Proceeding Paper

Bianchi Type I Cosmological Model in $f(R,T)$ Gravity [†]

Rishi Kumar Tiwari [1], Aroonkumar Beesham [2,3,*] , Soma Mishra [4] and Vipin Dubey [1]

[1] Department of Mathematics, Government Model Science College, Rewa 486001, India; rishitiwari59@rediffmail.com (R.K.T.); vipindubey.dubey@gmail.com (V.D.)

[2] Department of Mathematical Sciences, University of Zululand, Kwa-Dlangezwa 3886, South Africa

[3] Faculty of Natural Sciences, Mangosuthu University of Technology, Jacobs 4023, South Africa

[4] Mahatma Gandhi Chitrakoot Gramodaya Vishwavidyalaya, Satna 485001, India; somamishravits@gmail.com

[*] Correspondence: abeesham@yahoo.com

[†] Presented at the 1st Electronic Conference on Universe, 22–28 February 2021; Available online: https://ecu2021.sciforum.net/.

Abstract: Although the present universe is believed to be homogeneous and isotropic on large scales, there is some evidence of some anisotropy at early times. Hence, there is interest in the Bianchi models, which are homogeneous but anisotropic. In this presentation, the Bianchi type–I space-time in the framework of the $f(R,T)$ modified theory of gravity has been investigated for the specific choice of $f(R,T) = R + 2f(T)$, where $f(T) = -mT$, m = constant. The solution of the modified gravity field equations has been generated by assuming that the deceleration parameter q is a function of the Hubble parameter H, i.e., $q = b - n/H$ (where b and n are constants, and $n > 0$), which yields the scale factor $a = k[exp(nt) - 1]^{1/(1+b)}$ (where k is a constant). The model exhibits deceleration at early times, and is currently accelerating. It is also seen that the model approaches isotropy at late times. Expressions for the Hubble parameter in terms of red-shift, luminosity distance, and state-finder parameter are derived and their significance is described in detail. The physical properties of the cosmological model are also discussed. An interesting feature of the model is that it has a dynamic cosmological parameter, which is large during the early universe, decreases with time, and approaches a constant at late times. This may help in solving the cosmological constant problem.

Keywords: Bianchi type I universe; $f(R,T)$ theory; deceleration parameter

1. Introduction

Today's theoretical and experimental studies reveal that currently our universe is in its accelerating stage of expansion [1,2] and dark energy plays a significant role in driving this acceleration [3]. The most alluring entity of this dark energy is positive energy density but negative pressure.

From the nine-year results of the Wilkinson microwave anisotropy probe (WMAP) [4] and Plank, the universe is comprised of 68.5% dark energy, 26.5% dark matter and 5% baryonic matter. Dark energy can be expressed either by using the equation of state parameter (EOS) $\omega = \frac{p}{\rho}$, where p is the pressure and ρ is the energy density, or with respect to the cosmological constant.

The cosmological constant Λ, introduced by Albert Einstein in his field equations to obtain a static universe, is now treated as a suitable nominee for dark energy for explaining the increase in the acceleration of the universe. However, cosmological puzzles such as fine tuning and the cosmic coincidence problem are surrounding it currently [5].

In the last few years, to get around the mechanism of the late-time acceleration and also dark matter and dark energy, many modified theories of gravity have been studied, e.g., $f(R)$, $f(T)$, $f(G)$, and $f(R, T)$ gravity. These models are put forward to explore dark energy and other problems of cosmology. Noteworthy amongst them is $f(R)$ gravity, which has been broadly investigated by several authors [6,7]. Another recommendation is $f(T)$ gravity, which has been developed recently. The fascinating attribute of the theory is

that it can explain the current acceleration without involving dark energy. $f(R, T)$ gravity was introduced by Harko et al. [8], in which the gravitational Lagrangian is defined by an arbitrary function of the Ricci scalar R and the trace T of the energy momentum tensor. Some authors who have investigated this theory are Houndjo [9] and Myrzakulov [10].

In this paper, we have discussed the Bianchi type I cosmological model by assuming a particular form for the deceleration parameter as a function of the Hubble parameter. The field equations are presented in Section 2. The solution of the field equations are derived and discussed in Section 3. The observational parameters such as cosmological red-shift, luminosity distance, and state-finder parameters for the model are also discussed in Sections 3 and 4, which contain the conclusion.

2. Field Equations

The action of $f(R,T)$ gravity is given by:

$$S = \int \sqrt{-g} \left(\frac{-1}{16\pi G} f(R, T) + L_m \right) d^4 x \tag{1}$$

where the symbols have their usual meanings.

In the present study, we shall concentrate on the form $f(R, T) = R + 2f(T)$, and choose $f(T) = -\xi T$, where ξ is an arbitrary constant. The field equations are:

$$R_{ij} - \frac{1}{2} R g_{ij} = -(1 + 2\xi) T_{ij} + \xi(-T - 2p) g_{ij} \tag{2}$$

A comparison of (2) with Einstein's field equations:

$$R_{ij} - \frac{1}{2} R g_{ij} = -T_{ij} + \Lambda g_{ij} \tag{3}$$

suggests making the identification $\Lambda = \Lambda(T) = -\xi(T + 2p)$ and $-1 = -(1 + 2\xi)$. Therefore, in $f(R, T)$ gravity, the field equations with $\Lambda(T)$ can be expressed as:

$$R_{ij} - \frac{1}{2} R g_{ij} = -(1 + 2\xi) T_{ij} + \Lambda g_{ij} \tag{4}$$

It can clearly be seen from (4), which follows from (2), that the usual energy conservations law does not hold in the $f(R, T)$ theory.

The gravitational field for a spatially homogeneous and anisotropic Bianchi type-I space-time is given by the line element:

$$ds^2 = -dt^2 + A^2 dx^2 + B^2 dy^2 + C^2 dz^2 \tag{5}$$

where A, B, and C are metric functions of cosmic time t. For the Bianchi type-I space time (5), the field Equations (4) in $f(R, T)$ gravity yield the following dynamical equations:

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} = \Lambda - (1 + 2\xi) p \tag{6}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} = \Lambda - (1 + 2\xi) p \tag{7}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} = \Lambda - (1 + 2\xi) p \tag{8}$$

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC} = \Lambda + (1 + 2\xi) \rho \tag{9}$$

where an over-dot denotes the ordinary derivative with respect to cosmic time t.

We assume that the matter content obeys the equation of state:

$$p = \omega\rho, \quad 0 \le \omega \le 1 \tag{10}$$

From Equations (6)–(9), we can easily obtain the metric potentials A, B, and C as

$$A = m_1 a exp\left[\frac{2k_1 + k_2}{3}\int\frac{dt}{a^3}\right] \tag{11}$$

$$B = m_2 a exp\left[\frac{k_2 - k_1}{3}\int\frac{dt}{a^3}\right] \tag{12}$$

$$C = m_3 a exp\left[-\frac{k_1 + 2k_2}{3}\int\frac{dt}{a^3}\right] \tag{13}$$

where $m_1, m_2,$ and m_3 and k_1 and k_2 are arbitrary constants of integration satisfying $m_1 m_2 m_3 = 1$.

Equations (6)–(9) can be expressed in terms of $H, q,$ and σ as:

$$3H^2 - \sigma^2 = \Lambda + (1 + 2\xi)\rho \tag{14}$$

$$H^2(2q - 1) - \sigma^2 = (1 + 2\xi)p - \Lambda \tag{15}$$

3. Solution of Field Equations

Now Equations (6)–(10), which are obtained from field Equation (4), represent only five equations but with six unknown quantities, i.e., A, B, C, ρ, p, and Λ, respectively. Hence, system (4) is undetermined, and one extra equation is required to solve the system completely. There are many different assumptions that can be adopted to solve this system. In this investigation, we assume that the deceleration parameter q is a function of the Hubble parameter H [11]:

$$q = b - \frac{n}{H} \tag{16}$$

Here, b and n are constants, and $n > 0$.

This yields

$$a = k\left(e^{nt} - 1\right)^{\frac{1}{1+b}} \tag{17}$$

where k is a constant.

The spatial volume V, Hubble parameter H, expansion scalar θ, shear scalar σ^2, and deceleration parameter q take the form:

$$V = k^3\left(e^{nt} - 1\right)^{\frac{3}{1+b}} \tag{18}$$

$$H = \frac{ne^{nt}}{(1 + b)(e^{nt} - 1)} \tag{19}$$

$$\theta = \frac{3n}{(1 + b)(1 - e^{-nt})} \tag{20}$$

$$\sigma^2 = \frac{k_1^2 + k_2^2 + k_1 k_2}{3k^6\left(e^{nt} - 1\right)^{\frac{6}{1+b}}} \tag{21}$$

$$q = -1 + (1 + b)e^{-nt} \tag{22}$$

Equations (17)–(22) are determined essentially from (16), and are the kinematic quantities. Field Equation (4) on the other hand, is basically used to determine the dynamical quantities, viz., the energy density ρ, pressure p, and cosmological parameter Λ.

From Equations (8)–(10), we obtain the energy density ρ and pressure p as:

$$\rho = \frac{1}{(1+\omega)(1+2\xi)}\left[\frac{2n^2e^{nt}}{(1+b)(e^{nt}-1)^2} - \frac{2\left(k_1^2 + k_2^2 + k_1k_2\right)}{3k^6(e^{nt}-1)^{\frac{6}{1+b}}}\right] \tag{23}$$

$$p = \frac{\omega}{(1+\omega)(1+2\xi)}\left[\frac{2n^2e^{nt}}{(1+b)(e^{nt}-1)^2} - \frac{2\left(k_1^2 + k_2^2 + k_1k_2\right)}{3k^6(e^{nt}-1)^{\frac{6}{1+b}}}\right] \tag{24}$$

The cosmological parameter Λ is given by:

$$\Lambda = \xi\left[\frac{2(1-\omega)n^2e^{nt}}{(1+\omega)(1+2\xi)(1+b)(e^{nt}-1)^2} - \frac{2(1-\omega)\left(k_1^2 + k_2^2 + k_1k_2\right)}{3k^6(1+\omega)(1+2\xi)(e^{nt}-1)^2}\right] \tag{25}$$

For Model (5), we observe that the spatial volume V is zero and the expansion scalar θ is infinite at $t = 0$. Thus, the universe starts evolving with zero volume and infinite rate of expansion at $t = 0$. Equation (17) shows that the scale factors also vanish at $t = 0$, and hence the model has a "point type" singularity at the initial epoch. Initially at $t = 0$, the Hubble parameter H and shear scalar σ^2 are infinite. The energy density ρ, pressure p and cosmological constant Λ are also infinite. As t tends to infinity, V becomes infinitely large, whereas σ^2 approaches zero. However, the parameters H, θ are constant throughout the evolution of the universe. Now as t increases, the energy density ρ and pressure p converge to zero. The cosmological parameter Λ also approaches a constant at late times. The deceleration parameter q for the model is α at $t = 0$, and as t increases, i.e., when it is $(1/\beta)\log(1+\alpha)$, q is zero, which shows that there will be no more deceleration. It is equal to -1 when t tends to infinity, which shows that the model describes an accelerating phase of the universe. Since σ/θ tends to zero as $t \to \infty$, the model approaches isotropy for large t [12].

Some Cosmological Distance Parameters

(i) Cosmological red-shift: The age and size of the universe is defined by the Hubble parameter. With the help of Equation (19), we get

$$\frac{H}{H_0} = \frac{e^{nt}\left(e^{nt_0} - 1\right)}{e^{nt_0}\left(e^{nt} - 1\right)} \tag{26}$$

where H_0 is the present value of Hubble parameter and t_0 is the present time.

The following equation explains the relationship between the scale factor a and red-shift z:

$$a = \frac{a_0}{1+z} \tag{27}$$

where a_0 is the present value of scale factor. Here, we take $a_0 = 1$. The above Equation (27) can be rewritten as:

$$a = \frac{1}{1+z} = k\left(e^{nt}-1\right)^{\frac{1}{1+b}} \tag{28}$$

This gives:

$$H = H_0\left(1 - e^{-nt_0}\right)\left[\{k(1+z)\}^{1+b} + 1\right] \tag{29}$$

Equation (29) represents the value of the Hubble parameter in terms of the red shift parameter.

The distance modulus (μ) is given by

$$\mu(z) = 5\log d_L + 25 \tag{30}$$

where d_L stands for the luminosity distance defined by

$$d_L = r_1(1+z)a_0 \tag{31}$$

A source emits a photon at $r = r_1$ and $t = t_0$ and observer received at time, located it at $r = 0$, then we calculate r_1 by the following equations:

$$r_1 = \int_t^{t_0} \frac{dt}{a} = \int_t^{t_0} \frac{dt}{k(e^{nt} - 1)^{\frac{1}{1+b}}} \tag{32}$$

To solve this integral, we take $b = 0$ without any loss of generality. We obtain the value of r_1 as:

$$r_1 = \frac{1}{nk} log\left(\frac{1 - e^{-nt_0}}{1 - e^{-nt}} \right) \tag{33}$$

Hence, from Equations (31) and (33), we obtain the expression for the luminosity distance as:

$$d_L = \frac{1}{nk} log\left[\left(1 - e^{-nt_0}\right) \left(\{k(1+z)\}^{1+b} + 1 \right) \right] (1 + z). \tag{34}$$

(ii) State-finder parameters

The values of the state-finder parameter for our model are:

$$r = 1 - \frac{3n + (1+b)^2 \left(1 - e^{-2nt}\right)}{(e^{nt} - 1)}. \tag{35}$$

$$s = \frac{2(1+b)^2\left(1 - e^{-2nt}\right) - 6n}{(e^{nt} - 1)[6(1+b)e^{-nt} - 9]} \tag{36}$$

When $(r, s) = (1, 0)$ [13], we have the ΛCDM model, while for $(r, s) = (1, 1)$, we have the cold dark mater (CDM) limit. Additionally, when $r < 1$ we have a quintessence region and for $s > 0$ the phantom region.

We observe that when $t \rightarrow 0$, $\{r, s\} \rightarrow \{\infty, -\infty\}$ and as $t \rightarrow \infty$, $\{r, s\} \rightarrow \{1, 0\}$. This shows that our model starts from an Einstein static era and asymptotically approaches the ΛCDM universe.

4. Conclusions

In this paper, we have discussed a spatially homogeneous and anisotropic Bianchi type-I space time in the frame work of $f(R, T)$. gravity. A specific choice of $f(R, T) = R + 2f(T)$, where $f(T) = -\xi T$, has been considered to explore some exact solutions of an anisotropic and homogeneous Bianchi type-I space time. For obtaining deterministic solutions of the field equations, we have employed a variation law in which the deceleration parameter q is assumed to be a function of the Hubble parameter H, i.e., $q = b - \frac{n}{H}$, which gives the scale factor $a = k(e^{nt} - 1)^{\frac{1}{1+b}}$ (where b, n, and k are constants and $n > 0$). We find that the universe expands exponentially until late times and also it becomes isotropic at late times. The cosmological parameter Λ is very large at initial times, and approaches a constant as t tends to infinity. This agrees with the work of Amirhashchi [14] and Yadav [15]. For $t \rightarrow 0$, the deceleration parameter q is constant and gives the decelerating phase of expansion. We also discussed some cosmological distance parameters and state-finder parameters. Finally, we noticed from the state-finder parameters $\{r, s\}$ that the evolution of the universe starts from an Einstein static era ($r \rightarrow \infty, s \rightarrow -\infty$) and approaches the ΛCDM model ($r \rightarrow 1, s \rightarrow 0$) at late times [11].

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