized closed sets is denoted by $\mathcal{NWGC}(\mathcal{U})$. We set $\mathcal{NWGC}(\mathcal{U}, x) = \{\mathcal{M} \in \mathcal{NWGC}(\mathcal{U})$ such that $x \in \mathcal{M}\}$. The $\mathcal{N}w\text{-}g$ closure of a subset $\mathcal{A}$ of U is denoted by $\mathcal{N}w\text{-}g$-Cl($\mathcal{A}$). Similarly, the $\mathcal{N}w\text{-}g$ interior of subset $\mathcal{A}$ of U is denoted by $\mathcal{N}w\text{-}g$-Int($\mathcal{A}$).*

Definition 3. ([7]). *The function $f : (\mathcal{U}, \tau_{\mathcal{R}1}(\mathcal{X})) \to (\mathcal{V}, \sigma_{\mathcal{R}2}(\mathcal{Y}))$ is termed as follows:*

(i) *$\mathcal{N}w\text{-}g$ is continuous on $\mathcal{U}$ if the inverse image of every nano closed set in $\mathcal{V}$ is nano weakly generalized closed in $\mathcal{U}$.*
(ii) *$\mathcal{N}w\text{-}g$ is irresolute on $\mathcal{U}$ if the inverse image of every $\mathcal{N}w\text{-}g$-closed set in $\mathcal{V}$ is nano weakly generalized closed in $\mathcal{U}$.*
(iii) *$\mathcal{N}w\text{-}g$ is closed (open) on $\mathcal{U}$ if the image of every nano closed (open) set in $\mathcal{U}$ is an Nwg closed (open) set in $\mathcal{V}$.*

Definition 4. ([5]). *Let $f : (X, \tau) \to (Y, \sigma)$ be a function between two topological spaces, (X, τ) and (Y, σ). Then, the subset $\{(x, f(x))/x \in X\}$ of the product space $(X \times Y, \tau \times \sigma)$ is known as the graph of f and is written by $G(f)$.*

Definition 5. ([2]). *A function $f : (X, \tau) \to (Y, \sigma)$ is said to have a closed graph (resp. strongly closed graphs) if for each $(x, y) \in X \times Y - G(f)$ there exist open sets U and V containing x and y, respectively, such that $(U \times V) \cap G(f) = \varnothing$ (resp. $U \times Cl(V)) \cap G(f) = \varnothing$.*

Lemma 1. ([2]). *Let $f : (X, \tau) \to (Y, \sigma)$ be a function, and then the graph $G(f)$ is closed (resp. strongly closed) for graphs in $X \times Y$ if and only if for each $(x, y) \in X \times Y -$*

$G(f)$ there exist open sets U and V containing x and y, respectively, such that $f(U) \cap V = \varnothing$ (resp. $f(U) \cap Cl(V) = \varnothing$).

Definition 6. ([11]). *Let a graph $G = (V, E)$ and $v \in V(G)$; then $N(v) = \{v\} \cup \{u \in V(G) : \overrightarrow{uv} \in E(G)\}$ the neighbourhood of v.*

Definition 7. ([11]). *Let G be a graph with vertices (V, E), S be a subgraph of G and the neighbourhood of v is represented by $N(V(G))$ and $\in V$. Then*

$L_N(V(S)) = \{\{v_i\} \cup \{v_j\} : e_{ij} \in E(S); v_i, v_j \in E(S)\}$,
$H_N(V(S)) = \{\{v_i, v_j\} : e_{ij} \in E(S); v_i, v_j \in E(S)\} \cup \{v_k : v_k \in V(G - S) \text{ and } e_{ik} \in E(G)\}$
$B_N(V(S)) = H_N(V(S)) \backslash L_N(V(S))$

Definition 8. ([11]). *Let a graph $G = (V, E)$ and $v \in V(G)$, $N(v)$ be a neighbourhood of v in V and a subgraph S of G, and then $\tau_N(V(S)) = \{V(G), \varnothing, L_N(V(S)), H_N(V(S)), B_N(V(S))\}$ forms a topology called NTS on $V(G)$ with respect to $V(S)$. We call $\{V(G), \tau_N(V(S))\}$ as the NTS induced by a graph.*

Definition 9. ([15]). *According to graph theory, the graph that has two vertices (u, u') and (v, v') adjacent in $G \square H$ is a graph such that $u = v$ and u' is adjacent to v' in H, or $u' = v'$ and u is adjacent to v in G. Furthermore, the vertex set of $G \square H$ is the Cartesian product $V(G) \times V(H)$.*

3. Separation Axioms via Nano Weakly Generalized Closed Set

In this section, we explore the characterization of separation axioms with the aid of an $\mathcal{N}w g$-open set in an NTS.

Definition 10. *The space$(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{X}))$ is defined as follows:*

1. *$\mathcal{N}wg$-T_0 space ($\mathcal{N}wg$-Kolmogorov space) if for $x, y \epsilon \mathcal{U}$ and $x \neq y$ $\exists$ $\mathcal{N}wg$-OSs M such that $x \in M$ and $y \notin M$.*
2. *$\mathcal{N}wg$-T_1 space ($\mathcal{N}wg$-Fréchet space) if for $x, y \epsilon \mathcal{U}$ and $x \neq y$. $\exists$ $\mathcal{N}wg$-OSs M and N such that $x \in M$ and $y \notin M$ and $y \in N$ and $x \notin N$.*
3. *$\mathcal{N}wg$-T_2 space ($\mathcal{N}wg$-Harsdorf space) if for $x, y \epsilon \mathcal{U}$ and $x \neq y$. $\exists$ disjoint $\mathcal{N}wg$-OSs, M and N such that $x \in M$ and $y \in N$.*
4. *$\mathcal{N}wg$-$T_2\prime$ space ($\mathcal{N}wg$-Urysohn space) if for $x, y \epsilon \mathcal{U}$ and $x \neq y$. $\exists$ disjoint $\mathcal{N}wg$-OSs M and N, $x \in M$ and $y \in N$, such that $\mathcal{N}wg$-$Cl(M) \cap \mathcal{N}wg$- $Cl(N) = \varnothing$.*

Theorem 1. *Every $\mathcal{N}wg$-$T_2\prime$ space is an $\mathcal{N}wg$-T_2 space.*

Proof. Let $x, y \epsilon \mathcal{U}$ and $x \neq y$. Since $\mathcal{U}$ is the $\mathcal{N}wg$-$T_2\prime$ space, there exists disjoint $\mathcal{N}wg$-OSs $M \subset \mathcal{U}$ and $N \subset \mathcal{U}$, $x \in M$ and $y \in N$, such that $\mathcal{N}wg$-$Cl(M) \cap \mathcal{N}wg$-$Cl(N) = \varnothing$. Hence, $M \cap N = \varnothing$. Therefore, $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{X}))$ is an $\mathcal{N}wg$-T_2 space. □

Theorem 2. *Every $\mathcal{N}wg$-$T_2\prime$ space is an $\mathcal{N}wg$-T_1 space.*

Proof. Let $x, y \epsilon \mathcal{U}$ and $x \neq y$. The $\mathcal{U}$ is $\mathcal{N}wg$-$T_2\prime$ space, and there exists disjoint $\mathcal{N}wg$-*OSs* $M \subset \mathcal{U}$ and $N \subset \mathcal{U}$, $x \in M$ and $y \in N$, such that $\mathcal{N}wg$-$Cl(M) \cap \mathcal{N}wg$-$Cl(N) = \varnothing$. This indicates that $x \notin \mathcal{N}wg$-$Cl(N)$ *and* $y \notin \mathcal{N}wg$-$Cl(M)$. Now $\mathcal{N}wg$-$Cl(M)$,$\mathcal{N}wg$-$Cl(N) \in NWGC(M)$. Therefore, $\mathcal{U} - \mathcal{N}wg$-$Cl(M)$ *and* $\mathcal{U} - \mathcal{N}wg$-$Cl(N) \in NWGO(M)$ such that $x \in \mathcal{N}wg$-$Cl(M)$ *and* $y \in \mathcal{N}wg$-$Cl(N)$. Thus, $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{X}))$ is an $\mathcal{N}wg$- T_1 space. □

Example 1. *If $\mathcal{U} = \{\zeta_{a1}, \zeta_{b1}, \zeta_{c1}, \zeta_{d1}, \zeta_{e1}\}$, $\mathcal{X} = \{\zeta_{b1}, \zeta_{c1}\}$, $\mathcal{U}/\mathcal{R} = \{\{\zeta_{a1}, \zeta_{c1}\}, \{\zeta_{b1}\}, \{\zeta_{d1}\}, \{\zeta_{e1}\}\}$, $\tau_{\mathcal{R}}(\mathcal{X}) = \{U, \varphi, \{\zeta_{b1}\}, \{\zeta_{a1}, \zeta_{c1}\}, \{\zeta_{a1}, \zeta_{b1}, \zeta_{c1}\}\}$, then $\mathcal{N}wg$ -OSs are $\{U, \varphi, \{\zeta_{a1}\}, \{\zeta_{b1}\}, \{\zeta_{c1}\}, \{\zeta_{d1}\}, \{\zeta_{e1}\}, \{\zeta_{a1}, \zeta_{b1}\}, \{\zeta_{a1}, \zeta_{c1}\}, \{\zeta_{a1}, \zeta_{d1}\}, \{\zeta_{a1}, \zeta_{e1}\}, \{\zeta_{b1}, \zeta_{c1}\}, \{\zeta_{b1}, \zeta_{d1}\}, \{\zeta_{b1}, \zeta_{e1}\}, \{\zeta_{c1}, \zeta_{d1}\}, \{\zeta_{c1}, \zeta_{e1}\}, \{\zeta_{a1}, \zeta_{b1}, \zeta_{c1}\}, \{\zeta_a, \zeta_b, \zeta_d\}, \{\zeta_a, \zeta_b, \zeta_e\}, \{\zeta_{a1}, \zeta_{c1}, \zeta_{d1}\}$, $\{\zeta_{a1},$*

$\zeta_{c1}, \zeta_{d1}\}, \{\zeta_{a1}, \zeta_{c1}, \zeta_{d1}\}, \{\zeta_{a1}, \zeta_{c1}, \zeta_{d1}\}, \{\zeta_{a1}, \zeta_{c1}, \zeta_{e1}\}, \{\zeta_{b1}, \zeta_{c1}, \zeta_{e1}\}, \{\zeta_{b1}, \zeta_{c1}, \zeta_{e1}\}, \{\zeta_{a1}, \zeta_{b1}, \zeta_{c1}, \zeta_{d1}\}, \{\zeta_{a1}, \zeta_{b1}, \zeta_{c1}, \zeta_{e1}\}, \{\{\zeta_{a1}, \zeta_{b1}, \zeta_{d1}, \zeta_{e1}\}, \}, \{\zeta_{b1}, \zeta_{c1}, \zeta_{d1}, \zeta_{e1}\}\}$. *Thus, x, y in $\mathcal{U}$, $x \neq y$, there exists disjoint $\mathcal{N}wg$-OSs $M \subset \mathcal{U}$, $N \subset \mathcal{U}$ and $x \in M, y \in N$, such that $\mathcal{N}wg$-$Cl(M) \cap \mathcal{N}wg$- $Cl(N) = \varnothing$. Therefore, $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{X}))$ is an $\mathcal{N}wg$-$T_{2'}$ space, and $\mathcal{N}wg$-T_2, $\mathcal{N}wg$-T_1 and $\mathcal{N}wg$-T_0 space.*

Theorem 3. *Every open subspace of the $\mathcal{N}wg$-T_2space is $\mathcal{N}wg$ -T_2space.*

Proof. Let $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{X}))$ be an NTS. Suppose $(S,\ \tau_{\mathcal{R}}(\mathcal{X}))$ is the subspace of $\mathcal{U}$. Let x, y be two distinct points in S. Since x and y are also points of $\mathcal{U}$, which is given to be the $\mathcal{N}wg$-T_2 space, $\exists$ two disjoint $\mathcal{N}wg$-*OSs* G and H, such that G contains x and H contains y. Then, the sets $G\ \ S = S_1$, $H\ \ S = S_2$ are disjoint $\mathcal{N}wg$- *OSs* in S contains x S_1 and y S_2, such that $S_1 \cap S_2 = \varnothing$. Hence, S is the $\mathcal{N}wg$-T_2 space. □

Theorem 4. *If $f : (\mathcal{U}, \tau_{\mathcal{R}1}(\mathcal{X})) \to (\mathcal{V}, \sigma_{\mathcal{R}2}(\mathcal{Y}))$ is the $\mathcal{N}wg$-irresolute mapping and $\mathcal{V}$ is the $\mathcal{N}wg$-T_0 space, then $\mathcal{U}$ is $\mathcal{N}wg$-T_0.*

Proof. Let $x, y \in \mathcal{U}$ with $x \neq y$ and $\mathcal{V}$ be an $\mathcal{N}wg$-T_0 space. Then, $\exists \mathcal{N}wg$-*OS* P of $\mathcal{Y}$, such that either $f(x) \in P$ and $f(y) \in P$ with $f(x) \neq f(y)$. By using the injective $\mathcal{N}wg$-irresoluteness of f, $f^{-1}(P)$ is an $\mathcal{N}wg$-*OS* of $\mathcal{U}$ such that either $x \in f^{-1}(P)$ or $y \in f^{-1}(P)$. Therefore, $\mathcal{U}$ is an $\mathcal{N}wg$-T_0 Space. □

Lemma 2. *If the bijection function $f : (\mathcal{U}, \tau_{\mathcal{R}1}(\mathcal{X})\ ) \to (\mathcal{V}, \sigma_{\mathcal{R}2}(\mathcal{Y}))$ is the $\mathcal{N}wg$-open, then for any $M \in \mathcal{NWGC}(\mathcal{U})$, $f(M) \in \mathcal{NWGC}(\mathcal{V})$.*

Proof. The proof is obvious. □

Lemma 3. *Let $x \in \mathcal{U}$ and $A \subset \mathcal{U}$. The point $x \in \mathcal{N}wg$-Cl(A) if and only if $A \cap S \neq \varnothing$, for all S $\in \mathcal{NWGO}(\mathcal{U}, x)$.*

Proof. The proof is obvious. □

Theorem 5. *If $f : (\mathcal{U}, \tau_{\mathcal{R}1}(\mathcal{X})\ ) \to (\mathcal{V}, \sigma_{\mathcal{R}2}(\mathcal{Y}))$ is the bijective $\mathcal{N}wg$-open mapping and $(\mathcal{U}, \tau_{\mathcal{R}1}(\mathcal{X})\ )$ is the $\mathcal{N}wg$-$T_{2'}$ space, then $(\mathcal{V}, \sigma_{\mathcal{R}2}(\mathcal{Y}))$ is $\mathcal{N}wg$ -$T_{2'}$ space.*

Proof. Let $y_1, y_2 \in \mathcal{V}$ and $y_1 \neq y_2$. Since f is the bijection, $f^{-1}(y_1), f^{-1}(y_2) \in \mathcal{U}$ and $f^{-1}(y_1) \neq f^{-1}(y_2)$. The $\mathcal{N}wg - T_2'$ space property of $\mathcal{U}$ provides the existence of sets $M \in \mathcal{NWGO}(\mathcal{U}, f^{-1}(y_1))$, $N \in \mathcal{NWGO}(\mathcal{U}, f^{-1}(y_2))$ with the fact that $\mathcal{N}wg$-$Cl(M) \cap \mathcal{N}wg$- $Cl(N) = \varnothing$. By the Lemma 3, $\mathcal{N}wg$-$Cl(M)$ is the $\mathcal{N}wg$-*CS* in $\mathcal{U}$. By the Lemma 2, bijectivity and $\mathcal{N}wg$-openness reveals that $f(\mathcal{N}wg\text{-}Cl_{\mathcal{U}}(M)) \in \mathcal{NWGC}(\mathcal{V})$. Again, from $M \subset \mathcal{N}wg\text{-}Cl_{\mathcal{U}}(M)$, it follows that $f(M) \subset f(\mathcal{N}wg\text{-}Cl_{\mathcal{U}}(M))$. Since $\mathcal{N}wg$-clousure respects inclusion, $\mathcal{N}wg\text{-}Cl_{\mathcal{V}}(f(M)) \subset \mathcal{N}wg\text{-}Cl_{\mathcal{V}}(f(\mathcal{N}wg\text{-}Cl_{\mathcal{V}}(M)) = f(\mathcal{N}wg\text{-}Cl_{\mathcal{V}}(M))$. In like manner, $\mathcal{N}wg\text{-}Cl_{\mathcal{V}}(f(N)) \subset f(\mathcal{N}wg\text{-}Cl_{\mathcal{U}}(N))$. Therefore, by the injectivity of f, $\mathcal{N}wg\text{-}Cl_{\mathcal{V}}(f(M)) \bigcap \mathcal{N}wg\text{-}Cl_{\mathcal{V}}(f(N)) \subset f(\mathcal{N}wg\text{-}Cl_{\mathcal{U}}(M)) \bigcap f(\mathcal{N}wg\text{-}Cl_{\mathcal{U}}(N)) = f[(\mathcal{N}wg\text{-}Cl_{\mathcal{U}}(M)) \bigcap (\mathcal{N}wg\text{-}Cl_{\mathcal{U}}(N))] = f(\varnothing) = \varnothing$. Thus, the $\mathcal{N}wg$-openness of f gives the existence of two sets, $f(M) \in \mathcal{NWGO}(\mathcal{V}, y_1)$, $f(N) \in \mathcal{NWGO}(\mathcal{V}, y_2)$ with $\mathcal{N}wg\text{-}Cl_{\mathcal{V}}\ (f(M)) \bigcap \mathcal{N}wg\text{-}Cl_{\mathcal{V}}(f(N)) = \varnothing$. Hence, $\mathcal{V}$ is $\mathcal{N}wg$-$T_{2'}$ space. □

4. Discussion on Nano Weakly Generalized Closed Graphs

In this section, we introduce a weaker form of the closed graph, such as $\mathcal{N}wg$-closed graphs, with the aid of $\mathcal{N}wg$-*OS* in an NTS and investigate the functions and characterization of separation axioms along with $\mathcal{N}wg$-closed graphs. The example of the $\mathcal{N}wg$-closed graph via a simple graph with vertices is discussed.

Definition 11. *The function $f : (\mathcal{U}, \tau_{\mathcal{R}1}(\mathcal{X})) \to (\mathcal{V}, \sigma_{\mathcal{R}2}(\mathcal{Y}))$ is told to have a nano weakly generalized closed graph (briefly $\mathcal{N}wg$-CG) if for each $(x, y) \in \mathcal{U} \times \mathcal{V} - G(f)$, $\exists \mathcal{N}wg$-open sets M and N, $x \in M$ and $y \in N$, $(M \times N) \cap G(f) = \varnothing$.*

Lemma 4. *If $f : (\mathcal{U}, \tau_{\mathcal{R}1}(\mathcal{X})) \to (\mathcal{V}, \sigma_{\mathcal{R}2}(\mathcal{Y}))$ is the function, then the graph $G(f)$ is $\mathcal{N}wg$-CG in $\mathcal{U} \times \mathcal{V}$ if and only if for each $(x, y) \in \mathcal{U} \times \mathcal{V} - G(f)$, $\exists$ a $\mathcal{N}wg$-open set M and N, $x \in M$ and $y \in N$, such that $f(M) \cap N = \varnothing$.*

Proof. Necessity: Since f has a nano weakly generalized closed graph, for each $x \in \mathcal{U}$ and $y \in \mathcal{V}$ such that $y \neq f(x)$ $\exists$ a $\mathcal{N}wg$-OSs M and N, $x \in M$ and $y \in N$, in such a way that $(M \times N) \cap G(f) = \varnothing$. This implies that $f(M) \cap N = \varnothing$.

Sufficiency: Consider $(x, y) \notin G(f)$, and then there are two $\mathcal{N}wg$-OSs M and N, $x \in M$ and $y \in N$, such that $f(M) \cap N = \varnothing$. This indicates that $(M \times N) \cap G(f) = \varnothing$. As a result, f has an $\mathcal{N}wg$-CG. □

Example 2. *If $\mathcal{U} = \{\zeta_{a1}, \zeta_{b1}, \zeta_{c1}, \zeta_{d1}\}$, $\mathcal{X} = \{\zeta_{a1}\}$, $\mathcal{U}/\mathcal{R}1 = \{\{\zeta_{a1}\}, \{\zeta_{b1}, \zeta_{c1}\}, \{\zeta_{d1}\}\}$, $(\mathcal{U}, \tau_{\mathcal{R}1}(\mathcal{X})) = \{U, \emptyset, \{\zeta_{a1}\}\}$, then the $\mathcal{N}wg$-open sets are $\{\mathcal{U}, \varnothing, \{\zeta_{a1}\}, \{\zeta_{b1}\}, \{\zeta_{c1}\}, \{\zeta_{d1}\}$, $\{\zeta_{a1}, \zeta_{b1}\}, \{\zeta_{b1}, \zeta_{c1}\}, \{\zeta_{a1}, \zeta_{c1}\}, \{\zeta_{a1}, \zeta_{d1}\}$ $\{\zeta_{b1}, \zeta_{d1}\}\}, \{\zeta_{c1}, \zeta_{d1}\}, \{\zeta_{a1}, \zeta_{b1}, \zeta_{c1}\}$, $\{\zeta_{a1}, \zeta_{b1}, \zeta_{d1}\}$, $\{\zeta_{a1}, \zeta_{c1}, \zeta_{d1}\}\}$. If $\mathcal{V} = \{\eta_{a2}, \eta_{b2}, \eta_{c2}, \eta_{d2}\}$, $\mathcal{Y} = \{\eta_{a2}, \eta_{a2}\}$, $\mathcal{V}/\mathcal{R}2 = \{\{\eta_{a2}, \eta_{b2}\}, \{\eta_{c2}\}, \{\eta_{d2}\}\}$, $(\mathcal{V}, \sigma_{\mathcal{R}2}(\mathcal{Y})) = \{V, \emptyset, \{\eta_{d2}\}, \{\eta_{a2}, \eta_{b2}\}, \{\eta_{a2}, \eta_{b2}, \eta_{d2}\}\}$, then the $\mathcal{N}wg$-open sets are $\{\mathcal{V}, \varnothing$, $\{\eta_{a2}\}$, $\{\eta_{b2}\}$, η_{d2}, $\{\eta_{a2}, \eta_{b2}\}$, $\{\eta_{b2}, \eta_{d2}\}$, $\{\eta_{a2}, \eta_{d2}\}$, $\{\eta_{a2}, \eta_{b2}, \eta_{d2}\}$, $\{\eta_a, \eta_c, \eta_d\}$, $\{\eta_a, \eta_c, \eta_d\}\}$. Let $f : (\mathcal{U}, \tau_{\mathcal{R}1}(\mathcal{X})) \to (\mathcal{V}, \sigma_{\mathcal{R}2}(\mathcal{Y}))$ be a mapping defined by $f(\zeta_{a1}) = \eta_{a2}$, $f(\zeta_{b1}) = \eta_{b2}$, $f(\zeta_{c1}) = \eta_{c2}$ and $f(\zeta_{d1}) = \eta_{d2}$. Therefore, f has $\mathcal{N}wg$-CG.* □

Example 3. *In this example, we observed that an$\mathcal{N}wg$-closed graph will be induced by a general graph with vertices. The two distinct graphs are G and H, their vertices of G and H are $V(G) = \{\zeta_{a1}, \zeta_{a2}, \zeta_{a3}, \zeta_{a4}\}$and $V(H) = \{\eta_{b1}, \eta_{b2}, \eta b_3\}$ and the vertices of the Cartesian product of two graphs are $V(G \times H) = \{(\zeta_{a1}, \eta_{b1}), (\zeta_{a1}, \eta_{b2}), (\zeta_{a1}, \eta_{b3}), (\zeta_{a2}, \eta_{b1}), (\zeta_{a2}, \eta_{b2}), (\zeta_{a2}, \eta_{b3}), (\zeta_{a3}, \eta_{b1}), (\zeta_{a3}, \eta_{b2}), (\zeta_{a3}, \eta_{b3}), (\zeta_{a4}, \eta_{b1}), (\zeta_{a4}, \eta_{b2}), (\zeta_{a4}, \eta_{b3})\}$; this is shown in Figure 1.*

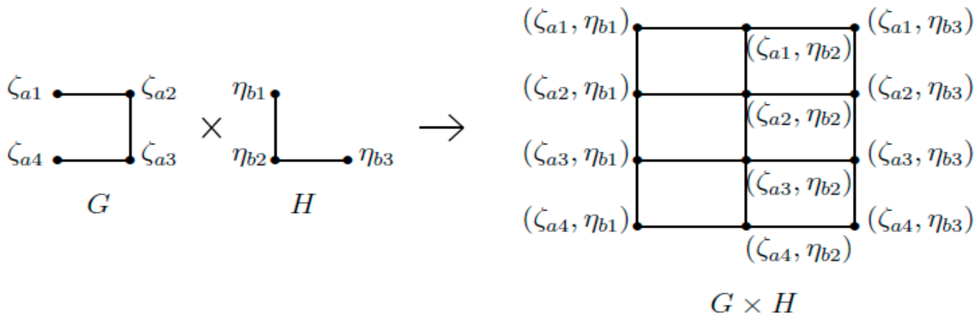

Figure 1. Graph representation of Cartesian product of two graphs forms a topology.

The neighbourhoods of the vertices of G are $N(\zeta_{a1}) = \{\zeta_{a1}, \zeta_{a2}\}$, $N(\zeta_{a2}) = \{\zeta_{a1}, \zeta_{a2}, \zeta_{a3}\}$, $N(\zeta_{a3}) = \{\zeta_{a2}, \zeta_{a3}, \zeta_{a4}\}$, $N(\zeta_{a4}) = \{\zeta_{a3}, \zeta_{a4}\}$. If the subgraph $\mathcal{X}$ of G such that $V(\mathcal{X}) = \{\zeta_{a1}, \zeta_{a2}, \zeta_{a3}\}$, then $L_N(V(\mathcal{X})) = \{\zeta_{a1}, \zeta_{a2}\}$, $U_N(V(\mathcal{X})) = \{\zeta_{a1}, \zeta_{a2}, \zeta_{a3}, \zeta_{a4}\}$, $B_N(V(\mathcal{X})) = \{\zeta_{a3}, \zeta_{a4}\}$ and an NTS $(G, \tau_N(\mathcal{X})) = \{G, \emptyset, \{\zeta_{a1}, \zeta_{a2}\}, \{\zeta_{a3}, \zeta_{a4}\}\}$.$\mathcal{N}wg$-open sets are $G, \emptyset$, $\{\zeta_{a1}\}, \{\zeta_{a2}\}, \{\zeta_{a3}\}, \{\zeta_{a4}\}, \{\zeta_{a1}, \zeta_{a2}\}$, $\{\zeta_{a2}, \zeta_{a3}\}, \{\zeta_{a1}, \zeta_{a3}\}, \{\zeta_{a1}, \zeta_{a4}\}, \{\zeta_{a2}, \zeta_{a4}\}\}, \{\zeta_{a3}, \zeta_{a4}\}$ $\{\zeta_{a1}, \zeta_{a2}, \zeta_{a3}\}, \{\zeta_{a1}, \zeta_{a2}, \zeta_{a4}\}, \{\zeta_{a1}, \zeta_{a3}, \zeta_{a4}\}, \{\zeta a_2, \zeta_{a3}, \zeta_{a4}\}$.

Similarly, the neighbourhoods of the vertices of H are $N(\eta_{b1}) = \{\eta_{b1}, \eta_{b2}\}$, $N(\eta_{b2}) = \{\eta_{b1}, \eta_{b2}, \eta_{b3}\}$, $N(\eta_{b3}) = \{\eta_{b2}, \eta_{b3}\}$. Assume that $\mathcal{Y}$ is a subgraph of H such that $V(\mathcal{Y}) = \{\eta_{b1}, \eta_{b2}\}$. So, $L_N(V(\mathcal{Y})) = \{\eta_1\}$, $U_N(V(\mathcal{Y})) = \{\eta_{b1}, \eta_{b2}, \eta_{b3}\}$, $B_N(V(\mathcal{Y})) = \{\eta_{b2}, \eta_{b3}\}$ and the NTS $(H, \sigma_N(\mathcal{Y})) = \{H, \emptyset, \{\eta_{b1}\}, \{\eta_{b2}, \eta_{b3}\}\}$. The $\mathcal{N}wg$-open sets are $\{H, \emptyset, \{\eta_{b1}\}, \{\eta_{b2}\}$, $\{\eta_{b3}\}$, $\{\eta b_1, \eta_{b2}\}, \{\eta_{b1}, \eta_{b3}\}, \{\eta_{b2}, \eta_{b3}\}\}$. The neighbourhoods of the vertices of $G \times H$ are

$N((\zeta_{a1}, \eta_{b1})) = \{(\zeta_{a1}, \eta_{b1}), (\zeta_{a1}, \eta_{b2}), (\zeta_{a2}, \eta_{b1})\}$, $N((\zeta_{a1}, \eta b_2)) = \{(\zeta_{a1}, \eta_{b1}), (\zeta_{a1}, \eta_{b2}), (\zeta_{a1}, \eta b_3), (\zeta_{a2}, \eta_{b2})\}$, $N((\zeta_{a1}, \eta_{b3})) = \{(\zeta_{a1}, \eta_{b2}), (\zeta_{a1}, \eta_{b3}), (\zeta_{a2}, \eta_{b3})\}$, $N((\zeta_{a2}, \eta_{b1})) = \{(\zeta_{a1}, \eta_{b1}), (\zeta_{a2}, \eta_{b1}), (\zeta_{a2}, \eta_{b2}), (\zeta_3, \eta_1)\}$, $N((\zeta_{a2}, \eta_{b2})) = \{(\zeta_{a1}, \eta_{b2}), (\zeta_{a2}, \eta_{b1}), (\zeta_{a2}, \eta_{b2}), (\zeta_{a2}, \eta_{b3}), (\zeta_{a3}, \eta_{b2})\}$, $N((\zeta_{a2}, \eta_{b3})) = \{(\zeta_{a1}, \eta_{b3}), (\zeta_{a2}, \eta_{b2}), (\zeta_{a2}, \eta_{b3}), (\zeta_{a3}, \eta_{b3})\}$, $N((\zeta_{a3}, \eta_{b1})) = \{(\zeta_{a2}, \eta_{b1}), (\zeta_{a3}, \eta_{b1}), (\zeta_3, \eta_{b2}), (\zeta_{a4}, \eta_{b1})\}$, $N((\zeta_{a3}, \eta_{b2})) = \{(\zeta_2, \eta_2), (\zeta_3, \eta_1), (\zeta_3, \eta_2), (\zeta_3, \eta_3), (\zeta_{a4}, \eta_{b2})\}$, $N((\zeta_{a3}, \eta_{b3})) = \{(\zeta_{aa2}, \eta_{b3}), (\zeta_{a3}, \eta_{b2}), (\zeta_{a3}, \eta_{b3}), (\zeta_{a4}, \eta_{b3})\}$, $N((\zeta_{a4}, \eta_{b1})) = \{(\zeta_{a3}, \eta_{b1}), (\zeta_{a4}, \eta_{b1}), (\zeta_{a4}, \eta_{b2})\}$, $N((\zeta_{a4}, \eta_{b2})) = \{(\zeta_{a3}, \eta_{b2}), (\zeta_{a4}, \eta_{b1}), (\zeta_{a4}, \eta_{b2}), (\zeta_{a4}, \eta_{b3})\}$, $N((\zeta_{a4}, \eta_{b3})) = \{(\zeta_{a3}, \eta_{b3}), (\zeta_{a4}, \eta_{b2}), (\zeta_{a4}, \eta_{b3})\}$.

Consider the subgraph $\mathcal{X} \times \mathcal{Y}$ of $G \times H$ such that $V(\mathcal{X} \times \mathcal{Y}) = \{(\zeta_{a1}, \eta_{b1}), (\zeta_{a2}, \eta_{b1}), (\zeta_{a3}, \eta_{b1}), (\zeta_{a1}, \eta_{b2})(\zeta_{a2}, \eta_{b2}), (\zeta_{a3}, \eta_{b2})\}$, $L_N(V(\mathcal{X} \times \mathcal{Y})) = \{(\zeta_{a1}, \eta_{b1}), (\zeta_{a2}, \eta_{b1})\}$, $U_N(V(\mathcal{X} \times \mathcal{Y})) = \{(\zeta_{a1}, \eta_{b1}), (\zeta_{a1}, \eta_{b2}), (\zeta_{a1}, \eta_{b3})(\zeta a_2, \eta_{b1}), (\zeta_{a2}, \eta_{b2}), (\zeta_{a2}, \eta_{a3}), (\zeta_{a3}, \eta_{b1}), (\zeta_{a3}, \eta_{b2}), (\zeta_{a3}, \eta_{b3}), (\zeta_{a4}, \eta_{b1}), (\zeta a_4, \eta_2)\}$ and $B_N(V(\mathcal{X} \times \mathcal{Y})) = \{(\zeta_{a1}, \eta_2), (\zeta a_1, \eta_3), (\zeta_{a2}, \eta_2), (\zeta_{a2}, \eta_3), (\zeta_{a3}, \eta_1), (\zeta_{a3}, \eta_{a2}), (\zeta_{a3}, \eta_{a3}), (\zeta_{a4}, \eta_{b1}), (\zeta_4, \eta_{b2})\}$ and a NTS $(G \times H, \delta_N(\mathcal{X} \times \mathcal{Y})) = \{G \times H, \varnothing, \{(\zeta_{a1}, \eta_{b1}), (\zeta_{a2}, \eta_{b1})\}, \{(\zeta_{a1}, \eta_{b1}), (\zeta_{a1}, \eta_{b2}), (\zeta_{a1}, \eta_{b3})(\zeta_{a2}, \eta_{b1}), (\zeta_{a2}, \eta_{b2}), (\zeta_{a2}, \eta_{b3}), (\zeta_{a3}, \eta_1), (\zeta_{a3}, \eta_{b2}), (\zeta_{a3}, \eta_{b3}), (\zeta_{a4}, \eta_{b1}), (\zeta a_4, \eta_{b2})\}, \{(\zeta_{a1}, \eta_{b2}), (\zeta_{a1}, \eta_{b3}), (\zeta_{a2}, \eta_{b2}), (\zeta_{a2}, \eta_{b3}), (\zeta_{a3}, \eta_{b1}), (\zeta_{a3}, \eta_{b2}), (\zeta_{a3}, \eta_{b3}), (\zeta_4, \eta_{b1}), (\zeta_{a4}, \eta_{b2})\}\}$.

If the function $f : (G, \tau_N(\mathcal{X})) \to (H, \sigma_N(\mathcal{Y}))$ is defined by $f(\zeta_{a1}) = \eta_{b1}, f(\zeta_{a2}) = \eta_{b1}$, $f(\zeta_{a3}) = \eta_{b1}$ and $f(\zeta_{a4}) = \eta_{b1}$, then f has $\mathcal{N}wg$-CG. □

Theorem 6. *If the function* $f : (\mathcal{U}, \tau_{\mathcal{R}1}(\mathcal{X})) \to (\mathcal{V}, \sigma_{\mathcal{R}2}(\mathcal{Y}))$ *is an injective with the* $\mathcal{N}wg$*-CG,* $G(f)$*, then* $\mathcal{U}$ *is* $\mathcal{N}wg$*-*T_1 *space.*

Proof. Let $x, y \epsilon\ \mathcal{U}$ and $x \neq y$. Since f is an injection, $f(x) \neq f(y)$ *in* $\mathcal{V}.(x, f(y)) \in (\mathcal{U} \times \mathcal{V}) - G(f)$. But $G(f)$ is the $\mathcal{N}wg$*-CG*, so that using the Lemma 4, $\exists$ *a* $\mathcal{N}wg$*-OSs* M and N, $x \in M$ and $f(y) \in N$, such that $f(M) \cap N = \varnothing$. Thus, $y \notin M$. Likewise, $\exists \mathcal{N}wg$-*OSs* P and Q containing y and $f(x)$, in such a way that $f(P) \cap Q = \varnothing$. As a result, $x \notin P$. $\mathcal{U}$ is $\mathcal{N}wg$-T_1 space. □

Theorem 7. *If the function* $f : (\mathcal{U}, \tau_{\mathcal{R}1}(\mathcal{X})) \to (\mathcal{V}, \sigma_{\mathcal{R}2}(\mathcal{Y}))$ *is surjective with respect to the* $\mathcal{N}wg$*-CG* $G(f)$*, then* $\mathcal{V}$ *is* $\mathcal{N}wg$*-*T_1 *space.*

Proof. Consider $y, z \epsilon\ \mathcal{V}$ and $y \neq z$. Given the function f is onto, $\exists$ a point x in $\mathcal{U}$ such that $f(x) = z$. Hence, $(x, y) \notin G(f)$, using the Lemma 4, there is $\mathcal{N}wg$ *-OSs* M and N, $x \in M$ and $y \in N$, because of which $f(M) \cap N = \varnothing$. Further, it implies $z \notin N$. Likewise, there exist $w \in \mathcal{U}$ such that $f(w) = y$. Thus, $(w, z) \notin G(f)$. Similarly, there exist $\mathcal{N}wg$-*OSs* P & Q, $w \in P$ and $z \in Q$, such that $f(P) \cap Q = \varnothing$. Hence, $y \notin Q$, and thus the space $\mathcal{V}$ is $\mathcal{N}wg$-T_1 space. □

Corollary 1. *If the bijective function* $f : (\mathcal{U}, \tau_{\mathcal{R}1}(\mathcal{X})) \to (\mathcal{V}, \sigma_{\mathcal{R}2}(\mathcal{Y}))$ *has the* $\mathcal{N}wg$*-CG* $G(f)$*, then both* $\mathcal{U}$*and* $\mathcal{V}$ *are* $\mathcal{N}wg$ *-*T_1 *space.*

Proof. That is obvious from Theorems 6 and 7. □

Theorem 8. *If* $f : (\mathcal{U}, \tau_{\mathcal{R}1}(\mathcal{X})) \to (\mathcal{V}, \sigma_{\mathcal{R}2}(\mathcal{Y}))$ *is the* $\mathcal{N}wg$*-open and onto function with the* $\mathcal{N}wg$*-CG* $G(f)$*, then* $\mathcal{V}$ *is* $\mathcal{N}wg$*-*T_2 *space.*

Proof. Let $y, w \epsilon\ \mathcal{V}$ and $y \neq w$. The points $x, z \epsilon\ \mathcal{U}$, $x \neq z$ and $f(x) = y, f(z) = w$. Since $(x, w) \notin G(f)$ and $G(f)$ is $\mathcal{N}wg$-CG, there exist $\mathcal{N}wg$-OSs M & N, $x \in M$ and $w \in N$, as a result of which $f(M) \cap N = \varnothing$. However, $f(M)$ is $\mathcal{N}wg$-open and contains y. Therefore, $\mathcal{V}$ is $\mathcal{N}wg$-T_2 space. □

Theorem 9. *If* $f : (\mathcal{U}, \tau_{\mathcal{R}1}(\mathcal{X})) \to (\mathcal{V}, \sigma_{\mathcal{R}2}(\mathcal{Y}))$*is injective and* $\mathcal{N}wg$*-continuous with* $\mathcal{N}wg$*-CG* $G(f)$*, and also* $\mathcal{V}$ *is* $\mathcal{N}wg$*-*T_2 *space, then* $\mathcal{U}$ *is* $\mathcal{N}wg$*-*T_2 *space.*

Proof. Let $x, y \epsilon \mathcal{U}$ be any two of points, Then, $\exists\, M$, $N \subseteq \mathcal{V} and\ M \neq N$ such that $f(x) \in M, f(y) \in N$. Since the function f is $\mathcal{N}wg$-continuous, f^{-1}(M), f^{-1}(N) are $\mathcal{N}wg$-open in $\mathcal{U}$, $x \in f^{-1}(M), y \in f^{-1}(N)$. By the $\mathcal{N}wg$-T_2 space, we obtain $f^{-1}(M) \cap f^{-1}(N) = \emptyset$. Thus, $\mathcal{U}$ is $\mathcal{N}wg$-T_2 space. □

Theorem 10. *If the function* $f : (\mathcal{U}, \tau_{\mathcal{R}1}(\mathcal{X})\ ) \to (\mathcal{V}, \sigma_{\mathcal{R}2}(\mathcal{Y}))$ *is the* $\mathcal{N}wg$*-homeomorphism with* $\mathcal{N}wg$*-CG G(*f*), then* $\mathcal{U}$ *and* $\mathcal{V}$ *are* $\mathcal{N}wg$*-*T_2 *space.*

Proof. It is implied by Theorems 8 and 9. □

5. Stronger Form of Nano Weakly Generalized Closed Graphs

We present a stronger form of the closed graph, such as strongly $\mathcal{N}wg$-closed graphs with the aid of $\mathcal{N}wg$-closed sets in an NTS examined with strongly $\mathcal{N}wg$-closed graphs with $\mathcal{N}wg$-irresolute, nano quasi wg-irresolute, nano θ-wg -irresolute, etc.

Definition 12. *The function* $f : (\mathcal{U}, \tau_{\mathcal{R}1}(\mathcal{X})\ ) \to (\mathcal{V}, \sigma_{\mathcal{R}2}(\mathcal{Y}))$ *is called strongly* $\mathcal{N}wg$*-CG if for each* $(x, y) \in \mathcal{U} \times \mathcal{V} - G(f)$, $\exists$ *an* $\mathcal{N}wg$*-OSs M and N,* $x \in M$ *and* $y \in N$, $(M \times \mathcal{N}wg\text{-}Cl(N))\ \cap G(f) = \emptyset$.

Lemma 5. *Let* $f : (\mathcal{U}, \tau_{\mathcal{R}1}(\mathcal{X})\ ) \to (\mathcal{V}, \sigma_{\mathcal{R}2}(\mathcal{Y}))$ *be the function. The graph* $G(f)$ *is strongly* $\mathcal{N}wg$*-closed in*$\mathcal{U} \times \mathcal{V} iff\ for\ each$ $(x, y) \in \mathcal{U} \times \mathcal{V} - G(f)$, $\exists$ *a* $\mathcal{N}wg$*-OSs M and N,* $x \in M$ *and* $y \in N$ *such that* $f(M) \cap \mathcal{N}wg\text{-}Cl(N) = \emptyset$.

Proof. The proof is evident from Definition 12. □

Example 4. *Let* $\mathcal{U} = \{\zeta_{\mathrm{a1}}, \zeta_{\mathrm{a2}}, \zeta_{\mathrm{a3}}\}, \mathcal{X} = \{\zeta_{\mathrm{a1}}, \zeta_{\mathrm{a2}}\}, \mathcal{U}/\mathcal{R}1 = \{\{\zeta_{\mathrm{a1}}\}, \{\zeta_{\mathrm{a2}}, \zeta_{\mathrm{a3}}\}\}, (\mathcal{U}, \tau_{\mathcal{R}1}(\mathcal{X})) = \{U, \emptyset, \{\zeta_{\mathrm{a1}}\}, \{\zeta_{\mathrm{a2}}, \zeta_{\mathrm{a3}}\}\}$. $\mathcal{N}wg$*-OSs are* $\{\mathcal{U}, \emptyset, \{\zeta_{\mathrm{a1}}\}, \{\zeta_{\mathrm{a2}}\}, \{\zeta_{\mathrm{a3}}\}, \{\zeta_{\mathrm{a1}}, \zeta_{\mathrm{a2}}\}, \{\zeta a_2, \zeta_{\mathrm{a3}}\}, \{\zeta_{\mathrm{a1}}, \zeta_{\mathrm{a3}}\}\}$. *Let* $\mathcal{V} = \{\eta_{\mathrm{b1}}, \eta_{\mathrm{b2}}, \eta_{\mathrm{b3}}\}, \mathcal{Y} = \{\eta_{b3}\}, \mathcal{V}/\mathcal{R}2 = \{\{\zeta_{b1}, \zeta_{b2}\}, \{\eta_{b3}\}\}, (\mathcal{V}, \sigma_{\mathcal{R}2}(\mathcal{Y})) = \{V, \emptyset, \{\eta_{b3}\}\}$ *and* $\mathcal{N}wg$*-OSs* be $\{\mathcal{V},\ \emptyset, \{\eta_{b1}\}, \{\eta_{b2}\}, \{\eta_{b3}\}, \{\eta_{b2}, \eta_{b3}\}, \{\eta_{b1}, \eta_{b3}\}\}$. *Let* $f : (\mathcal{U}, \tau_{\mathcal{R}1}(\mathcal{X})) \to (\mathcal{V}, \sigma_{\mathcal{R}2}(\mathcal{Y}))$ *be a mapping defined by* $f(\zeta_{a1}) = \eta_{b1}$, $f(\zeta_{a2}) = \eta_{b2}$ *and* $f(\zeta_3) = \eta_{b3}$. *Then,* f *has strongly* $\mathcal{N}wg$*-closed graph.* □

Remark 1. *The previous example is* $\mathcal{N}wg$*-CG. But Example 2 is not strongly* $\mathcal{N}wg$*-CG. Thus, strongly* $\mathcal{N}wg$*-CG is* $\mathcal{N}wg$*-CG. The converse, however, is not necessarily true.* □

Theorem 11. *If* $f : (\mathcal{U}, \tau_{\mathcal{R}1}(\mathcal{X})\ ) \to (\mathcal{V}, \sigma_{\mathcal{R}2}(\mathcal{Y}))$ *is* $\mathcal{N}wg$*-irresolute and* $\mathcal{V}$ *is* $\mathcal{N}wg$*-*T_2*, then G(*f*) is strongly* $\mathcal{N}wg$*-CG.*

Proof. Let $(x, y) \in \mathcal{U} \times \mathcal{V} - \mathrm{G}(f)$. Since $\mathcal{V}$ is $\mathcal{N}wg$-T_2 space, there exist $N \in \mathcal{NWGO}(\mathcal{V}$, $y)$ such that $f(x) \notin \mathcal{N}wg$-$Cl(N)$. Then, $\mathcal{V} - \mathcal{N}wg$-$Cl(N) \in \mathcal{NWGO}(\mathcal{V}, f(x))$. Since f is $\mathcal{N}wg$-irresolute, there exist M $\in \mathcal{NWGO}(\mathcal{U}, x)$ such that $f(M) \subseteq \mathcal{V} - \mathcal{N}wg$-$Cl$(N). Then, $f(M) \cap \mathcal{N}wg$-$Cl(N) = \emptyset$. Hence, G(f) is strongly $\mathcal{N}wg$-CG. □

Theorem 12. *If* $f : (\mathcal{U}, \tau_{\mathcal{R}1}(\mathcal{X})\ ) \to (\mathcal{V}, \sigma_{\mathcal{R}2}(\mathcal{Y}))$ *is one to one and G(*f*) is strongly* $\mathcal{N}wg$*-CG, then* $\mathcal{U}$ *is* $\mathcal{N}wg$*-*T_1 *space.*

Proof. Given that f is one to one, $x, y \in \mathcal{U}$ and $x \neq y$, $f\ (x) \neq f\ (y)$. Since G(f) is strongly $\mathcal{N}wg$-closed, as per Definition 12, $(x, f(y)) \in \mathcal{U} \times \mathcal{V} - \mathrm{G}(f)$, $\exists$ a $\mathcal{N}wg$ -*OSs* M and N, $x \in M$ and $y \in N$, and thus $f(M) \cap \mathcal{N}wg$-$Cl(N) = \emptyset$. Therefore, $y \notin M$. Consequently, $\exists$ a $\mathcal{N}wg$ -*OS* W contains $f\ (y)$, $x \notin$ W. As a result, $\mathcal{U}$ is $\mathcal{N}wg$-T_1. □

Theorem 13. *If* $f : (\mathcal{U}, \tau_{\mathcal{R}1}(\mathcal{X})\ ) \to (\mathcal{V}, \sigma_{\mathcal{R}2}(\mathcal{Y}))$ *is surjection along with strongly* $\mathcal{N}wg$*-CG, then* $\mathcal{V}$ *is both* $\mathcal{N}wg$*-*T_2 *and* $\mathcal{N}wg$*-*T_1 *space.*

Proof. Let $\mathcal{y}_1,\ \mathcal{y}_2\ \in \mathcal{V}$. Since f is surjective, there exist $\mathcal{x}_1 \in \mathcal{U}$ such that $f(\mathcal{x}_1) = \mathcal{y}_1$. Since G($f$) is strongly $\mathcal{N}w\text{-}g$-CG, by the Lemma 5 ($\mathcal{x}_1$, $\mathcal{y}_2$) $\in \mathcal{U} \times \mathcal{V} -$ G(f), there is an $\mathcal{N}wg$-OS M, N and $\mathcal{x}_1 \in M$, $\mathcal{y}_2 \in N$, and hence $f(M) \cap \mathcal{N}wg\text{-}Cl(N) = \emptyset$. As a result, $\mathcal{y}_1 \notin \mathcal{N}wg\text{-}Cl(N)$. This implies that $\exists$ W $\in \mathcal{NWGO}(\mathcal{V}, \mathcal{y}_1)$, W $\bigcap$ N = $\emptyset$. Thus, $\mathcal{V}$ is $\mathcal{N}wg$-T$_2$. Thus, $\mathcal{V}$ is $\mathcal{N}wg$-T$_1$ space. □

Theorem 14. *If* $f : (\mathcal{U}, \tau_{\mathcal{R}1}(\mathcal{X})\ ) \to (\mathcal{V}, \sigma_{\mathcal{R}2}(\mathcal{Y}))$ *is one to one and onto with strongly* $\mathcal{N}wg$*-CG, then* $\mathcal{U}$ *and* $\mathcal{V}$ *are* $\mathcal{N}wg$*-T*$_1$ *spaces.*

Proof. Theorems 12 and 13 directly lead to the proof. □

Theorem 15. *The space* $\mathcal{U}$ *is* $\mathcal{N}wg$*-T*$_2$ *space if and only if the identity function has strongly* $\mathcal{N}wg$*-CG.*

Proof. Necessary: *Consider* $\mathcal{U}$ *to be* $\mathcal{N}wg$*-T*$_2$*. According to Theorem 11, the identity function is* $\mathcal{N}wg$*-irresolute, and G(f) is strongly* $\mathcal{N}wg$*-CG.*

Sufficiency: *Let G (f) be strongly* $\mathcal{N}wg$*-CG. Since the function f is onto using Theorem 13,* $\mathcal{U}$ *is* $\mathcal{N}wg$*-T*$_2$ *space.* □

Definition 13. *A function* $f : (\mathcal{U}, \tau_{\mathcal{R}1}(\mathcal{X})\ ) \to (\mathcal{V}, \sigma_{\mathcal{R}2}(\mathcal{Y}))$ *is called nano quasi* wg*-irresolute if* $\forall \mathcal{x} \in \mathcal{U}$ *and for each* $\mathcal{N}wg$*-OS* $f(\mathcal{x})$ *is a subset of N,* $\exists$ *a* $\mathcal{N}wg$*-OS* $\mathcal{x} \in M$*, in such a way that* $f(M) \subset \mathcal{N}wg\text{-}Cl(N)$. □

Remark 2. *Every* $\mathcal{N}wg$*-irresolute is nano quasi* wg*-irresolute. However, the contrary is not always true, as demonstrated by the given example.* □

Example 5. *Let* $\mathcal{U} = \{\zeta_{a1}, \zeta_{a2}, \zeta_{a3}\}, \mathcal{X} = \{\zeta_{a3}\}, \mathcal{U}/\mathcal{R}1 = \{\{\zeta_{a1}, \zeta_{a2}\}, \{\zeta_{a3}\}\}, (\mathcal{U}, \tau_{\mathcal{R}1}(\mathcal{X})) = \{U, \emptyset, \{\zeta_{a3}\}\}$*and* $\mathcal{N}wg$*-open sets are* $\{\mathcal{U}, \varnothing, \{\zeta_{a1}\}, \{\zeta_{a2}\}, \{\zeta_{a3}\}, \{\zeta_{a2}, \zeta_{a3}\}, \{\zeta_{a1}, \zeta_{a3}\}\}$*. Let* $\mathcal{V} = \{\eta_{b1}, \eta_{b2}, \eta_{b3}\}, \mathcal{Y} = \{\eta_{b1}, \eta_{b2}\}, \mathcal{V}/\mathcal{R}2 = \{\{\eta_{b1}\}, \{\eta_{b2}, \eta_{b3}\}\}$, $(\mathcal{V}, \sigma_{\mathcal{R}2}(\mathcal{Y})) = \{V, \emptyset, \{\eta_{b1}\}, \{\eta_{b2}, \eta_{b3}\}\}$ *and* $\mathcal{N}wg$*-open sets are* $\{\mathcal{V}, \emptyset, \{\eta_{b1}\}, \{\eta_{b2}\}, \{\eta_{b3}\}, \{\eta_{b1}, \eta_{b2}\}, \{\eta_{b1}, \eta_{b3}\}, \{\eta_{b2}, \eta_{b3}\}\}$*. Let* $f : (\mathcal{U}, \tau_{\mathcal{R}1}(\mathcal{X})) \to (\mathcal{V}, \sigma_{\mathcal{R}2}(\mathcal{Y}))$ *be a mapping defined by* $f(\zeta_{a1}) = \eta_{b1}$*,* $f(\zeta_{a2}) = \eta_{b2}$ *and* $f(\zeta_{a3}) = \eta_{b3}$*. Then f is nano quasi* wg*-irresolute. However, in* $\mathcal{U}$*,* $f^{-1}(\{\zeta_{a3}\}) = \{\eta_{b3}\}$ *is not* $\mathcal{N}wg$*-CS in* $\mathcal{U}$*. As a result, f is not* $\mathcal{N}wg$*-irresolute function.* □

Theorem 16. *If* $f : (\mathcal{U}, \tau_{\mathcal{R}1}(\mathcal{X})\ ) \to (\mathcal{V}, \sigma_{\mathcal{R}2}(\mathcal{Y}))$ *is nano quasi* wg*-irresolute, a one-to-one function along with strongly* $\mathcal{N}wg$*-CG G(f), then* $\mathcal{U}$ *is* $\mathcal{N}wg$*-T*$_2$ *space.*

Proof. Since f is one to one, for any two separate points $\mathcal{x}_1, \mathcal{x}_2 \in \mathcal{U}$, $f(\mathcal{x}_1) \neq f(\mathcal{x}_2)$. Therefore ($\mathcal{x}_1$, $f(\mathcal{x}_2)$) $\in \mathcal{U} \times \mathcal{V} -$ G(f). The $\mathcal{N}wg$- closedness of G(f) gives $M \in \mathcal{NWGO}(\mathcal{U}$, $\mathcal{x}_1$) and $N \in \mathcal{NWGO}(\mathcal{V}$, $f(\mathcal{x}_2)$) such that $f(M) \bigcap \mathcal{N}wg\text{-}Cl_{\mathcal{V}}(N) = \emptyset$. Therefore, we obtain $M \bigcap f^{-1}(\mathcal{N}wg\text{-}Cl_{\mathcal{V}}(N)) = \emptyset$. Consequently, $f^{-1}(\mathcal{N}wg\text{-}Cl_{\mathcal{V}}(N)) \subset \mathcal{U} - M$. Since f is nano quasi wg- irresolute, this is applicable at $\mathcal{x}_2$. Then there exists W $\in \mathcal{NWGO}(\mathcal{U}, \mathcal{x}_2)$ in such a way that f(W) $\subset \mathcal{N}wg\text{-}Cl_{\mathcal{V}}(N)$. It follows that W $\subset f^{-1}(\mathcal{N}wg\text{-}Cl_{\mathcal{V}}(N)) \subset \mathcal{U} - M$. Thus, it may be shown that W $\bigcap M = \emptyset$. As a result of this, $\mathcal{U}$ is a $\mathcal{N}wg$-T$_2$ space. □

Theorem 17. *If* $\mathcal{V}$ *is a nano weakly generalized Urysohn space and* $f : (\mathcal{U}, \tau_{\mathcal{R}1}(\mathcal{X})\ ) \to (\mathcal{V}, \sigma_{\mathcal{R}2}(\mathcal{Y}))$ *is nano quasi* wg*-irresolute, then* $\mathcal{U}$ *is* $\mathcal{N}wg$*-T*$_2$ *space.*

Proof. As the function f is one to one, $\mathcal{x}_1, \mathcal{x}_2 \in \mathcal{U}, \mathcal{x}_1 \neq \mathcal{x}_2$, $f(\mathcal{x}_1) \neq f(\mathcal{x}_2)$. The nano weakly generalized Urysohn property implies that there exist $\mathcal{H}_i \in \mathcal{NWGO}(\mathcal{V}, f(\mathcal{x}_i))$, $i = 1,\ 2$ such that $\mathcal{N}wg\text{-}Cl_{\mathcal{V}}(\mathcal{H}_1) \bigcap \mathcal{N}wg\text{-}Cl_{\mathcal{V}}(\mathcal{H}_2) = \emptyset$. Hence, $f^{-1}(\mathcal{N}wg\text{-}Cl_{\mathcal{V}}(\mathcal{H}_1)) \bigcap f^{-1}(\mathcal{N}wg\text{-}Cl_{\mathcal{V}}(\mathcal{H}_2)) = \emptyset$. Since f is nano quasi wg-irresolute, there exists $\mathcal{G}_i \in \mathcal{NWGO}(\mathcal{U}$, $\mathcal{x}_i$), $i = 1,\ 2$ such that $f(\mathcal{G}_i) \subset \mathcal{N}wg\text{-}Cl_{\mathcal{V}}(\mathcal{H}_i)$, $i = 1,\ 2$. Then, it follows that $\mathcal{G}_i \subset f^{-1}(\mathcal{N}wg\text{-}Cl_{\mathcal{V}}(\mathcal{H}_i))$, $i = 1,\ 2$. Hence, $\mathcal{G}_1 \bigcap \mathcal{G}_2 \subset f^{-1}(\mathcal{N}wg\text{-}Cl_{\mathcal{V}}(\mathcal{H}_1) \bigcap f^{-1}(\mathcal{N}wg\text{-}Cl_{\mathcal{V}}(\mathcal{H}_2)) = \emptyset$. This implies that $\mathcal{U}$ is $\mathcal{N}wg$-T$_2$ space. □

Definition 14. *A function $f : (\mathcal{U}, \tau_{\mathcal{R}1}(\mathcal{X})) \to (\mathcal{V}, \sigma_{\mathcal{R}2}(\mathcal{Y}))$ is called nano θ-wg-irresolute if for each $\mathcal{N}wg$-neighbourhood N of $f(x)$ there is a $\mathcal{N}wg$-neighbourhood M of x such that $f(\mathcal{N}wg$-$Cl_{\mathcal{V}}(M)) \subseteq \mathcal{N}wg$-$Cl_{\mathcal{V}}(N)$.*

Remark 3. *Every $\mathcal{N}wg$-irresolute function is nano θ-wg-irresolute. However, as the following example demonstrates, the contrary need not be true.*

Example 6. *Let $\mathcal{U} = \{\zeta_{a1}, \zeta_{a2}, \zeta_{a3}, \zeta_{a4}\}, \mathcal{X} = \{\zeta_{a1}\}, \mathcal{U}/\mathcal{R}1 = \{\{\zeta_{a1}\}, \{\zeta_{a2}, \zeta_{a3}\}, \{\zeta_{a4}\}\}$, $(\mathcal{U}, \tau_{\mathcal{R}1}(\mathcal{X})) = \{U, \emptyset, \{\zeta_{a1}\}\}$ $\mathcal{N}wg$ -open sets be $\left\{\begin{array}{l} U, \emptyset, \{\zeta_{a1}\}, \{\zeta_{a2}\}, \{\zeta_{a3}\}, \{\zeta_{a4}\}, \{\zeta_{a1}, \zeta_{a2}\}, \{\zeta_{a2}, \zeta_{a3}\}, \{\zeta_{a1}, \zeta_{a3}\}, \{\zeta_{a1}, \zeta_{a4}\}, \\ \{\zeta_{a2}, \zeta_{a4}\}, \{\zeta_{a3}, \zeta_{a4}\}, \{\zeta_{a1}, \zeta_{a2}, \zeta_{a3}\}, \{\zeta_{a1}, \zeta_{a2}, \zeta_{a4}\}, \{\zeta_{a1}, \zeta_{a3}, \zeta_{a4}\} \end{array}\right\}$. Let $\mathcal{V} = \{\eta_{b1}, \eta_{b2}, \eta_{b3}, \eta_{b4}\}, \mathcal{Y} = \{\eta_{b2}, \eta_{b4}\}, \mathcal{V}/\mathcal{R}2 = \{\{\eta_{b1}, \eta_{b2}\}, \{\eta_{b3}\}, \{\eta_{b4}\}\}$, $(\mathcal{V}, \sigma_{\mathcal{R}2}(\mathcal{Y})) = \{V, \emptyset, \{\eta_{b4}\}, \{\eta_{b1}, \eta_{b2}\}, \{\eta_{b1}, \eta_{b2}, \eta_{b4}\}\}$. $\mathcal{N}wg$-open sets are $\{\mathcal{V}, \emptyset, \{\eta_{b1}\}, \{\eta_{b2}\}, \{\eta_{b4}\}, \{\eta_{b1}, \eta_{b2}\}, \{\eta_{b2}, \eta_{b4}\}, \{\eta_{b1}, \eta_{b4}\}, \{\eta_{b1}, \eta_{b2}, \eta_{b4}\}, \{\eta_{b1}, \eta_{b3}, \eta_{b4}\}, \{\eta_{b2}, \eta_{b3}, \eta_{b4}\}\}$. Let $f : (\mathcal{U}, \tau_{\mathcal{R}1}(\mathcal{X})) \to (\mathcal{V}, \sigma_{\mathcal{R}2}(\mathcal{Y}))$ be a mapping defined by $f(\zeta_1) = \eta_1$, $f(\zeta_2) = \eta_2$, $f(\zeta_3) = \eta_3$ and $f(\zeta_{a4}) = \eta_{b4}$. Then, f is nano θ-wg-irresolute. Nonetheless, in $\mathcal{U}$, $f^{-1}(\{\eta_{a1}\}) = \{\zeta_{b1}\}$ is not $\mathcal{N}wg$-CS. Thus, f is not an $\mathcal{N}wg$-irresolute function.*

Corollary 2. *If $f : (\mathcal{U}, \tau_{\mathcal{R}1}(\mathcal{X})) \to (\mathcal{V}, \sigma_{\mathcal{R}2}(\mathcal{Y}))$ is nano θ-wg-irresolute, a one-to-one function with strongly $\mathcal{N}wg$-CG $G(f)$, then $\mathcal{U}$ is $\mathcal{N}wg$-T_2 space.*

Proof. Given that nano θ-wg-irresoluteness is nano quasi wg-irresoluteness, Theorem 16 provides the basis for the proof. □

Theorem 18. *If the bijective function $f : (\mathcal{U}, \tau_{\mathcal{R}1}(\mathcal{X})) \to (\mathcal{V}, \sigma_{\mathcal{R}2}(\mathcal{Y}))$ is nano quasi wg-irresolute (resp. nano θ-wg-irresolute) with strongly $\mathcal{N}wg$-CG $G(f)$, then $\mathcal{U}$ and $\mathcal{V}$ are $\mathcal{N}wg$-T_2 space.*

Proof. The proof is a direct result of Theorem 16 and Theorem 13 (resp. Corollary 2 and Theorem 13). □

Corollary 3. *If $f : (\mathcal{U}, \tau_{\mathcal{R}1}(\mathcal{X})) \to (\mathcal{V}, \sigma_{\mathcal{R}2}(\mathcal{Y}))$ is $\mathcal{N}wg$-irresolute, a one-to-one function with $\mathcal{N}wg$-CG $G(f)$, then $\mathcal{U}$ is $\mathcal{N}wg$-T_2 space.*

Proof. Theorem 16 as well as the fact that every $\mathcal{N}wg$-irresoluteness is nano quasi wg-irresoluteness provide the proof. □

Definition 15. *A function $f : (\mathcal{U}, \tau_{\mathcal{R}1}(\mathcal{X})) \to (\mathcal{V}, \sigma_{\mathcal{R}2}(\mathcal{Y}))$ is almost $\mathcal{N}wg$-irresolute if for each $x \in \mathcal{U}$ and each $\mathcal{N}wg$-neighbourhood $\mathcal{V}$of $f(x)$, $\mathcal{N}wg$-Cl$(f^{-1}(\mathcal{V}))$ is the $\mathcal{N}wg$-neighbourhood of x.*

Theorem 19. *If $f : (\mathcal{U}, \tau_{\mathcal{R}1}(\mathcal{X})) \to (\mathcal{V}, \sigma_{\mathcal{R}2}(\mathcal{Y}))$ is nano almost wg-irresolute, a one-to-one function with $\mathcal{N}wg$-CG $G(f)$, then $\mathcal{U}$ is $\mathcal{N}wg$-T_2 space.*

Proof. By using Theorem 16, we get $f(M) \cap \mathcal{N}wg$-Cl $(N) = \emptyset$. Therefore, $f^{-1}(\mathcal{N}wg$-Cl$(N)) \subseteq \mathcal{U} - M$. Since $\mathcal{U} - M$ is a nano wg-closed set containing $f^{-1}(\mathcal{N}wg$-Cl$(N))$, any $\mathcal{N}wg$-Cl$(f^{-1}(\mathcal{N}wg$-Cl$(N)))$ is the smallest $\mathcal{N}wg$-closed that contains $f^{-1}(\mathcal{N}wg$-Cl$(N))$ as the result of $\mathcal{N}wg$-Cl $(f^{-1}(\mathcal{N}wg$-Cl$(N))) \subseteq \mathcal{U} - M$. The nano almost wg-irresoluteness of f confirms that $f^{-1}(\mathcal{N}wg$-Cl$(N))$ and hence $\mathcal{N}wg$-Cl$(f^{-1}(\mathcal{N}wg$-Cl$(N)))$ is a $\mathcal{N}wg$-neighbourhood of x_2. This implies that there exist $H \in \mathcal{NWGO}(\mathcal{U}, x_2)$ such that $H \subseteq \mathcal{N}wg$-Cl $(f^{-1}(\mathcal{N}wg$ -Cl$(N))) \subseteq \mathcal{U} - M$. From this we can obtain $M \bigcap H = \emptyset$. Therefore, $\mathcal{U}$ is $\mathcal{N}wg$-T_2 space. □

6. Conclusions

In this paper, we explored the characterization of separation axioms with the aid of $\mathcal{N}wg$-OS in an NTS. We presented a weaker form of a closed graph, such as $\mathcal{N}wg$-closed graphs, and a stronger form of a closed graph, such as strongly $\mathcal{N}wg$-closed graphs, with the aid of $\mathcal{N}wg$-closed sets in an NTS and examined the characterization of strongly $\mathcal{N}wg$-closed graphs with $\mathcal{N}wg$-irresolute, nano quasi wg-irresolute, nano θ-wg-irresolute, etc. The example of an $\mathcal{N}wg$-closed graph via a simple graph with vertices was discussed. We investigated some separation properties, especially $\mathcal{N}wg$-Harsdorf space and $\mathcal{N}wg$-Urysohn space, induced by both closed graph functions such as $\mathcal{N}wg$-closed graphs and strongly $\mathcal{N}wg$-closed graphs either on its domain, range, or on both spaces.

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