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# Students' Epistemological Framings When Solving an Area Problem of a Degenerate Triangle: The Influence of Presence and Absence of a Drawing

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**Abstract:** This study explores the epistemological framings of undergraduate students when solving an area problem of a degenerate triangle, without or with a triangle drawing. Through mixed research with a triangulation design, the resolution processes and responses of students were analyzed. The aim was to analyze how students' epistemological framing changes during the problem-solving process depending on whether the task contains the drawing of the triangle or not. Quantitative results show significant differences between students who solve the problem without a triangle drawing and those who do. Qualitative results evidence that students who solved the problem with the drawing established an initial epistemological framing that contained an "obvious fact": the non-zero area of the triangle. They hardly modified this epistemological framing during the solving process, forcing the response to be a positive number. In contrast, students who solved the problem without the drawing easily modified their initial epistemological framing by observing that the area of the triangle was zero. Students' perceptions of the level of difficulty of the problem are discussed, too.

**Keywords:** degenerate triangle; impossible triangle; epistemological framing

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## 1. Introduction

Triangles are the simplest form of polygons, and their basic geometrical properties have been known for many centuries. Heron (10–70 d. C.) knew how to calculate the area of the triangle with side lengths 13, 14, and 15 in two different ways [1,2]. One way was to find out that the height on side 14 is 12, giving the area a value of 84. The other way was to use an algorithm, now known as "Heron's formula", which can be applied when the lengths of the triangle sides are known as:

$$Area = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{a+b+c}{2}$, $a$, $b$, and $c$ are the lengths of the triangle sides.

In mathematics at school, students are supposed to learn: (a) which lengths can or cannot form the sides of a triangle [3]; (b) how to calculate the area of different kinds of triangles [4]; and (c) how to calculate the angles of a triangle if their side lengths are given [5,6]. The first objective is learned by students if they comprehend and know how to apply in different tasks the so-called "third side rule": The sum of any two sides of a triangle must be longer than the third side. Combining the first objective with the third objective leads to a classification of triangles into three types: regular triangles, degenerate triangles, and impossible triangles.

Fisher et al. [5] gave examples of these triangle types, asking students to use the "Law of cosines" to find all angles of the triangles with the following side lengths: (a) 2, 4, and 5; (b) 2, 3, and 5; (c) 2, 3, and 6. It turns out that the first triangle has angles of 22°, 108°, and

50°, making it a "common triangle". The second triangle should have two zero angles and one angle equal to 180°, making it a "degenerate triangle". The third triangle would not have real angles, being an "impossible triangle".

Applying Heron's formula to these examples, one finds additional characteristics: the area of the "common triangle" is a positive number (square root from 14.4375 or, approximately, 3.7997); the area of the "degenerate triangle" is zero; and the area of the "impossible triangle" is an imaginary number (square root from 24.0625 or, approximately, 4.9*i* (the number 4.9*i* is a complex number, more precisely a purely imaginary number, where $i = \sqrt{-1}$)).

In the literature, we can find examples where students are asked to calculate the angles of degenerate or impossible triangles. For example, in Barton [6], the degenerate triangle of sides lengths 7, 9, and 16 and the impossible triangle of sides lengths 17, 29, and 48 are provided; in Foerster [7], the impossible triangle with sides lengths 3, 5, and 10; and in Carter [8], the impossible triangle of sides lengths 5, 12, and 18 is provided.

Martin Gardner also used the notion of a "degenerate triangle" with zero area in his answers to two corresponding problems. The first example is shown in the following dialog [9] (p. 17):

> Which has the larger area? A triangle with sides of three, four, five, or a triangle with sides of 300, 400, and 700?
>
> The second one, naturally.
>
> Wrong!... The second triangle is degenerate. It's a straight line. Its area is zero.

The second example is shown in Figure 1 [10] (p. 102).

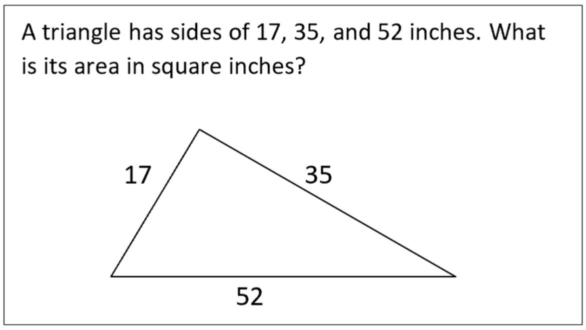



**Figure 1.** Figure adapted from Gardner [10] (p. 102).

Since we have used this problem in our study, it is important to point out that Gardner [10] has emphasized that these problems are short and easy, but each of them contains a funny twist that gives an unexpected turn to the answer. Gardner justifies the inclusion of these problems as follows: "A truly creative mathematician of scientist must have a mind that is constantly on the alert for surprising, off-beat angles" [10] (p. 100).

Gardner's answer to the problem in question is as follows [10] (p. 108): "A 'triangle' with the sides given would be a straight line (mathematicians sometimes call this a "degenerate triangle"), so it would have no area at all. This result shows that the students' perception of the connections between two problem parts is not a simple "attention" process but rather depends on the activation of adequate or inadequate knowledge or conceptual and procedural resources.

It is important that students become familiar with this type of problem in order to learn what degenerate figures are all about and that it is a mistake to draw them as if they were ordinary figures, because this type of figure is often found in the problems that are posed in textbooks [11–15] and teachers often do not recognize them either [16]. It would be very beneficial if these degenerate figures were easily recognized by teachers, appearing more in everyday geometry lessons and serving as an opportunity for learning and reflection.

An appropriate conceptual perspective from which to examine the process of solving this type of problem is the "resource framework", in particular the "epistemological framing". The resource framework is a model based on previous work in sociology by Goffman [17] and cognitive science by Minsky [18] and Schank [19]. It has been used in studies in science education [20] and specifically in engineering education [21], physics teaching [22,23], and mathematics education [24–26]. We will explain these concepts in detail below.

Thinking and learning are enormously complex processes. Thought is dynamic, with fragments of knowledge being freely associated and appearing in ways that may depend heavily on the individual's perception of the context and environment [21].

A particular theoretical way of thinking about student thinking and learning is the resource framework. It is a framework rather than a theory because it provides ontologies (classes of structural elements and the way they behave) and allows for a variety of possible structures and interactions built from these elements [23].

In phenomenological terms, by frame, we mean a set of expectations that an individual has of the situation in which he finds himself and that affect what he notices and how he intends to act. It is a person's framing of a situation, which can have many aspects, including social ("Who do I expect to interact with here and how?"), affective ("How do I expect to feel about this?"), epistemological ("What do I hope to do to answer questions and construct new knowledge?"), and others [27].

The basic ontology of the resource framework is that of an associative network based on the well-established metaphor of neurons in the brain. Everything that individuals have learned—a thought or the perception of a particular object—is stored in the state of a group of neurons, specifically in the strength of the synaptic connections of that specific group of neurons [21]. These groups of neurons can be considered a unit from the point of view of perception and functionality of the thinker. We refer to the cognitive elements of each group of neurons as a resource [27].

Just as the activation of one group of neurons leads to the activation of other groups of associated neurons, thoughts are represented as resources or groups of resources with strong associations that are not isolated, as one resource or group of resources leads to the activation of other resources [23].

As Bing and Redish [23] explain, this enables the creation of networks of higher-level structures that are perceived as unified by the user. This interconnection can occur at many levels and can be narrow or loose. They are narrow when they almost always manifest together, e.g., when we hear the word "cat" and the associated image of that feline appears in our mind; and loose when they are not so closely connected, e.g., when we can hear an orchestral performance as a whole or individually of a particular instrument. It is an extremely dynamic process, as associations between resources and groups of resources are activated and inhibited depending on the context.

Analyzing thinking in terms of associated resources or groups of resources leads us to suggest that students construct alternative associative pathways when misconceptions are treated as unary and instruction focuses on replacing these misconceptions with correct ones [21].

The resource framework also has a control structure. Control refers to the process by which certain resources are selected for activation rather than other resources that may be relevant [22]. From the wealth of information that reaches individuals and given their limited attentional and processing resources, they develop schemas that determine where they focus most of their attention, which parts of their memory and long-term skills they activate and apply, and which they suppress [23]. This control structure is highly context-dependent and controls which resources we make available under certain circumstances [21].

Following the structure proposed by Bing and Redish [23], epistemological resources are a component of the control structure of the model. They can be thought of as a package of information that, when activated, leads the person to interpret the knowledge in question from a particular perspective and controls which conceptual resources are used. They are a variety of resources that are used to build new knowledge and solve problems [22]. Epistemological resources, like other resources, are dynamic; they can be turned on and off during the course of a cognitive activity [23].

The process of reducing all epistemological and conceptual options to a manageable size that can be considered by the individual is called an epistemological frame [22]. Bing and Redish [23] explain it as a student's perception or judgment of the kind of knowledge that is appropriate to apply in a given situation.

The difference between a framework and a framing from MacLachlan and Reid's perspective [28] is that a framework is an individual's interpretation of "what is going on here?" and the form of gerund framing emphasizes interpretation as an ongoing process. An analysis in terms of epistemological framing focuses on the moment-to-moment changes observed in students' reasoning, as their interpretation of the task and the knowledge in question may change [23], for example, when a topic is discussed in class or a task is solved.

Finally, Felicity's condition is the tacit premise naturally adopted by an individual that incoming information, whether spoken, read, observed, and so on, comes from a rational source and depends on the individual's attempt to contextualize and interpret that incoming information [23].

In this study, we want to investigate how students deal with a verbal problem that contains an incorrect figure and how this differs from students who solve the verbal problem without such a figure. We ask how the figure affects students' epistemological framing when they are confronted with these situations and how it affects or does not affect their resolution processes.

For this reason, our research question in this paper is: How does students' epistemological framing change during the problem resolution process depending on whether the verbal problem contains the triangle drawing or not? The aim is to analyze how students' epistemological framing changes during the problem resolution process depending on whether the verbal problem contains the drawing of the triangle or not. The results will allow us to better understand how students deal epistemologically with ill-posed problems.

The hypothesis is that students who solve the verbal problem without the triangle drawing will perform better than students who solve the verbal problem with the triangle drawing.

## 2. A Brief Review of Problem Statements Involving Uncommon Triangles

The problems of degenerate or impossible triangles have been studied for a long time. In the first case of degenerate triangles, we can establish that, as far as we know, the first author to use a degenerate triangle in a mathematics textbook was the Italian mathematician Lorenzo Forestani in 1603 [29] (pp. 285–285A) as follows:

> And saying, it is the triangle A.B.C. with the base B.C. which is 20 "canna" and A.B. of 12 "canna", and A.C. of 8 "canna", it is asked how much is the surface [i.e., area]. You should know, that if the side A.B. is summed with the side A.C., and their sum is not longer than the base B.C., this question cannot be answered, however similar questions might be given to people with low knowledge.

The problem was complemented by the author with a drawing like Figure 2.

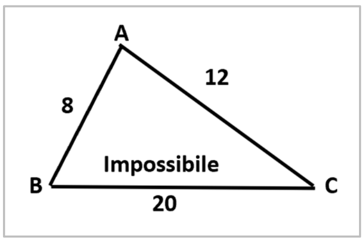



**Figure 2.** Drawing adapted from Forestani's illustration [28].

According to the terminology introduced above, the triangle presented by Forestani is not impossible but a degenerate one. As Forestani stated the problem, he did not pay too much attention to the drawing details. The text says that AB is equal to 12, but in the drawing, it is 8. Similarly, the text says the side AC is 8, but in the drawing, it is 12. This error was not corrected even in the revised edition, published in the year 1682 [30].

Another interesting problem, related to a triangle that becomes degenerate when two of its sides are doubled, was given to high school students in an Annual Contest organized by the Mathematical Association of America in 1961 [31] (p. 10):

> In triangle ABC, AB = 12, AC = 7, and BC = 10. If sides AB and AC are doubled while BC remains the same, then: (A) the area is doubled, (B) the altitude is doubled, (C) the area is four time the original area, (D) the median is unchanged, (E) the area of the triangle is 0.

A special kind of impossible triangle is one that has a supposed element that contradicts its basic characteristics. An example is the triangle in the following problem: "Find the area of the right-angled triangle if its hypotenuse is 10 cm and the height dropped on the hypotenuse is 6 cm" [32] (p. 3). This triangle is impossible because the maximum height on the 10 cm hypotenuse of a right-angle triangle is 5 cm.

In the same way, questions related to the area of degenerate triangles appear often in puzzle books. Table 1 brings three examples in which the sought answer is "zero", with or without an argument. The first two examples show a rather common phenomenon of formulating a new problem with only minimal changes.

**Table 1.** Three puzzles related to the area of degenerate triangles.

| Puzzle | Authors' Answer |
|---|---|
| "A triangle has sides of 17, 42, and 59 inches. What is its area?" [33] (p. 66). | "Since 17 + 42 = 59, this is a very skinny triangle. Its area is zero" [33] (p. 187). |
| "A triangle has sides 17 cm, 42 cm, and 59 cm. Find its area" [34] (p. 26). | "Since 17 + 42 = 59, what results is a very skinny triangle. Its area is zero" [34] (p. 184). |
| "Which triangle is larger—one with sides measuring 200, 300, and 400 cm or one with sides measuring 300, 400, and 700 cm?" [35] (p. 171). | "The first triangle is larger-one with sides measuring 200, 300, and 400 cm. The triangle with sides measuring 300, 400, and 700 cm has an area of zero!" [35] (p. 273). |

In the second case regarding impossible triangles, in contemporary mathematics textbooks, it is observed, in practical activities, how the triangle inequality is learned through this type of problem. For example, Tanton [36] (p. 26) provides the following problems: "(a) Draw a triangle with sides 6 inches, 3 inches, and 2 inches, and (b) Draw a triangle with sides 67 inches, 23 inches, and 95 inches".

Some suggested problems have a less clear learning path and gain for example, in Francis [37] (p. 9), the following problem is established: "The sides of a triangle measure 3, 4, and 8. Find the perimeter and the area of this triangle". Although the author states that the problem "relates to the triangle-inequality and the assertion that a straight line is the shortest path between two points" (p. 9), it is quite uncertain what result students would give for the area of the triangle.

For Nitsche, a degenerate triangle is one "whose vertices lie on a straight line and whose area is zero" [38] (p. 202). A similar definition is given by Pogonowski more recently [39] (p. 79): "A degenerate triangle is one with collinear vertices and zero area." O'Rourke uses the same elements to define a degenerate triangle [40] (p. 148): "If you collapse a triangle so that one angle becomes 180° and the other two 0°, the resulting shape (which would appear visually as a line segment) can be considered a zero-area degenerate triangle."

Some authors give both answers ("the area is zero" or "the triangle does not exist"). MacHale is one of the advocates of two answers for the problem related to the area of a degenerate triangle. In 2006, his problem had the formulation [41] (p. 29): "What is the area of a triangle whose sides measure three meters, four meters and seven meters?" The answer was [41] (p. 72): "Its area is zero (or there is no such triangle)."

In 2015, the problem formulation differed in numbers and units, as established by Sloan and MacHale [42] (p. 7): "What is the area of a triangle whose sides measure 400 feet, 700 feet, and 300 feet? Two-answer approach was much more elaborated [42] (p. 48):

> There are really two answers to this problem, but they come to pretty same thing. Sticklers in mathematics would say there is no such triangle because the sum of the lengths on any two side of a triangle must be greater than the length of the third side, and here 700 = 400 + 300. Less fussy people would say that the area of the triangle is zero because it has collapsed to a line segment. You can take your choice.

In his extensive psychological research project on the mathematical abilities of schoolchildren, Krutetskii [43] used impossible triangles in some tasks in the area of so-called "unrealistic problems". These are the problems "whose numerical facts make the problem meaningless" [43] (p. 132). Two examples were: "The perimeter of a right triangle is equal to 3.72 m. Two of its sides are 1.24 m each. Find the third side" [43] (p. 132). And "What is the area of an isosceles right triangle with leg equal to 5a cm and hypotenuse equal to 12a cm?" [43] (p. 133). Krutetskii did not report on the performance of the children tested in solving these "unrealistic problems".

## 3. Method

### *3.1. Research Focus*

The research conducted in this paper was a mixed-methods study with a triangulation design, i.e., the researcher uses quantitative and qualitative methods to investigate the same phenomenon and determine whether the two perspectives converge to a unified understanding of the research problem [44]. The qualitative part consisted of a content analysis of the students' written productions. The quantitative part dealt with an experimental design with two groups, posttest only, in which Fraenkel et al. [44] explain that there are two groups, both randomly assigned. One group receives the experimental treatment, the other does not, and both groups are then subjected to a posttest.

### *3.2. Population and Sample*

The population consisted of first-year students at the Facultad de Ciencias de la Electrónica of the Benemérita Universidad Autónoma de Puebla, in Mexico. The experimental and control groups were determined by two-stage cluster random sampling. In the first stage, four groups were randomly selected from a total of 22 groups in the population. In the second stage, each of these four groups was again randomly divided into two subgroups. The experimental group consisted of students belonging to the union of half of the subgroups, and the control group consisted of the union of the other half of the subgroups. The experimental group of students solved the problem without the triangle drawing (hereafter referred to as "Group A"), and the control group solved the problem with the triangle drawing (hereafter referred to as "Group B"). Group A consisted of 37 students with an average age of 19.30 years, with the highest age being 26 years and eight months and the lowest age being 18 years (SD 1.41). Group B consisted of 41 students with an average age of 19.33 years, a maximum age of 21 years and 9 months, and a minimum age of 18 years (SD 0.94).

### *3.3. Instrument*

The data collection instrument consisted of Gardner's degenerate triangle problem [10] in two versions, with the drawing of the triangle (see Figure 1) and without it. In addition, in both versions of the instrument, students were asked to:

(a) Describe verbally, without using a formula, the procedure you will use;

(b) Execute the plan mathematically and state the solution; and, once the problem is solved;
(c) Rate the difficulty of the task, with one option to choose from: *very difficult*, *difficult*, *normal*, *easy*, *and very easy*.

### 3.4. Data Collection and Analysis Procedure

The data collection procedure was carried out as follows: The experimental group was asked to solve the task without the triangle drawing (Figure 1), and the control group was asked to solve the task with the triangle drawing. The time required to solve the task was one hour.

The data analysis was carried out as follows: First, the qualitative analysis was carried out, followed by the quantitative analysis. The qualitative analysis was inductive. First, the students' responses were classified according to the type of solution method. Then, the solution processes of each type were analyzed in detail, both for the students who solved the problem with the drawing of the triangle and for those who solved it without it. As a result of the analysis, two categories were formed: students who had changed their epistemological framing during the solution process and students who had not. Finally, the quantitative study was conducted with a hypothesis test on the students' performance in solving the problem. The analysis was first conducted by one of the researchers and then reviewed by the other investigator to ensure the objectivity of the analysis. The agreement rate was approximately 90%.

## 4. Results and Analysis

The analysis of the results is presented in this order: first the quantitative results, then the qualitative analysis. We refer to the students from group A as A1, A2, . . ., A37, and the students from group B as B1, B2, . . ., B41. "The area of the triangle is zero" was scored as the correct answer.

### 4.1. Quantitative Analysis

As for the overall results, Table 2 shows the percentage of correct and incorrect answers in each group. The group of students who solved the problem without the triangle drawing (group A) had a better performance than the group who solved the problem with it (group B). First, it was found that the presence of the triangle drawing influenced the solution processes of the students in group B as they tried to obtain a non-zero area value. In contrast, the students in Group A, who solved the problem without drawing the triangle, had more freedom to reason with the data and, of course, to realize that it was not possible to obtain a triangle with these side lengths.

**Table 2.** Percentages of correct and incorrect answers in each group.

| | Group A (No Triangle) | Group B (With Triangle) |
|---|---|---|
| Correct answers | 67.6% | 43.9% |
| Incorrect answers | 32.4% | 56.1% |

In order to find out whether these results show significant differences, a hypothesis test for two independent proportions was carried out. At a significance level of $\alpha = 0.05$, we obtain a test statistic $z = 2.1$ and a $p$-value of 0.02. Therefore, conclude that there is sufficient evidence to support the claim that the proportion of correct answers to the task without the triangle drawing is significantly higher than the proportion of correct answers to the task with the triangle drawing.

As far as the solution strategies chosen by the students are concerned, Table 3 shows the most common.

**Table 3.** Solution strategies used by students to solve the problem.

| Resolution Strategy | Group A (No Triangle) | Group B (With Triangle) |
|---|---|---|
| Heron's formula | 43.2% | 39% |
| Triangle inequality | 32.4% | 22% |
| Pythagoras theorem | 8.1% | 22% |
| Trigonometric ratios | 0% | 7.3% |
| Area formula | 8.1% | 7.3% |
| Other | 8.1% | 2.4% |

In both groups, the solution strategy with the highest incidence was Heron's formula. If we look closely at the specific results of the students' application of this formula, we see that the students in group A (without the triangle drawing) had more correct answers (81.8%) than incorrect answers (18.2%). For the students in group B (with a triangle drawing), however, the result was the opposite: 28.6% correct answers and 71.4% incorrect answers.

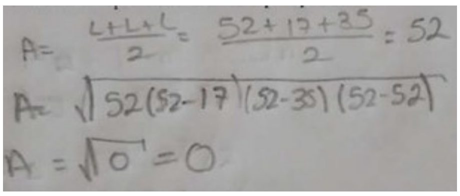
Regarding the students' perception of the difficulty of the task, the frequency obtained by each group is shown in Table 4. We first noted that the students in group A had higher percentages in regular and difficult categories, namely 86.5% in both categories, while group B obtained 65.9% in the same categories. It follows that the students in group A perceived the difficulty of the task as greater but ended up with more correct answers. In group B, on the other hand, the percentage was lower, but more students answered incorrectly. Note that 12.2% of the students in group B rated the problem as *easy* and 17.1% did not answer, which was not the case in group A.

**Table 4.** Frequency by difficulty level of the problem.

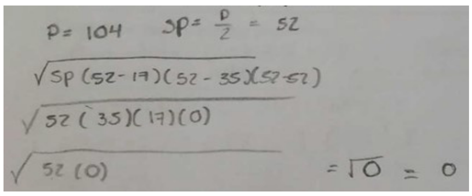
| | Group A (Without Triangle Drawing) | Group B (With Triangle Drawing) |
|---|---|---|
| Very difficult | 2.7% | 4.9% |
| Difficult | 40.5% | 29.3% |
| Regular | 45.9% | 36.6% |
| Easy | 5.4% | 12.2% |
| Very easy | 5.4% | 0% |
| Unanswered | 0% | 17.1% |

### 4.2. Qualitative Analysis

The results of the qualitative analysis of the students who solved the problem with the Heron formula are as follows: The analysis showed that the students who solved the task without drawing the triangle readily accepted the arithmetic result equal to zero, but the students who solved the task with drawing the triangle did not, forcing a non-zero result. Figure 3 shows the performance of students A9 and A14 from group A as an example. These students established an initial epistemological framing that contained no figure. The epistemological resource associated with the solution method of Heron's formula allowed them to develop a fluid and unchanging solution process. In this way, it is interpreted that they did not change their initial epistemological framing.



A9



A14

**Figure 3.** Correct procedures of two students in Group A.

In contrast, the students in group B with the triangle drawing mostly made the same math error, as can be seen in Figure 4. In the case of student B6, he eliminates the factor (52–52) to obtain a result that is not zero. In the case of student B22, he ignores the factor zero in the next step, even though he already had it. These students created an initial epistemological framing that includes the figure. In other words, they recognized that they had to find an area value greater than zero. By applying the epistemological resource of Heron's formula method, most developed a resolution process with the error of omitting a zero factor because they must obtain a non-zero result. In this way, it is interpreted that they could not change their initial epistemological framing by providing an erroneous result.

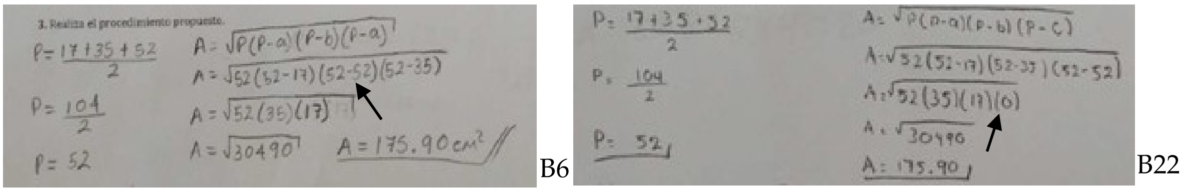



**Figure 4.** Incorrect procedures of two students in Group B.

Another solution strategy that was used more frequently in both groups was triangular inequality, with a frequency of 32.4% in group A and 22% in group B. In this strategy, the students added the value of the two smaller sides of the triangle with lengths 17 and 35 to obtain the length of the third side, 52, from which they concluded that the triangle could not be formed. Figure 5 shows the graph of one student from each group, both with correct results and almost the same drawing. Of note is the change in the epistemological framing of the students in Group B, who ignore the triangle drawing in the task and conclude that the triangle is impossible. These Group B students changed their initial epistemological framing during the solution process. Despite the figure in the problem, they recognized that the triangle had been reduced to a line and that the triangle had degenerated. This result suggests that some students may change their initial epistemological framing during the solution process, even if it is only in a few cases.

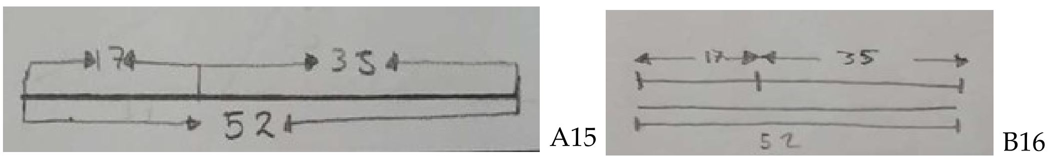
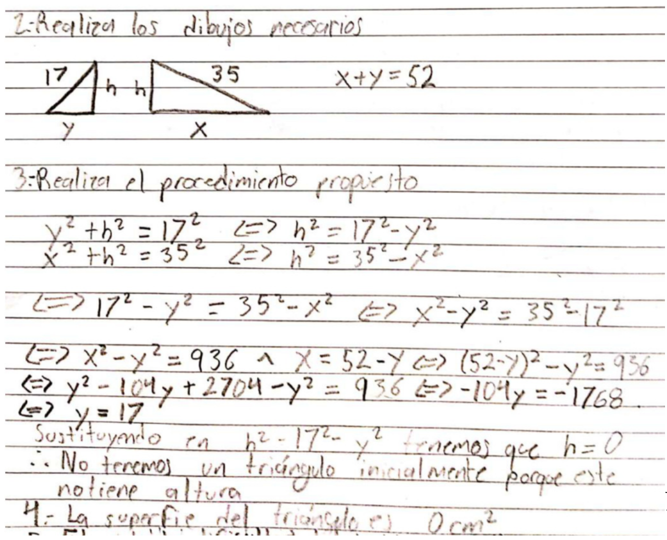



**Figure 5.** Graphical representation of a student from each group in the triangle inequality strategy.

The Pythagorean Theorem strategy was used more frequently by students in group B (22%) than by those in group A (8.1%). We interpreted that this higher frequency in group B was due to the presence of the figure. We interpret that the students of Group B, who had the figure of the triangle in the presentation of the problem, were able to activate more easily the epistemological resource of the method of Pythagoras' theorem due to their initial epistemological framing. This situation did not occur in the group of students who did not have a triangle drawing.

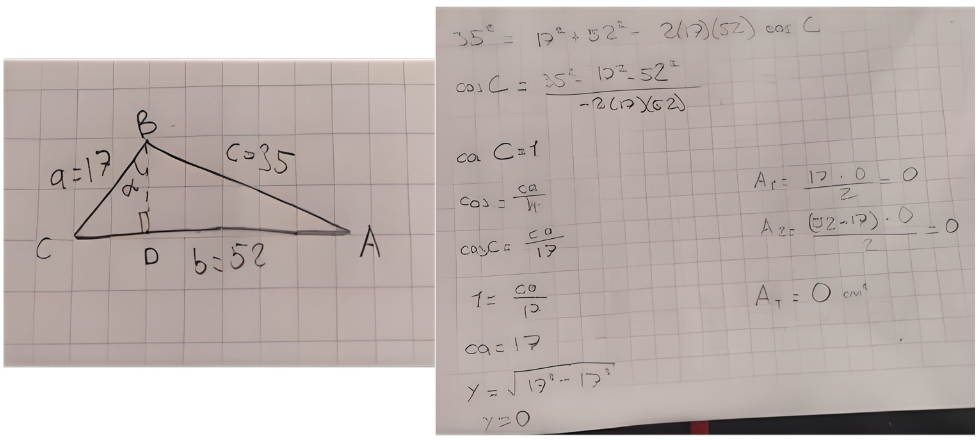
Of all the students in both groups who used the Pythagorean Theorem strategy, only one student from Group B developed the correct procedure, as can be seen in Figure 6. The student performs the whole procedure to draw the correct conclusion: "We do not have a triangle to begin with because it has no height [...] The surface area of the triangle is 0 cm$^2$". This student has a first epistemological framing with the triangular figure. He realized the mathematical procedure with this figure, and during the process, he changed his epistemological framing to accept that the triangle does not exist. In this way, he changed his initial epistemological framing to one that admitted that the triangle does not exist. Furthermore, this student classifies the problem as easy and trusts his solution.




B4

**Figure 6.** A student's procedure when using the strategy of the Pythagorean theorem.

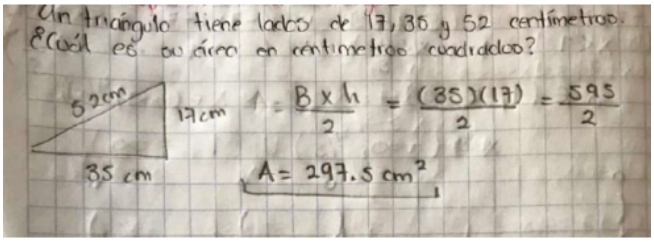
Very few students chose trigonometric ratios or the usual area formula as a solution strategy. Students in both groups arrived at incorrect results, with the exception of one student from Group B who correctly applied the cosine law and trigonometric ratios and obtained an area result equal to zero, as shown in Figure 7. This Group B student has also changed his initial epistemological framing and now accepts that there is no triangle. He said that the difficulty of the task was high, so we interpreted it to mean that he found it difficult to accept that the triangle had no height and its area was therefore zero.




B37

**Figure 7.** Solution strategy by trigonometric functions performed per student B37.

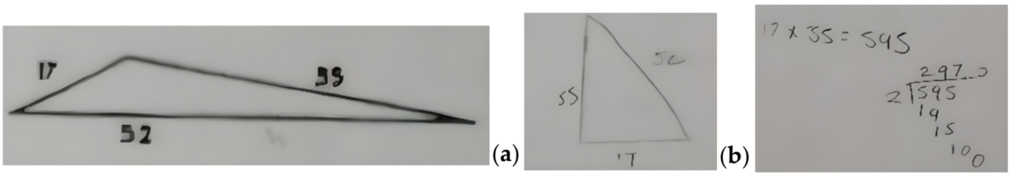
For the area formula method, Figure 8 shows the performance of student A27, who incorrectly draws a right triangle and incorrectly obtains an area of 297.5 cm$^2$. However, she says that she found the problem difficult because "I was unsure [. . .] as I had doubts about how to arrange the data of the triangle, and it is the case that using different base and height data gives a different area". In this way, we could observe that the student set up an incorrect initial epistemological framing by assuming that there was a right-angled triangle. During her solution process, she was unsure how to classify the length of the sides. In the end, she was not sure if her answer was correct. She also did not realize that the lengths of the sides contradicted Pythagoras' theorem.




A5

**Figure 8.** Performance of student A5 with the area formula strategy.

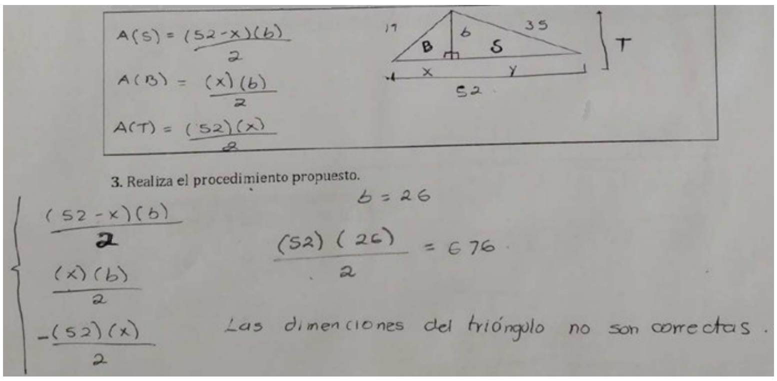
In the case of student B18, she also places the data in a right-angled triangle (Figure 9b) and calculates the area in exactly the same way as student A5 (Figure 9c). She also classifies the task as difficult and argues that she had doubts about the correctness of her answer because she could no longer remember some things. We interpreted that the student rotated the triangle given in the task (Figure 9a) on the assumption that it was a right-angled triangle in order to use a formula that she remembered. This student had difficulty selecting an appropriate epistemological resource to solve the problem. She only remembers the formula base times height over two, applies it, but is always unsure how to proceed. This is the reason she thinks the problem is difficult. She changes her initial epistemological framing by replacing the figure with one that she recognizes.




(**a**) (**b**) (**c**) B18

**Figure 9.** Response from student B18.

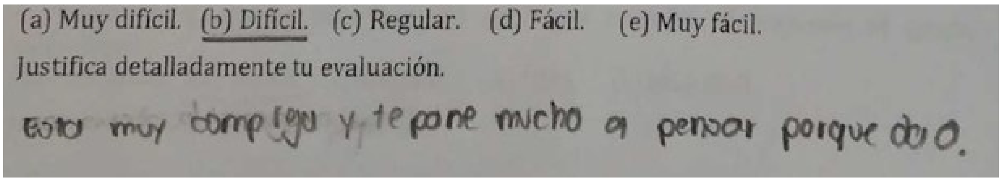
In the case of student B11, she suggests the following solution strategy: "To split the triangle into two parts, we get two new [triangles] with the bases $x$ and $y$ and the height b. From this data, we form a system of equations". Her procedure is shown in Figure 10. She correctly calculates the area of the right-angled triangles B and S, but not that of the entire triangle. Then she writes b = 26 without justification and calculates the area to be 676 cm$^2$. At the end, however, she says "The dimensions of the triangle are not correct." This means that she has somehow realized that something is wrong, but she cannot see the reason. She classifies the task as regular.




B11

**Figure 10.** Performance of student B11 in the strategy area formula.

Regarding the students' perception of the difficulty of the task, Figure 11 shows the answer of student A14, who solved the task correctly using the Heron formula. He rated the task as difficult because, in his own words, "it is very complex and makes you think a lot because it gives 0". This comment is evidence that this student had a conflict with his original epistemological framing, which he needed to change. As he went through his solution process, he realized that the area was zero and wondered if that made sense. Since this student belonged to group A, in which the problem did not contain the illustration, he was able to modify his epistemological framing more easily during the solution process.


(a) Muy difícil. (b) Difícil. (c) Regular. (d) Fácil. (e) Muy fácil.

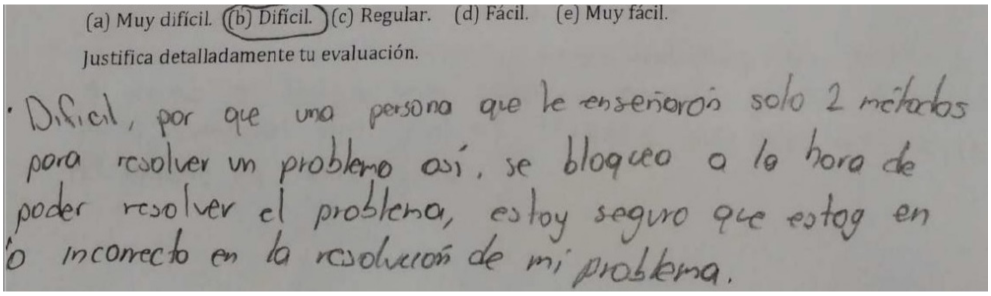
Justifica detalladamente tu evaluación.

Esta muy complejo y te pone mucho a pensar porque da 0.

A14

**Figure 11.** Student comment A14 about the degree of difficulty of the problem.

Another comment from a student in group A can be seen in Figure 12. Student A2 solved the problem correctly using the Heron formula, but he states: "[. . .] I am sure that I am wrong in solving my problem" and rates it as *difficult*. This result is evidence that his original epistemological framing remains, because although he has carried out the correct procedure, he does not believe in the result he has obtained.


(a) Muy difícil. (b) Difícil. (c) Regular. (d) Fácil. (e) Muy fácil.

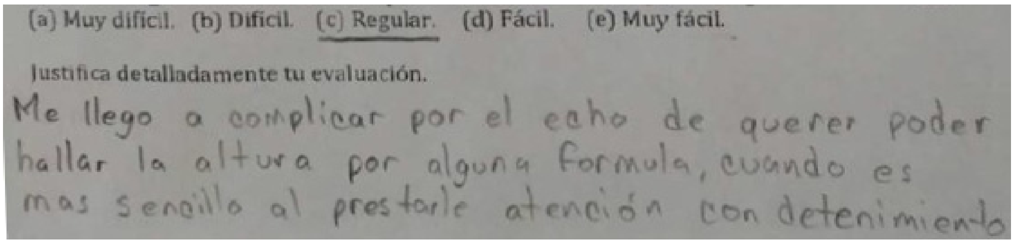
Justifica detalladamente tu evaluación.

· Dificil, por que una persona que le enseñaron solo 2 metodos para resolver un problema así, se bloquea a la hora de poder resolver el problema, estoy seguro que estoy en lo incorrecto en la resolución de mi problema.

A2

**Figure 12.** Student comment A2 about the degree of difficulty of the problem.

On the other hand, Figure 13 shows the comment of student B16 from Group B, who tried to solve the problem by unsuccessfully calculating the height of the triangle and then solved it when he realized that the sum of the two small sides was equal to the larger side. This student established the problem difficulty as *regular*. He said, "It was complicated for me by the fact that I wanted to find out the height by a formula, although it is easier to observe it exactly". This student was able to successfully change his initial epistemological framing by trying to find the height of the triangle and only later realizing that the triangle did not exist.


(a) Muy difícil. (b) Difícil. (c) Regular. (d) Fácil. (e) Muy fácil.

Justifica detalladamente tu evaluación.

Me llego a complicar por el echo de querer poder hallar la altura por alguna formula, cuando es mas sencillo al prestarle atención con detenimiento

B16

**Figure 13.** Student comment B16 about the degree of difficulty of the problem.

## 5. Discussion and Conclusions

The aim of this study was to analyze the epistemological framing that occurs in students during the solution process, depending on whether the problem has a triangle drawing or not. Based on the epistemological framing, we conclude that there are significant differences in students' solution processes as well as in students' perceptions of the difficulty of the problem. In the following, we will explain these two results in more detail.

The quantitative analysis showed that the percentage of students who solved the problem correctly without the triangle (67.6%) was significantly higher than the group of students who solved the problem correctly with the triangle drawing (43.9%). This means that the presence of the drawing with the triangle caused a difference in the initial epistemological framing that the students developed when reading the task, and that this framing influenced their solution process and final answer to the task. Students who solved the task with the triangle were more likely to give incorrect answers than those who did not have this element in the task. This result is consistent with the study by Juárez López et al. [16], in which 69 primary school teachers in training were given the task of calculating the perimeter of an irregular pentagonal surface from a fifth-grade math textbook. The task contained the figure of the pentagon with certain values on each side, but it was an impossible pentagon. The teachers were then asked: Can the terrain described in the task exist in reality? They were also asked to support their answer with arguments. Of the total number of teachers surveyed, 62 answered that *it can exist*. Only three teachers answered that *it cannot exist*, but of these, only one provided the correct argument.

Regarding the qualitative content analysis of the answers given by the students, the results show that students who solved the problem without the triangle drawing were able to perform a better solution process than students who solved the problem with the triangle drawing. In Group A of students, it can be observed that the initial epistemological framing, when the students read the problem without the triangle drawing, allowed them to preferentially rely on two solution strategies (Heron's formula and triangle inequality, with 75.6% between the two). It also allowed them not to make intentional arithmetic errors during their solution process and to obtain an area value equal to zero, which did not put them in conflict with their original epistemological framing. For the Heron formula, 81.8% of students solved the problem correctly by accepting that the area was zero, while 18.2% of students did not accept this result because they ignored the zero during their solution process. This means that in group A, the epistemological framework was naturally modified to accept that the area of the triangle was zero for most students.

Conversely, those students who solved the problem with the illustration made more errors in their solution processes. The most notable error was found in the Heron formula, where 71.4% of students made the same intentional math error by eliminating the partial result equal to zero and forcing a positive answer. Felicity's condition occurred when students interpreted that the drawing of a triangle in the task exists because such a triangle necessarily exists. Their initial epistemological framing, as expected by Gardner, led the students away from the correct solution approach. For the Heron formula, however, only 28.6% of the students in this group carried out the correct solution process. These students were able to change their original epistemological framing and regard the triangle as degenerate.

Furthermore, the drawing of the triangle in the task might encourage a solution procedure that leads some students to transform the original drawing, for which they do not have a ready-to-use formula to calculate the area, into a drawing to which they can apply a known formula (e.g., a right-angled triangle) (see, for example, the case of student B18 and Figure 9).

Generally speaking, during the solution process, students enter a cognitive conflict that leads them to two possible answer options:

i. They reconsider their original epistemological framing, change it considering the new situation, and give the correct answer;
ii. They deny the stated result, do not change their epistemological framing, and force a non-zero result.

In the last case, the students in Group B do not worry about whether the drawing is possible, i.e., whether the length of the sides matches the dimensions of the triangle, because they assume that the given graphical representation is correct.

In the case of Group A, where the problem statement does not contain the triangle drawing, the illustration was only useful to the students when they had to construct it themselves or not construct it, as happened in several cases when they realized that the data did not allow them to form it. In this case, students had the opportunity to explore the data provided by the task more freely. They changed their epistemological framing fluidly and adapted it to the new form during the solution process so that they could obtain the correct answer.

When analyzing the results of some students in Group B, it is noticeable that their epistemological framing changed as they solved the problem to find the correct answer. Some of them readily accepted the solution by trusting their mathematical procedure, while others did not trust their correct answer and stated that they were sure they had done something wrong. From this, we can conclude that all possibilities are present in the students. Some can change their epistemological framing more easily, and others cannot.

Regarding the students' perception of the difficulty of the problem, the students who solved the problem without the triangle drawing tended to mark the problem as difficult and regular with 86.4%, while the students who solved the problem with the drawing marked the same options with 65.9%. From this, we conclude that the students who solved the problem without the triangle generally found it more difficult than the students who solved the problem with the drawing, even though the latter ultimately gave more incorrect answers.

To summarize, this study shows the importance of giving students problems that do not contain erroneous drawings or warning them about them so that students can more easily change their epistemological framing during the solution process. It is important to see these types of problems as an opportunity for students to learn, reflect, and develop more flexible and adaptable epistemological framing during the mathematical problem-solving process.

A possible follow-up to this work could be the development of more flexible epistemological framings that are adaptable to the implausible conditions that may arise when solving mathematical problems, as an opportunity to recognize that Felicity's condition may not be so.

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