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A Unified Representation of q- and h-Integrals and Consequences in Inequalities

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Abstract: This paper aims to unify q-derivative/q-integrals and h-derivative/h-integrals into a single definition, called $q-h$-derivative/$q-h$-integral. These notions are further extended on the finite interval $[a,b]$ in the form of left and right $q-h$-derivatives and $q-h$-integrals. Some inequalities for $q-h$-integrals are studied and directly connected with well known results in diverse fields of science and engineering. The theory based on q-derivatives/q-integrals and h-derivatives/h-integrals can be unified using the $q-h$-derivative/$q-h$-integral concept.

Keywords: q-derivative; q-integral; h-derivative; h-integral; $q-h$-derivative; $q-h$-integral; inequalities

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1. Introduction

The subject of q-calculus is based on the quotient $\frac{f(qx)-f(x)}{(q-1)x}$ involved in the derivative of a function. This motivates researchers to consider whether the results and theory that hold for usual derivatives can be further developed by analyzing this quotient. Euler (1707–1783) was the first to work in this direction, introducing the number q in the infinite series defined by Newton. Jackson [1,2] continued the work of Euler and defined q-derivatives and q-integrals. Roughly speaking, q-calculus analyzes q-analogues of mathematical concepts and formulas that can be recaptured by the limit $q \to 1$. The concepts of q-calculus are extensively applied in various subjects of physics and mathematics, including combinatorics, number theory, orthogonal polynomials, geometric function theory, quantum theory and mechanics, and the theory of relativity; see [3–9].

Nowadays, many authors are applying quantum calculus theory in their fields of research. Consequently, they have contributed plenty of articles in this active field. For instance, theory of fractional calculus, optimal control problems, q-difference, and q-integral equations are studied in q-analysis; see [10–13] and references therein. In [14,15], Tariboon et al. defined quantum calculus on finite intervals and extended some important integral inequalities using this concept.

Here, our goal is to unify quantum calculus (q-calculus) and plank calculus (h-calculus). For this purpose, the notion of $q-h$-derivatives is introduced and basic calculus formulas are presented. Moreover, a $q-h$-binomial is constructed and the $q-h$-integral is defined on a finite interval. Using the $q-h$-integral, Hermite–Hadamard type inequalities can be constructed which combine the inequalities for q- and h-integrals in implicit form. By imposing the symmetric condition, a correct proof of an already published inequality (the first inequality in Equation (12)) is provided. Next, we begin to lay out well known initial concepts which are useful for the reader in understanding the findings of this paper.

The h-derivative and the q-derivative of a function ν are defined by the quotients

$$\frac{\nu(\gamma+h)-\nu(\gamma)}{h} \text{ and } \frac{\nu(q\gamma)-\nu(\gamma)}{(q-1)\gamma},$$

respectively. The h-derivative is usually denoted by $\mathcal{D}_h\nu(\gamma)=\frac{d_h\nu(\gamma)}{d_h\gamma}$ and the q-derivative is denoted by $\mathcal{D}_q\nu(\gamma)=\frac{d_q\nu(\gamma)}{d_q\gamma}$, where $d_h\nu(\gamma)=\nu(\gamma+h)-\nu(\gamma)$ is called the h-differential and $d_q\nu(\gamma)=\nu(q\gamma)-\nu(\gamma)$ is called the q-differential for the function ν. As an example, the h-derivative and the q-derivative of γ^n can be computed in the forms $\frac{(\gamma+h)^n-\gamma^n}{h}=n\gamma^{n-1}+\frac{n(n-1)}{2}\gamma^{n-2}h+\ldots+h^{n-1}$ and $\frac{q^n-1}{q-1}\gamma^{n-1}=(q^{n-1}+\ldots+1)\gamma^{n-1}$, respectively. For the sake of simplicity, the notation $[n]_q$ is used instead of $\frac{q^n-1}{q-1}$; thus, $\mathcal{D}_q\gamma^n=[n]_q\gamma^{n-1}$. Because $\lim\limits_{q\to 1}\mathcal{D}_q\nu(\gamma)=\lim\limits_{h\to 0}\mathcal{D}_h\nu(\gamma)=\frac{d\nu(\gamma)}{d\gamma}$, the h-derivative and the q-derivative are generalizations of ordinary derivative. The q-derivative leads to the subject of q-calculus; see [16] for details.

The sum and product formula of q-derivatives for functions ν_1 and ν_2 are provided by

$$\mathcal{D}_q\{\nu_1(\gamma)+\nu_2(\gamma)\}=\mathcal{D}_q\nu_1(\gamma)+\mathcal{D}_q\nu_2(\gamma) \tag{1}$$

and

$$\mathcal{D}_q\{\nu_1(\gamma)\nu_2(\gamma)\}=\nu_1(q\gamma)\mathcal{D}_q\nu_2(\gamma)+\nu_2(\gamma)\mathcal{D}_q\nu_1(\gamma), \tag{2}$$

respectively. Because $\nu_1(\gamma)\nu_2(\gamma)=\nu_2(\gamma)\nu_1(\gamma)$, Equation (2) is equivalent to the formula

$$\mathcal{D}_q\{\nu_1(\gamma)\nu_2(\gamma)\}=\nu_1(\gamma)\mathcal{D}_q\nu_2(\gamma)+\nu_2(q\gamma)\mathcal{D}_q\nu_1(\gamma). \tag{3}$$

In view of Equation (2), the quotient formula of q-derivatives is provided by

$$\mathcal{D}_q\left(\frac{\nu_1(\gamma)}{\nu_2(\gamma)}\right)=\frac{\nu_2(\gamma)\mathcal{D}_q\nu_1(\gamma)-\nu_1(\gamma)\mathcal{D}_q\nu_2(\gamma)}{\nu_2(\gamma)\nu_2(q\gamma)}. \tag{4}$$

In view of Equation (3), the quotient formula of q-derivatives is provided by

$$\mathcal{D}_q\left(\frac{\nu_1(\gamma)}{\nu_2(\gamma)}\right)=\frac{\nu_2(q\gamma)\mathcal{D}_q\nu_1(\gamma)-\nu_1(q\gamma)\mathcal{D}_q\nu_2(\gamma)}{\nu_2(\gamma)\nu_2(q\gamma)}. \tag{5}$$

The formulae for the h-derivatives are as follows:

$$\mathcal{D}_h\{\nu_1(\gamma)+\nu_2(\gamma)\}=\mathcal{D}_h\nu_1(\gamma)+\mathcal{D}_h\nu_2(\gamma), \tag{6}$$

$$\mathcal{D}_h\{\nu_1(\gamma)\nu_2(\gamma)\}=\nu_1(\gamma)\mathcal{D}_h\nu_2(\gamma)+\nu_2(\gamma+h)\mathcal{D}_h\nu_1(\gamma), \tag{7}$$

and

$$\mathcal{D}_h\left(\frac{\nu_1(\gamma)}{\nu_2(\gamma)}\right)=\frac{\nu_2(\gamma)\mathcal{D}_h\nu_1(\gamma)-\nu_1(\gamma)\mathcal{D}_h\nu_2(\gamma)}{\nu_2(\gamma)\nu_2(\gamma+h)}. \tag{8}$$

Next, we provide the definition of a q-derivative on a finite interval.

Definition 1 ([15])**.** *Let $\mu: I=[a,b]\to\mathbb{R}$ be a continuous function. For $0<q<1$, the q-derivative ${}_a\mathcal{D}_q\mu$ on I is provided by*

$${}_a\mathcal{D}_q\mu(\xi):=\frac{\mu(q\xi+(1-q)a)-\mu(\xi)}{(q-1)(\xi-a)},\ \xi\neq a,\ {}_a\mathcal{D}_q\mu(a)=\lim_{\xi\to a}{}_a\mathcal{D}_q\mu(\xi). \tag{9}$$

Function μ is called q-differentiable on $[a,b]$ if ${}_a\mathcal{D}_q\mu(\xi)$ exists for all $\xi \in [a,b]$. For $a=0$, we have ${}_0\mathcal{D}_q\mu(\xi) = \mathcal{D}_q\mu(\xi)$; moreover, $\mathcal{D}_q\mu(\xi)$ is the q-derivative of μ at $\xi \in [a,b]$, defined as follows:

$$\mathcal{D}_q\mu(\xi) := \frac{\mu(q\xi) - \mu(\xi)}{(q-1)\xi},\ \xi \neq 0. \tag{10}$$

The q-integral of the function μ on interval $[a,b]$ is defined below.

Definition 2 ([15]). *Let $\mu : I = [a,b] \to \mathbb{R}$ be a function. For $0 < q < 1$, the q-definite integral on I is provided by*

$$\int_a^\xi \mu(\gamma)\, {}_a d_q\gamma = (1-q)(\xi - a)\sum_{n=0}^{\infty} q^n \mu(q^n\xi + (1-q^n)a),\ \xi \in [a,b]. \tag{11}$$

In the following we provide a q-integral inequality published in [15].

Theorem 1 ([15]). *Let $\mu : [a,b] \to \mathbb{R}$ be a convex continuous function on $[a,b]$ and let $0 < q < 1$; then, we have*

$$\mu\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b \mu(\gamma)\, {}_a d_q\gamma \le \frac{q\mu(a) + \mu(b)}{q+1}. \tag{12}$$

In (11), setting $a=0$, the Jackson q-definite integral in [16] is deduced as follows:

$$\int_0^\xi \mu(\gamma)\, {}_0 d_q\gamma = \int_0^\xi \mu(\gamma) d_q\gamma = (1-q)\xi\sum_{n=0}^{\infty} q^n\mu(q^n\xi),\ \xi \in [a,b]. \tag{13}$$

If $c \in (a,\xi)$, then the q-definite integral on $[c,\xi]$ is calculated as follows:

$$\int_c^\xi \mu(\gamma)\, {}_a d_q\gamma = \int_a^\xi \mu(\gamma)\, {}_a d_q\gamma - \int_a^c \mu(\gamma)\, {}_a d_q\gamma. \tag{14}$$

We intend to unify the q-derivative and h-derivative into a single notion, which we name the $q-h$-derivative. We provide sum/difference, product, and quotient formulas for $q-h$-derivatives, along with the definition of the $q-h$-integral. Further, we define the $q-h$-derivative and $q-h$-integral on a finite interval. The composite derivatives and integrals provide the opportunity to simultaneously study theoretical and practical concepts and problems from different fields related to q-derivatives and h-derivatives. For instance, in Theorem 3 we prove the generalization of the inequality in (12) via the $q-h$-integral.

2. Generalization of q- and h-Derivatives

We define the $(q-h)$-differential of a real valued function μ as follows:

$${}_h d_q\mu(\xi) = \mu(q(\xi+h)) - \mu(\xi). \tag{15}$$

Then, for $h=0$ and $q \to 1$ in (15), we have

$${}_0 d_q\mu(\xi) = \mu(q\xi) - \mu(\xi) = d_q\mu(\xi)$$

and

$${}_h d_1\mu(\xi) = \mu(\xi+h) - \mu(\xi) = {}_h d\mu(\xi).$$

In particular,

$${}_h d_q(\xi) = q\xi + qh - \xi = (q-1)\xi + qh. \tag{16}$$

Then, for $h = 0$ and $q \to 1$ in (16), we have

$$ {}_0d_q(\xi) = (q-1)\xi = d_q(\xi) \quad \textit{and} \quad {}_hd_1(\xi) = h = d_h(\xi). \tag{17} $$

For $u(\xi) = \mu(\xi) + \nu(\xi)$, the $(q-h)$-differential of u is provided by

$$ {}_hd_q(u(\xi)) = {}_hd_q(\mu(\xi) + \nu(\xi)) = (\mu+\nu)(q(\xi+h)) - (\mu+\nu)(\xi) = {}_hd_q\mu(\xi) + {}_hd_q\nu(\xi). \tag{18} $$

For $\alpha \in \mathbb{R}$, the $(q-h)$-differential of $\alpha\mu$ is provided by

$$ {}_hd_q(\alpha\mu)(\xi) = {}_hd_q(\alpha\mu)(\xi) = (\alpha\mu)(q(\xi+h)) - (\alpha\mu)(\xi) = \alpha\, {}_hd_q\mu(\xi). \tag{19} $$

From (18) and (19), it can be seen that the $(q-h)$-differential is linear. If $p(\xi) = \mu(\xi)\nu(\xi)$, then the $(q-h)$-differential is calculated as follows:

$$ \begin{aligned} {}_hd_q(p(\xi)) = {}_hd_q(\mu(\xi)\nu(\xi)) &= \mu(q(\xi+h))\nu(q(\xi+h)) - \mu(\xi)\nu(\xi) \\ &= \mu(q(\xi+h))\nu(q(\xi+h)) + \mu(q(\xi+h))\nu(\xi) \\ &\quad - \mu(q(\xi+h))\nu(\xi) - \mu(\xi)\nu(\xi) \\ &= \mu(q(\xi+h))[\nu(q(\xi+h)) - \nu(\xi)] \\ &\quad + \nu(\xi)[\mu(q(\xi+h)) - \mu(\xi)]. \end{aligned} $$

Hence, we obtain

$$ {}_hd_q(\mu(\xi)\nu(\xi)) = \mu(q(\xi+h))\,{}_hd_q\nu(\xi) + \nu(\xi)\,{}_hd_q\mu(\xi). \tag{20} $$

For $h = 0$ and $q \to 1$ in (20), we have

$$ \begin{aligned} {}_0d_q(\mu(\xi)\nu(\xi)) = d_q(\mu(\xi)\nu(\xi)) &= \mu(q\xi)\,{}_0d_q\nu(\xi) + \nu(\xi)\,{}_0d_q\mu(\xi) \\ &= \mu(q\xi)d_q\nu(\xi) + \nu(\xi)d_q\mu(\xi) \end{aligned} $$

and

$$ \begin{aligned} {}_hd_1(\mu(\xi)\nu(\xi)) = d_h(\mu(\xi)\nu(\xi)) &= \mu(\xi+h)\,{}_hd_1\nu(\xi) + \nu(\xi)\,{}_hd_1\mu(\xi) \\ &= \mu(\xi+h)d_h\nu(\xi) + \nu(\xi)d_h\mu(\xi), \end{aligned} $$

respectively. Next, we define the $q-h$-derivative as follows:

Definition 3. *Let $0 < q < 1$ and $h \in \mathbb{R}$, and let $\mu : I \to \mathbb{R}$ be a continuous function. Then, the $q-h$-derivative of μ is defined by*

$$ \begin{aligned} &\mathcal{C}_h\mathcal{D}_q\mu(\xi) = \frac{{}_hd_q\mu(\xi)}{{}_hd_q\xi} = \frac{\mu(q(\xi+h)) - \mu(\xi)}{(q-1)\xi + qh}, \; \xi \neq \frac{qh}{1-q} := \xi_\circ \\ &\mathcal{C}_h\mathcal{D}_q\mu(\xi_\circ) = \lim_{\xi\to\xi_\circ} \mathcal{C}_h\mathcal{D}_q\mu(\xi) \end{aligned} \tag{21} $$

provided that $q(\xi+h) \in I$.

For $h = 0$ and $q \to 1$ in (21), we have

$$ \mathcal{C}_0\mathcal{D}_q\mu(\xi) = \mathcal{D}_q\mu(\xi) = \frac{d_q\mu(\xi)}{d_q\xi} = \frac{\mu(q\xi) - \mu(\xi)}{(q-1)\xi} \tag{22} $$

and

$$ \mathcal{C}_h\mathcal{D}_1\mu(\xi) = \mathcal{D}_h\mu(\xi) = \frac{d_h\mu(\xi)}{d_h\xi} = \frac{\mu(\xi+h) - \mu(\xi)}{h}. \tag{23} $$

By setting $h = 0$, $q \to 1$ in (21), we obtain the ordinary derivative of μ, provided that the limit exists.

Example 1. *Consider $P(x) = \xi^n$, $n \in \mathbb{N}$; then,*

$$\mathcal{C}_h\mathcal{D}_q(P(x)) = \frac{q^n(\xi+h)^n - \xi^n}{(q-1)\xi + qh} = \frac{(q^n-1)\xi^n}{(q-1)\xi+qh} + \frac{q^n(n\xi^{n-1}h + \ldots + h^n)}{(q-1)\xi+qh}. \tag{24}$$

For $h = 0$ and $q \to 1$ in (24), we have

$$\mathcal{C}_0\mathcal{D}_q(\xi^n) = \frac{q^n\xi^n - \xi^n}{(q-1)\xi} = \frac{q^n-1}{q-1}\xi^{n-1} = [n]_q\xi^{n-1} = \mathcal{D}_q(\xi^n) \tag{25}$$

and

$$\mathcal{C}_h\mathcal{D}_1(\xi^n) = \frac{(\xi+h)^n - \xi^n}{h} = n\xi^{n-1} + \frac{n(n-1)}{2}\xi^{n-2}h + \ldots\ldots + h^{n-1}. \tag{26}$$

In particular, we have $\lim_{h\to 0}\mathcal{C}_h\mathcal{D}_1(\xi^n) = n\xi^{n-1}$.

2.1. Linearity

The $q-h$-derivative is linear, i.e., for $\alpha, \beta \in \mathbb{R}$ and using the linearity of $(q-h)$-differentials, we have

$$\mathcal{C}_h\mathcal{D}_q(\alpha\mu(\xi) + \beta\nu(\xi)) = \alpha\,\mathcal{C}_h\mathcal{D}_q\mu(\xi) + \beta\,\mathcal{C}_h\mathcal{D}_q\nu(\xi).$$

2.2. Product Formula

The following formula for a product of functions can be obtained using (20):

$$\begin{aligned}\mathcal{C}_h\mathcal{D}_q(\mu(\xi)\nu(\xi)) = \frac{{}_hd_q(\mu(\xi)\nu(\xi))}{{}_hd_q\xi} &= \frac{\mu(q(\xi+h))\,{}_hd_q\nu(\xi) + {}_h d_q\mu(\xi)\nu(\xi)}{{}_hd_q\xi} \\ &= \mu(q(\xi+h))\mathcal{C}_h\mathcal{D}_q\nu(\xi) + \nu(\xi)\mathcal{C}_h\mathcal{D}_q\mu(\xi).\end{aligned} \tag{27}$$

The product formula for q-derivatives and h-derivatives can be obtained as follows.

By setting $h = 0$ in (27), the following q-derivative formula for products of functions is yielded:

$$\begin{aligned}\mathcal{C}_0\mathcal{D}_q(\mu(\xi)\nu(\xi)) = \frac{d_q(\mu(\xi)\nu(\xi))}{d_q\xi} &= \mathcal{D}_q(\mu(\xi)\nu(\xi)) \\ &= \mu(q\xi)\mathcal{C}_0\mathcal{D}_q\nu(\xi) + \nu(\xi)\mathcal{C}_0\mathcal{D}_q\mu(\xi) \\ &= \mu(q\xi)\mathcal{D}_q\nu(\xi) + \nu(\xi)\mathcal{D}_q\mu(\xi).\end{aligned} \tag{28}$$

By taking $q \to 1$ in (27), the following h-derivative formula for products of functions is yielded:

$$\begin{aligned}\mathcal{C}_h\mathcal{D}_1(\mu(\xi)\nu(\xi)) = \frac{d_h(\mu(\xi)\nu(\xi))}{d_h\xi} &= \mathcal{D}_h(\mu(\xi)\nu(\xi)) \\ &= \mu(\xi+h)\mathcal{C}_h\mathcal{D}_1\nu(\xi) + \nu(\xi)\mathcal{C}_h\mathcal{D}_1\mu(\xi) \\ &= \mu(\xi+h)\mathcal{D}_h\nu(\xi) + \nu(\xi)\mathcal{D}_h\mu(\xi).\end{aligned} \tag{29}$$

Using symmetry, from (27) we have the following:

$$\mathcal{C}_h\mathcal{D}_q(\nu(\xi)\mu(\xi)) = \nu(q(\xi+h))\mathcal{C}_h\mathcal{D}_q\mu(\xi) + \mu(\xi)\mathcal{C}_h\mathcal{D}_q\nu(\xi). \tag{30}$$

Both (27) and (30) are equivalent.

2.3. Quotient Formula

Using (27) and (30), the quotient formula of $q-h$-derivatives is calculated as follows. For $\nu(\xi)\neq 0$, we have

$$\nu(\xi)\frac{\mu(\xi)}{\nu(\xi)}=\mu(\xi). \tag{31}$$

Using definition of $q-h$-derivatives and (27), we have

$$\mathcal{C}_h\mathcal{D}_q\left(\nu(\xi)\frac{\mu(\xi)}{\nu(\xi)}\right)=\mathcal{C}_h\mathcal{D}_q(\mu(\xi)), \tag{32}$$

$$\nu(q(\xi+h))\mathcal{C}_h\mathcal{D}_q\left(\frac{\mu(\xi)}{\nu(\xi)}\right)+\frac{\mu(\xi)}{\nu(\xi)}\mathcal{C}_h\mathcal{D}_q\nu(\xi)=\mathcal{C}_h\mathcal{D}_q(\mu(\xi)). \tag{33}$$

Now,

$$\begin{aligned}\mathcal{C}_h\mathcal{D}_q\left(\frac{\mu(\xi)}{\nu(\xi)}\right)&=\frac{\mathcal{C}_h\mathcal{D}_q(\mu(\xi))-\frac{\mu(\xi)}{\nu(\xi)}\mathcal{C}_h\mathcal{D}_q(\nu(\xi))}{\nu(q(\xi+h))}\\&=\frac{\nu(\xi)\mathcal{C}_h\mathcal{D}_q(\mu(\xi))-\mu(\xi)\mathcal{C}_h\mathcal{D}_q(\nu(\xi))}{\nu(q(\xi+h))\nu(\xi)}.\end{aligned} \tag{34}$$

Using (30), we can obtain

$$\frac{\mu(q(\xi+h))}{\nu(q(\xi+h))}\mathcal{C}_h\mathcal{D}_q\Big(\nu(\xi)\Big)+\nu(\xi)\mathcal{C}_h\mathcal{D}_q\left(\frac{\mu(\xi)}{\nu(\xi)}\right)=\mathcal{C}_h\mathcal{D}_q\Big(\mu(\xi)\Big),$$

that is,

$$\mathcal{C}_h\mathcal{D}_q\left(\frac{\mu(\xi)}{\nu(\xi)}\right)=\frac{\mathcal{C}_h\mathcal{D}_q(\mu(\xi))\nu(q(\xi+h))-\mu(q(\xi+h))\mathcal{C}_h\mathcal{D}_q(\nu(\xi))}{\nu(q(\xi+h))\nu(\xi)}. \tag{35}$$

Remark 1. *By putting $h=\frac{\omega}{q}$ for $\omega>0$, Equation (27) produces product formulas and Equation (34) produces quotient formulas for the (q,ω)-derivatives in [17].*

Next, let us define the $q-h-$binomial $(\xi-a)^n_{h,q}$ analogue to $(\xi-a)^n$ as follows:

$$(\xi-a)^n_{h,q}=\begin{cases}1, & n=0,\\(\xi-a)(\xi-q(a+h))(\xi-q^2(a+2h))...(\xi-q^{n-1}(a+(n-1)h), & n\geq 1.\end{cases} \tag{36}$$

Then, it is clear that for $h=0$ we have $(\xi-a)^n_{0,q}=(\xi-a)^n_q$, i.e., the q-analogue of $(\xi-a)^n$ is obtained, which is defined in ([16], Page 8, Definition) as follows:

$$(\xi-a)^n_q=\begin{cases}1, & n=0,\\(\xi-a)(\xi-qa)...(\xi-q^{n-1}a), & n\geq 1.\end{cases} \tag{37}$$

In addition, from (36) we have $(\xi-a)^n_{h,1}=(\xi-a)^n_h$ for $q\to 1$, i.e., the h-analogue of $(\xi-a)^n$ is obtained, which is defined in ([16], Page 80, Definition) as follows:

$$(\xi-a)^n_h=\begin{cases}1, & n=0,\\(\xi-a)(\xi-a-h)...(\xi-a-(n-1)h), & n\geq 1.\end{cases} \tag{38}$$

Next, we find the $q-h-$derivative of the $q-h-$binomial $(\xi-a)^n_{h,q}$ as follows.
For $n=1$, we have

$${}_h\mathcal{D}_q((\xi-a)^1_{h,q})={}_h\mathcal{D}_q(\xi-a)=1.$$

For $n = 2$, we have

$$
\begin{aligned}
&{}_h\mathcal{D}_q((\xi - a)^2_{h,q}) = {}_h\mathcal{D}_q((\xi - a)(\xi - q(a+h))) = (q(\xi + h) - q(a+h)).1 + (\xi - a) \\
&= (\xi - a)(1+q) = [2]_q(\xi - a)^1_{h,q}.
\end{aligned}
$$

As $h \to 0$, we have ${}_0\mathcal{D}_q((\xi - a)^2_{0,q}) = \mathcal{D}_q((\xi - a)^2_q) = [2]_q(\xi - a)^1_q$, while because $q \to 1$ we have ${}_h\mathcal{D}_1((\xi - a)^2_{h,1}) = \mathcal{D}_h((\xi - a)^2_h) = 2(\xi - a)^1_h$.
For $n = 3$, we have

$$
\begin{aligned}
&{}_h\mathcal{D}_q((\xi - a)^3_{h,q}) = {}_h\mathcal{D}_q((\xi - a)^2_{h,q}(\xi - q^2(a+2h))) \\
&= (q(\xi + h) - q^2(a+2h))\{(q+1)(\xi - a)\} + (\xi - a)^2_{h,q}.1 \\
&= q(q+1)(\xi - a)(\xi - q(a+h)) + q(1-q^2)(\xi - a)h + (\xi - a)^2_{h,q} \\
&= q(q+1)(\xi - a)^2_{h,q} + (\xi - a)^2_{h,q} + q(1-q^2)(\xi - a)h \\
&= (q^2 + q + 1)(\xi - a)^2_{h,q} + q(1-q^2)(\xi - a)h = [3]_q(\xi - a)^2_{h,q} + q(1-q^2)h(\xi - a)^1_{h,q}.
\end{aligned}
$$

As $h \to 0$, we have ${}_0\mathcal{D}_q((\xi - a)^3_{0,q}) = \mathcal{D}_q((\xi - a)^3_q) = [3]_q(\xi - a)^2_q$, while because $q \to 1$ we have ${}_h\mathcal{D}_1((\xi - a)^3_{h,1}) = \mathcal{D}_h((\xi - a)^3_h) = 3(\xi - a)^2_h$.
For $n = 4$, we have

$$
\begin{aligned}
&{}_h\mathcal{D}_q((\xi - a)^4_{h,q}) = {}_h\mathcal{D}_q((\xi - a)^3_{h,q}(\xi - q^3(a+3h))) \\
&= (q(\xi + h) - q^3(a+3h))\left\{[3]_q(\xi - a)^2_{h,q} + q(1-q^2)h(\xi - a)^1_{h,q}\right\} + (\xi - a)^3_{h,q}.1 \\
&= [3]_q q(\xi - a)^2_{h,q}(\xi - q^2(a+2h)) + hq^2(1-q^2)(\xi - a)(\xi - q^2(a+2h)) \\
&+ [3]_q qh(1-q^2)(\xi - a)^2_{h,q} + q^2(1-q^2)^2h^2(\xi - a) + (\xi - a)^3_{h,q} \\
&= (1 + [3]_q q)(\xi - a)^3_{h,q} + [3]_q q(1-q^2)h(\xi - a)^2_{h,q} + hq^2(1-q^2)(\xi - a) \\
&\{x - q^2(a+3h) + h\} \\
&= [4]_q(\xi - a)^3_{h,q} + [3]_q qh(1-q^2)(\xi - a^2)_{h,q} + hq^2(1-q^2)(\xi - a)(\xi - q(a+h)) \\
&+ hq^2(1-q^2)(q(a+h) - q^2(a+3h) + h)(\xi - a) \\
&= [4]_q(\xi - a)^3_{h,q} + q(1+q)^2(1-q^2)h(\xi - a)^2_{h,q} \\
&+ hq^2(1-q^2)(q(a+h) - q^2(a+3h) + h)(\xi - a).
\end{aligned}
$$

As $h \to 0$, we have ${}_0\mathcal{D}_q((\xi - a)^4_{0,q}) = \mathcal{D}_q((\xi - a)^4_q) = [4]_q(\xi - a)^3_q$, while because $q \to 1$ we have ${}_h\mathcal{D}_1((\xi - a)^4_{h,1}) = \mathcal{D}_h((\xi - a)^4_h) = 4(\xi - a)^3_h$.
Inductively, it can be seen that this leads to the following results.
As $h \to 0$, we have ${}_0\mathcal{D}_q((\xi - a)^n_{0,q}) = \mathcal{D}_q((\xi - a)^n_q) = [n]_q(\xi - a)^{n-1}_q$.
As $q \to 1$, we have ${}_h\mathcal{D}_1((\xi - a)^n_{h,1}) = \mathcal{D}_h((\xi - a)^n_h) = n(\xi - a)^{n-1}_h$.
If μ is the $q - h$-derivative of μ, i.e., $\mu(\xi) = \mathcal{C}_h\mathcal{D}_q\mu(\xi)$, then μ is called the $q - h$-antiderivative of μ. The $q - h$-antiderivative is denoted by $\int \mu(\xi)\, {}_hd_qx$.

3. $q - h$-Derivative on a Finite Interval

Throughout this section, $I := [a, b]$ for $a, b \in \mathbb{R}$. The $q - h-$derivative on I is provided in the upcoming definition.

Definition 4. *Let $0 < q < 1$, $h \in \mathbb{R}$, and $\xi \in I$, and let $\mu : I \to \mathbb{R}$ be a continuous function. Then, the left $q - h-$derivative $\mathcal{C}_h\mathcal{D}_q^{a^+}\mu$ and right $q - h-$derivative $\mathcal{C}_h\mathcal{D}_q^{b-}\mu$ on I are defined by*

$$\mathcal{C}_h\mathcal{D}_q^{a^+}\mu(\xi) := \frac{\mu((1-q)a+q(\xi+h))-\mu(\xi)}{(1-q)(a-\xi)+qh};\ \xi \neq \frac{qh+(1-q)a}{1-q} := u, \tag{39}$$

$$\mathcal{C}_h\mathcal{D}_q^{b^-}\mu(\xi) := \frac{\mu((1-q)\xi+q(b+h))-\mu(b)}{(1-q)(\xi-b)+qh};\ \xi \neq \frac{-qh+(1-q)b}{1-q} := v, \tag{40}$$

provided that $(1-q)a+q(\xi+h) \in [a,\xi]$ *and* $(1-q)\xi+q(b+h) \in [\xi,b]$. *Also,* $\mathcal{C}_h\mathcal{D}_q^{a^+}\mu(u) = \lim_{\xi\to u}\mathcal{C}_h\mathcal{D}_q^{a^+}\mu(\xi)$ *and* $\mathcal{C}_h\mathcal{D}_q^{b^-}\mu(v) = \lim_{\xi\to v}\mathcal{C}_h\mathcal{D}_q^{b^-}\mu(\xi)$.

We say that μ is left $q-h$-differentiable on $(a, x+h)$ if $\mathcal{C}_h\mathcal{D}_q^{a^+}\mu(\xi)$ exists for each of its points, and we say that μ is right $q-h$-differentiable on $(\xi+h, b)$ if $\mathcal{C}_h\mathcal{D}_q^{b^-}\mu(\xi)$ exists at each of its points. It can be seen that $\mathcal{C}_h\mathcal{D}_q^{a^+}\mu(b) = \mathcal{C}_h\mathcal{D}_q^{b^-}\mu(a)$. In (39), by setting $h=0$, it is possible to obtain the q-derivative defined in Definition 1, i.e., $\mathcal{C}_0\mathcal{D}_q^{a^+}\mu(\xi) = {}_a\mathcal{D}_q\mu(\xi)$. Similarly, for $a=0$ we can have $\mathcal{C}_h\mathcal{D}_q^{0^+}\mu(\xi) = \mathcal{C}_h\mathcal{D}_q\mu(\xi)$, i.e., the $q-h$-derivative in (21) is deduced; for $h=0=a$, we can have $\mathcal{C}_0\mathcal{D}_q^{0^+}\mu(\xi) = \mathcal{D}_q\mu(\xi)$, i.e., the q-derivative is deduced; for $a=0$, $q=1$, we can have $\mathcal{C}_h\mathcal{D}_1^{0^+}\mu(\xi) = \mathcal{D}_h\mu(\xi)$, i.e., the h-derivative is deduced; and for $h=0=a$, taking the limit $q\to 1$, we can obtain the usual derivative for a differentiable function μ, i.e., $\lim_{q\to 1}\mathcal{C}_0\mathcal{D}_q^{0^+}\mu(\xi) = \frac{d}{d\xi}\mu(\xi)$. It is possible to obtain similar results from Equation (40). The definition of left and right $q-$derivatives defined on I can be obtained from (40) by setting $h=0$, as follows.

Definition 5. *Let* $0<q<1$, $h\in\mathbb{R}$, *and* $\xi \in I$, *and let* $\mu : I \to \mathbb{R}$ *be a continuous function. Then, the left* $q-$*derivative* $\mathcal{D}_q^{a^+}\mu$ *and right* $q-$*derivative* $\mathcal{D}_q^{b^-}\mu$ *on* I *are defined as follows:*

$$\mathcal{D}_q^{a^+}\mu(\xi) := \frac{\mu(q\xi+(1-q)a)-\mu(\xi)}{(1-q)(a-\xi)};\ \xi > a, \tag{41}$$

$$\mathcal{D}_q^{b^-}\mu(\xi) := \frac{\mu(qb+(1-q)\xi)-\mu(b)}{(1-q)(\xi-b)};\ \xi < b. \tag{42}$$

It is notable that from (41) we have $\mathcal{D}_q^{0^+}\mu(\xi) = \mathcal{D}_q\mu(\xi)$, i.e., the left $q-$derivative coincides with the $q-$derivative defined in Definition 1.

Definition 6. *Let* $0<q<1$ *and* $\mu : I=[a,b] \to \mathbb{R}$ *be a continuous function. Then, the left* $q-h$*-integral* $I_{q,h}^{a+}\mu$ *and right* $q-h$*-integral* $I_{q-h}^{b}\mu$ *on* I *are defined as follows:*

$$\begin{aligned} I_{q,h}^{a+}\mu(\xi) &:= \int_a^\xi \mu(\gamma)\,{}_hd_q\gamma \\ &= ((1-q)(\xi-a)+qh)\sum_{n=0}^{\infty} q^n\mu(q^na+(1-q^n)\xi+nq^nh),\ \xi>a, \end{aligned} \tag{43}$$

$$\begin{aligned} I_{q,h}^{b-}\mu(\xi) &:= \int_\xi^b \mu(\gamma)\,{}_hd_q\gamma \\ &= ((1-q)(b-\xi)+qh)\sum_{n=0}^{\infty} q^n\mu(q^n\xi+(1-q^n)b+nq^nh),\ \xi<b. \end{aligned} \tag{44}$$

Example 2. *Consider* $\mu(\gamma)=\gamma-a$ *and* $\nu(\gamma)=b-\gamma$. *The left and right* $q-h$*-integrals are calculated as follows:*

$$I_{q,h}^{a+}\mu(\xi) = \int_a^\xi (\gamma-a)\,{}_hd_q\gamma = \frac{(1-q)(\xi-a)+qh}{1-q}\left(\frac{q(\xi-a)}{1+q}+(1-q)h\sum_{n=0}^{\infty}nq^{2n}\right) \tag{45}$$

and

$$I_{q,h}^{b-}\nu(\xi) = \int_{\xi}^{b} (b-\gamma)\,{}_h d_q\gamma = \frac{(1-q)(b-\xi)+qh}{1-q}\left(\frac{b-\xi}{1+q} + (1-q)h\sum_{n=0}^{\infty} nq^{2n}\right), \tag{46}$$

where 1 *is the radius of convergence of the series involved in the above integrals.*

Example 3. *Let $\mu(\gamma) = \xi - \gamma$ and $\nu(\gamma) = \gamma - \xi$; then, we have*

$$I_{q,h}^{a+}\mu(\xi) = \int_{a}^{\xi} (\xi-\gamma)\,{}_h d_q\gamma = \frac{(1-q)(\xi-a)+qh}{1-q}\left(\frac{\xi-a}{1+q} - (1-q)h\sum_{n=0}^{\infty} nq^{2n}\right) \tag{47}$$

and

$$I_{q,h}^{b-}\nu(\xi) = \int_{\xi}^{b} (\gamma-\xi)\,{}_h d_q\gamma = \frac{(1-q)(b-\xi)+qh}{1-q}\left(\frac{q(b-\xi)}{1+q} + (1-q)h\sum_{n=0}^{\infty} nq^{2n}\right), \tag{48}$$

where 1 *is the radius of convergence of the series involved in the above integrals.*

By setting $h = 0$, the left and right q-integrals can be obtained and defined as follows.

Definition 7. *Let $0 < q < 1$ and $\mu : I = [a,b] \to \mathbb{R}$ be a continuous function. Then, the left q-integral $I_q^{a+}\mu$ and right q-integral $I_q^{b}\mu$ on I are provided by*

$$I_{q-0}^{a+}\mu(\xi) = I_q^{a+}\mu(\xi) = \int_a^{\xi} \mu(\gamma)d_q\gamma = (1-q)(\xi-a)\sum_{n=0}^{\infty} q^n\mu(q^n a + (1-q^n)\xi),\ \xi > a, \tag{49}$$

$$I_{q-0}^{b-}\mu(\xi) = I_q^{b-}\mu(\xi) = \int_{\xi}^{b} \mu(\gamma)d_q\gamma = (1-q)(b-\xi)\sum_{n=0}^{\infty} q^n\mu(q^n\xi + (1-q^n)b),\ \xi < b. \tag{50}$$

The left q-integral is the same as the q_a-definite integral defined in [15], while the right q-integral is the same as the q^b-definite integral defined in [18].

Example 4. *Consider $\mu(\gamma) = \gamma - a$ and $\nu(\gamma) = b - \gamma$. By setting $h = 0$ in Example 2, we have $I_{q-0}^{a+}\mu(\xi) = I_q^{a+}\mu(\xi) = \int_a^{\xi}(\gamma-a)d_q\gamma = \frac{q(\xi-a)^2}{1+q}$ and $I_{q-0}^{b-}\nu(\xi) = I_q^{b-}\mu(\xi) = \int_{\xi}^{b}(b-\gamma)d_q\gamma = \frac{(b-\xi)^2}{1+q}$.*

By considering $q \to 1$, we can include the left and right h-integrals in the upcoming definition.

Definition 8. *Let $\mu : I = [a,b] \to \mathbb{R}$ be a continuous function; then, the left h-integral $I_h^{a+}\mu$ and right h-integral $I_h^{b}\mu$ on I are defined as follows:*

$$I_h^{a+}\mu(\xi) = \lim_{q\to 1} I_{q,h}^{a+}\mu(\xi),\ \xi > a, \tag{51}$$

$$I_h^{b-}\mu(\xi) = \lim_{q\to 1} I_{q,h}^{b-}\mu(\xi),\ \xi < b. \tag{52}$$

Note that from Definition 6 we have $I_{q,h}^{a+}\mu(b) = I_{q,h}^{b-}\mu(a) = \int_a^b \mu(\gamma)\,{}_h d_q t$.

4. Some $q-h$-Integral Inequalities for Convex Functions

In this section, we provide inequalities for $q-h$-integrals of convex functions. A function $\mu : [a,b] \to \mathbb{R}$ is called convex if the following inequality holds for all $u, v \in [a,b]$ and $\lambda \in [0,1]$:

$$\mu(\lambda u + (1-\lambda)v) \leq \lambda\mu(u) + (1-\lambda)\mu(v). \tag{53}$$

Theorem 2. *Let $\mu : J \to \mathbb{R}$ be a convex function and let $a, b \in J^{\circ}$ (the interior of J). The left and right $q-h$-integrals satisfy the following inequalities:*

$$I_{q,h}^{a+}\mu(\xi) \leq \frac{(1-q)(\xi-a)+qh}{(1-q)(\xi-a)}\left\{\mu(a)\left(\frac{\xi-a}{1+q}-(1-q)hS\right)\right. \\ \left. +\mu(\xi)\left(\frac{q(\xi-a)}{1+q}+(1-q)hS\right)\right\} \tag{54}$$

and

$$I_{q,h}^{b-}\mu(\xi) \leq \frac{(1-q)(b-\xi)+qh}{(1-q)(b-\xi)}\left\{\mu(\xi)\left(\frac{b-\xi}{1+q}+(1-q)hS\right)\right. \\ \left. +\mu(b)\left(\frac{q(b-\xi)}{1+q}+(1-q)hS\right)\right\}, \tag{55}$$

where $S = \sum_{n=0}^{\infty} nq^{2n}$.

Proof. For $\gamma \in [a, \xi]$, we have $\frac{\xi-\gamma}{\xi-a} \in [0,1]$. By selecting $\lambda = \frac{\xi-\gamma}{\xi-a}$, $u = a$, $v = \xi$ in (53), we obtain the following inequality:

$$\mu(\gamma) \leq \frac{\xi-\gamma}{\xi-a}\mu(a) + \frac{\gamma-a}{\xi-a}\mu(\xi).$$

By taking the $q-h$-integral over $[a, \xi]$, we have

$$\int_a^{\xi} \mu(\gamma)\, {}_h d_q\gamma \leq \frac{\mu(a)}{\xi-a}\int_a^{\xi}(\xi-\gamma)\, {}_h d_q\gamma + \frac{\mu(\xi)}{\xi-a}\mu\int_a^{\xi}(\gamma-a)\, {}_h d_q\gamma.$$

Using the values of the integrals involved in the above inequality from (45) and (47), we can obtain the required inequality (54). On the other hand, for $\gamma \in [\xi, b]$ we have $\frac{b-\gamma}{b-\xi} \in [0,1]$; by selecting $\lambda = \frac{b-\gamma}{b-\xi}$, $u = \xi$, $v = b$ in (53), we obtain the following inequality:

$$\mu(\gamma) \leq \frac{b-\gamma}{b-\xi}\mu(\xi) + \frac{\gamma-\xi}{b-\xi}\mu(b).$$

By taking the $q-h$-integral over $[\xi, b]$, we have

$$\int_{\xi}^{b} \mu(\gamma)\, {}_h d_q\gamma \leq \frac{\mu(\xi)}{b-\xi}\int_{\xi}^{b}(b-\gamma)\, {}_h d_q\gamma + \frac{\mu(b)}{b-\xi}\int_{\xi}^{b}(\gamma-\xi)\, {}_h d_q\gamma.$$

Using the values of the integrals involved in the above inequality from (46) and (48), we can obtain the required inequality (55). □

Corollary 1. *As an application of the above theorem, the following inequalities for left and right q-integrals hold:*

$$I_q^{a+}\mu(\xi) \leq \mu(a)\left(\frac{\xi-a}{1+q}\right) + \mu(\xi)\left(\frac{q(\xi-a)}{1+q}\right) \tag{56}$$

and

$$I_q^{b-}\mu(\xi) \leq \mu(\xi)\left(\frac{b-\xi}{1+q}\right) + \mu(b)\left(\frac{q(b-\xi)}{1+q}\right). \tag{57}$$

Remark 2. *By taking $\xi = b$ in (56) or $\xi = a$ in (57), we can obtain the following inequality:*

$$\frac{1}{b-a}\int_a^b \mu(\gamma)\, {}_a d_q \gamma \leq \frac{\mu(a) + q\mu(b)}{1+q}. \tag{58}$$

The above inequality (58) is independently proved in ([15], Theorem 12).

The following lemma is required to prove the next result.

Lemma 1 ([19])**.** *Let $\mu : [a,b] \to \mathbb{R}$ be a convex function. If μ is symmetric about $\frac{a+b}{2}$, then the inequality*

$$\mu\left(\frac{a+b}{2}\right) \leq \mu(\xi), \tag{59}$$

holds for all $\xi \in [a,b]$.

Theorem 3. *If μ is symmetric about $\frac{a+b}{2}$ and the assumptions of Theorem 2 are satisfied, then the following inequality holds:*

$$\mu\left(\frac{a+b}{2}\right) \leq \frac{1-q}{(1-q)(\xi-a)+qh}\int_a^\xi \mu(\gamma)\, {}_h d_q \gamma + \frac{1-q}{(1-q)(b-\xi)+qh}\int_\xi^b \mu(\gamma)\, {}_h d_q \gamma, \quad \xi \in [a,b]. \tag{60}$$

Proof. A convex function that is symmetric about $\frac{a+b}{2}$ satisfies the inequality in (59); therefore, by taking $q-h$-integration of (59) over $[a,\xi]$ we have

$$\mu\left(\frac{a+b}{2}\right)\frac{(1-q)(\xi-a)+qh}{1-q} \leq \int_a^\xi \mu(\gamma)\, {}_h d_q \gamma. \tag{61}$$

On the other hand, by taking the $q-h$-integration of (59) over $[\xi, b]$, we have

$$\mu\left(\frac{a+b}{2}\right)\frac{(1-q)(b-\xi)+qh}{1-q} \leq \int_\xi^b \mu(\gamma)\, {}_h d_q \gamma. \tag{62}$$

By adding (61) and (62), we obtain the inequality in (60). □

Remark 3. *By taking $x = b$ in (61) or $x = a$ along with $h = 0$ in (62), we can obtain the following inequality:*

$$\mu\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \mu(\gamma)\, d_q \gamma. \tag{63}$$

The above inequality (63) is independently proved in ([15] Theorem 3.2). Unfortunately, the proof is not correct; see ([20] Example 5). Here, we have imposed an additional symmetric function condition to ensure the result. Hence, if we impose a condition of symmetry in addition to the assumptions in ([15] Theorem 3.2), we obtain the correct result.

5. Conclusions

This article provides a base for unifying the theory of $q-$ and h-derivatives provided in [16] by Kac and Cheung. The notion of a $q-h$-derivative that generates the q-derivative and h-derivative is introduced. The $q-h-$binomial $(\xi-a)^n_{h,q}$ analogue to $(\xi-a)^n$ is defined, which generates the $q-$binomial $(\xi-a)^n_q$ and $h-$binomial $(\xi-a)^n_h$ in particular. The $q-h$-derivatives of the $q-h-$binomial $(\xi-a)^n_{h,q}$ are found, which generate the q-derivative of the $q-$binomial $(\xi-a)^n_q$ and h-derivative of the $h-$binomial $(\xi-a)^n_h$ in particular. The rest of the theory in [16] needs further attention from researchers, as it may be unified in a similar way to the $q-h$-derivative and $q-h-$binomial. In addition, the $q-h$-

derivatives and integrals are defined on an interval $[a, b]$, which is used to establish some inequalities linked to recent research and provide a corrected proof of the inequality in [15]. "The composite derivatives and integrals create the opportunity to study theoretical and practical concepts and problems of different fields related to q-derivative and h-derivative simultaneously".

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