physical sciences forum

MDPI

Proceeding Paper

De Sitter Solutions in Models with the Gauss-Bonnet Term †

Sergey Vernov *,‡ and Ekaterina Pozdeeva ‡

Skobeltsyn Institute of Nuclear Physics, Lomonosov Moscow State University, Leninskie Gory 1, 119991 Moscow, Russia; pozdeeva@www-hep.sinp.msu.ru

* Correspondence: svernov@theory.sinp.msu.ru

† Presented at the 1st Electronic Conference on Universe, 22–28 February 2021; Available online: https://ecu2021.sciforum.net/.

‡ These authors contributed equally to this work.

Abstract: De Sitter solutions play an important role in cosmology because the knowledge of unstable de Sitter solutions can be useful in describing inflation, whereas stable de Sitter solutions are often used in models of the late-time acceleration of the Universe. Einstein–Gauss–Bonnet models are actively used as both inflationary models and dark energy models. To modify the Einstein equations, one can add a nonlinear function of the Gauss–Bonnet term or a function of the scalar field multiplied on the Gauss–Bonnet term. The effective potential method essentially simplifies the search and stability analysis of de Sitter solutions, because the stable de Sitter solutions correspond to the minima of the effective potential.

Keywords: Einstein–Gauss–Bonnet gravity; de Sitter solution; stability

Citation: Vernov, S.; Pozdeeva, E. De Sitter Solutions in Models with the Gauss-Bonnet Term. *Phys. Sci. Forum* **2021**, *2*, 53. https://doi.org/10.3390/ECU2021-09305

Academic Editor: Pedro Avelino

Published: 22 February 2021

1. Introduction

It is well-known that one can add the Gauss–Bonnet term to the Hilbert–Einstein Lagrangian of General Relativity, and this does not change the equations of motion. On the other hand, this term, when multiplied by some nonconstant function of a scalar field, modifies the equations of motion. Additionally, models with a non-linear function of the Gauss–Bonnet term can be rewritten in the equivalent form, which includes a scalar field without a kinetic term.

The cosmological models with the Gauss–Bonnet term are motivated by string theory [1–8] and are actively used for describing of both the early Universe evolution [9–24] and the current dark energy dominated epoch [5–7,25–30].

Note that these studies of the Universe's evolution are characterized by the quasi de Sitter accelerated expansion of the Universe. Therefore, it is important to have an effective method to sear the de Sitter solutions and for the study of their stability. For the Gauss–Bonnet model with the standard scalar field, such a method has been proposed in [31]. This is a generalization of the effective potential method [32,33]. In this paper, we generalize this method on a model with nonlinear functions of the Gauss–Bonnet term. We also consider the case of a phantom scalar field and show that, in this case, the situation is more difficult.

2. Models the Gauss-Bonnet Term

Let us consider the model with the Gauss–Bonnet term, described by the following action:

$$S = \int d^4x \sqrt{-g}\Big[UR - \frac{c}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V - F\mathcal{G}\Big], \tag{1}$$

where the functions $U(\phi)$, $V(\phi)$, and $F(\phi)$ are double differentiable ones, c is a constant, R is the Ricci scalar and $\mathcal{G}$ is the Gauss–Bonnet term,

$$\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}.$$

Note that the action

$$S = \int d^4x \sqrt{-g} W(\mathcal{G}), \tag{2}$$

where $W(\mathcal{G})$ is a double differentiable function, can be rewritten in the following form [8,27]:

$$S = \int d^4x \sqrt{-g} \left[W'(\phi)(\mathcal{G} - \phi) + W(\phi) \right], \tag{3}$$

where a prime denotes the derivatives with respect to ϕ. Varying action (3) over ϕ, one gets $\phi = \mathcal{G}$ and the initial $W(\mathcal{G})$ model. Therefore, action (1) with $c = 0$ describes $W(\mathcal{G})$ models.

In the spatially flat Friedmann–Lemaître–Robertson–Walker metric with

$$ds^2 = -dt^2 + a^2(t)\left(dx_1^2 + dx_2^2 + dx_3^2\right), \tag{4}$$

one obtains the following evolution equations:

$$6H^2U + 6HU'\dot{\phi} = \frac{c}{2}\dot{\phi}^2 + V + 24H^3F'\dot{\phi}, \tag{5}$$

$$4\left(U - 4H\dot{F}\right)\dot{H} = -c\dot{\phi}^2 - 2\ddot{U} + 2H\dot{U} + 8H^2\left(\ddot{F} - H\dot{F}\right), \tag{6}$$

$$c\ddot{\phi} + 3cH\dot{\phi} - 6\left(\dot{H} + 2H^2\right)U' + V' + 24H^2F'\left(\dot{H} + H^2\right) = 0, \tag{7}$$

where $H = \dot{a}/a$ is the Hubble parameter; dots and primes denote the derivatives with respect to the cosmic time and the scalar field ϕ, respectively. At $c = 1$, these equations have been investigated in many papers (see, for example, [11,31]).

To find de Sitter solutions with a constant ϕ in the model (2), we substitute $\phi = \phi_{dS}$ and $H = H_{dS}$ into Equations (5) and (7). A de Sitter solution does not depend on the value of c, so we obtain the same results as in the case $c = 1$ considered in [31]:

$$H_{dS}^2 = \frac{V_{dS}}{6U_{dS}} \tag{8}$$

and

$$F'_{dS} = \frac{3U_{dS}(2U'_{dS}V_{dS} - V'_{dS}U_{dS})}{2V_{dS}^2}, \tag{9}$$

where $A_{dS} \equiv A(\phi_{dS})$ for any function A. Therefore, for arbitrary functions $U(\phi)$ and $V(\phi)$ with $V_{dS}U_{dS} > 0$, we can choose $F(\phi)$ such that the corresponding point becomes a de Sitter solution, with the Hubble parameter defined by Equation (8). We always choose that $H_{dS} > 0$.

3. Stability of de Sitter Solutions

To analyze the stability of a de Sitter solution, we transform Equations (6) and (7) into the following dynamical system:

$$\begin{aligned}
\dot{\phi} =& \psi, \\
\dot{\psi} =& \frac{1}{2(\tilde{B} - 4cF'H\psi)}\Big\{2H\Big[3B + 4F'V' - 6U'^2 - 6cU\Big]\psi - 2\frac{V^2}{U}X \\
&+ \Big[12H^2\left[(2U'' + 3c)F' + 2U'F''\right] - 96F'F''H^4 - 3(2U'' + c)U'\Big]\psi^2\Big\}, \\
\dot{H} =& \frac{1}{4(\tilde{B} - 4cF'H\psi)}\Big\{8c\left(U' - 4F'H^2\right)H\psi \\
&- 2\frac{V^2}{U^2}\left(4F'H^2 - U'\right)X + \left(8F''H^2 - 2U'' - c\right)c\psi^2\Big\},
\end{aligned} \tag{10}$$

where

$$\tilde{B} = 3\Big(4H^2F' - U'\Big)^2 + cU, \tag{11}$$

$$X = \frac{U^2}{V^2}\Big[24H^4F' - 12H^2U' + V'\Big]. \tag{12}$$

In the case $c = 0$, the last equation is essentially simplified:

$$\dot{H} = \frac{24H^4F' - 12H^2U' + V'}{6(U' - 4H^2F')}. \tag{13}$$

At a de Sitter point system (10) is

$$\dot{\phi} = 0, \quad \dot{\psi} = 0, \quad \dot{H} = 0,$$

that corresponds to $X_{dS} = 0$.

In Reference [31], the effective potential has been proposed for cosmological models with the Gauss–Bonnet term:

$$V_{eff} = -\frac{U^2}{V} + \frac{2}{3}F. \tag{14}$$

Using Equation (8), we obtain

$$X_{dS} = \frac{2}{3}F'_{dS} - 2\frac{U'_{dS}U_{dS}}{V_{dS}} + \frac{V'_{dS}U^2_{dS}}{V^2_{dS}} = V'_{eff}(\phi_{dS}) = 0, \tag{15}$$

therefore, de Sitter solutions correspond to extremum points of the effective potential V_{eff}.

To investigate the Lyapunov stability of a de Sitter solution, we use the following expansions:

$$H(t) = H_{dS} + \varepsilon H_1(t) \quad \phi(t) = \phi_{dS} + \varepsilon\phi_1(t), \quad \psi(t) = \varepsilon\psi_1(t), \tag{16}$$

where ε is a small parameter. Therefore,

$$X = \varepsilon(X_{,H}H_1 + X_{,\phi}\phi_1) + \mathcal{O}(\varepsilon^2), \tag{17}$$

where

$$X_{,H} = \frac{\partial X}{\partial H}\bigg|_{\phi=\phi_{dS}} = \frac{4\sqrt{6}}{V_{dS}^{5/2}}U_{dS}^{3/2}(U'_{dS}V_{dS} - V'_{dS}U_{dS}),$$

$$X_{,\phi} = \frac{\partial X}{\partial \phi}\bigg|_{\phi=\phi_{dS}} = \frac{1}{V^2_{dS}}\left(\frac{2}{3}F''_{dS}V^2_{dS} - 2U''_{dS}U_{dS}V_{dS} + V''_{dS}U^2_{dS}\right).$$

The functions $H_1(t)$, $\phi_1(t)$, and $\psi_1(t)$ are connected by Equation (5):

$$H_1(t) = \frac{V'_{dS}U_{dS} - U'_{dS}V_{dS}}{2U_{dS}V_{dS}}(H_{dS}\phi_1(t) - \psi_1(t)). \tag{18}$$

This expression does not depend on the value of c and coincides with the corresponding expression obtained in Reference [31].

Substituting (16)–(18) into Equation (10) in the first order of ε, we obtain the following system of two linear differential equations:

$$\dot{\phi}_1 = \psi_1, \tag{19}$$

$$\dot{\psi}_1 = -\frac{\Big[2F''_{dS}V^3_{dS} - 6U''_{dS}U_{dS}V^2_{dS} + 3V''_{dS}U^2_{dS}V_{dS} - 6\big(U'_{dS}V_{dS} - V'_{dS}U_{dS}\big)^2\Big]}{3U_{dS}V_{dS}B_{dS}}\phi_1 - \frac{\sqrt{6U_{dS}V_{dS}}}{2U_{dS}}\psi_1, \tag{20}$$

where

$$B_{dS} = \frac{3}{V_{dS}^2}\left(V'_{dS}U_{dS} - U'_{dS}V_{dS}\right)^2 + cU_{dS}. \tag{21}$$

This system can be rewritten in the matrix form:

$$\begin{pmatrix} \dot{\phi}_1 \\ \dot{\psi}_1 \end{pmatrix} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{21} \\ \tilde{A}_{12} & \tilde{A}_{22} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \psi_1 \end{pmatrix} \tag{22}$$

where

$$\tilde{A} = \begin{Vmatrix} 0, & 1 \\ -\frac{V_{dS}^2 V''_{eff}(\phi_{dS})}{U_{dS}B_{dS}}, & -3H_{dS} \end{Vmatrix}$$

The general solution of system (22) has the following form

$$\phi_1 = c_{11}\mathrm{e}^{-\lambda_- t} + c_{21}\mathrm{e}^{-\lambda_+ t}, \tag{23}$$

$$\psi_1 = c_{21}\mathrm{e}^{-\lambda_- t} + c_{22}\mathrm{e}^{-\lambda_+ t}, \tag{24}$$

where c_{ij} are some constants. Solving the characteristic equation:

$$\det(\tilde{A} - \lambda \cdot I) = \lambda^2 - 3H_{dS}\lambda + \frac{V_{dS}^2 V''_{eff}(\phi_{dS})}{U_{dS}B_{dS}} = 0, \tag{25}$$

we obtain the following roots:

$$\lambda_\pm = -\frac{3}{2}H_{dS} \pm \sqrt{\frac{9}{4}H_{dS}^2 - \frac{V_{dS}^2}{U_{dS}B_{dS}}V''_{eff}(\phi_{dS})}\,. \tag{26}$$

A de Sitter solution is stable if the real parts of both λ_- and λ_+ are negative. We consider the case $H_{dS} = \sqrt{\frac{V}{6U}} > 0$, hence, $\Re e(\lambda_-) < 0$.

In the case of a positive U_{dS}, we see that $B_{dS} > 0$ for $c \geqslant 0$ and the condition $\Re e(\lambda_+) < 0$ is equivalent to $V''_{eff}(\phi_{dS}) > 0$. In the cases $c > 0$ and $c = 0$, a de Sitter solution is stable if $V''_{eff}(\phi_{dS}) > 0$ and unstable if $V''_{eff}(\phi_{dS}) < 0$.

In the case $c < 0$, we see that B_{dS} can be negative. Therefore, in this case, the de Sitter solution is stable if the $V''_{eff}(\phi_{dS})B_{dS} > 0$. Therefore, the main result of Reference [31] can be generalized on the case $c = 0$ without any correction, whereas the condition should be changed to $V''_{eff}(\phi_{dS})B_{dS} > 0$ in the case of $c < 0$, which corresponds to a phantom scalar field ϕ.

4. Conclusions

In this paper, we consider de Sitter solutions in models with the Gauss–Bonnet term, including $W(\mathcal{G})$ models. We show that the effective potential proposed [31] for model with the Gauss–Bonnet term, multiplied on a function of the scalar field, can be used in $W(\mathcal{G})$ models as well. To find de Sitter solutions in a $W(\mathcal{G})$ model, we rewrite the action of this model in the form (3) and construct the corresponding effective potential V_{eff}. A stable de Sitter solution corresponds to $V''_{eff}(\phi_{dS}) > 0$, where the values of the scalar field at the de Sitter point ϕ_{dS} are determined by the condition $V'_{eff}(\phi_{dS}) = 0$. Examples of de Sitter solutions in $W(\mathcal{G})$ gravity models and in more complicated models with the $\mathcal{L}(R+\mathcal{G})$ function in the action are given in [34].

Note that the effective potential provides a useful tool for the construction of inflationary scenarios in models where the Gauss–Bonnet term is multiplied to a function of the scalar field [24,35,36]. We plan to generalize this approach to inflationary scenarios in $W(\mathcal{G})$ models.

The Supplementary Materials include the presentation on the first Electronic Conference on Universe.

Supplementary Materials: The presentation is available online at https://www.mdpi.com/article/10.3390/ECU2021-09305/s1.

Funding: This work is partially supported by the Russian Foundation for Basic Research grant No. 20-02-00411.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Antoniadis, I.; Rizos, J.; Tamvakis, K. Singularity-free cosmological solutions of the superstring effective action. *Nucl. Phys. B* **1994**, *415*, 497. [CrossRef]
2. Kawai, S.; Soda, J. Evolution of fluctuations during graceful exit in string cosmology. *Phys. Lett. B* **1999**, *460*, 41. [CrossRef]
3. Cartier, C.; Hwang, J.C.; Copeland, E.J. Evolution of cosmological perturbations in nonsingular string cosmologies. *Phys. Rev. D* **2001**, *64*, 103504. [CrossRef]
4. Hwang, J.C.; Noh, H. Classical evolution and quantum generation in generalized gravity theories including string corrections and tachyon: Unified analyses. *Phys. Rev. D* **2005**, *71*, 063536. [CrossRef]
5. Calcagni, G.; Tsujikawa, S.; Sami, M. Dark energy and cosmological solutions in second-order string gravity. *Class. Quant. Gravity* **2005**, *22*, 3977. [CrossRef]
6. Tsujikawa, S.; Sami, M. String-inspired cosmology: Late time transition from scaling matter era to dark energy universe caused by a Gauss-Bonnet coupling. *J. Cosmol. Astropart. Phys.* **2007**, *0701*, 006. [CrossRef]
7. Cognola, G.; Elizalde, E.; Nojiri, S.; Odintsov, S.; Zerbini, S. String-inspired Gauss-Bonnet gravity reconstructed from the universe expansion history and yielding the transition from matter dominance to dark energy. *Phys. Rev. D* **2007**, *75*, 086002. [CrossRef]
8. Nojiri, S.; Odintsov, S.D.; Oikonomou, V.K. Modified Gravity Theories on a Nutshell: Inflation, Bounce and Late-time Evolution. *Phys. Rep.* **2017**, *692*, 1. [CrossRef]
9. Guo, Z.; Schwarz, D.J. Slow-roll inflation with a Gauss-Bonnet correction. *Phys. Rev. D* **2010**, *81*, 123520. [CrossRef]
10. Koh, S.; Lee, B.H.; Lee, W.; Tumurtushaa, G. Observational constraints on slow-roll inflation coupled to a Gauss-Bonnet term. *Phys. Rev. D* **2014**, *90*, 063527. [CrossRef]
11. Bruck, C.V.; Longden, C. Higgs Inflation with a Gauss-Bonnet term in the Jordan Frame. *Phys. Rev. D* **2016**, *93*, 063519. [CrossRef]
12. Wu, Q.; Zhu, T.; Wang, A. Primordial Spectra of slow-roll inflation at second-order with the Gauss-Bonnet correction. *Phys. Rev. D* **2018**, *97*, 103502. [CrossRef]
13. Nozari, K.; Rashidi, N. Perturbation, nonGaussianity, and reheating in a Gauss-Bonnet α-attractor model. *Phys. Rev. D* **2017**, *95*, 123518. [CrossRef]
14. Oikonomou, V.K. Autonomous dynamical system approach for inflationary Gauss-Bonnet modified gravity. *Int. J. Mod. Phys. D* **2018**, *27*, 1850059. [CrossRef]
15. Chakraborty, S.; Paul, T.; SenGupta, S. Inflation driven by Einstein-Gauss-Bonnet gravity. *Phys. Rev. D* **2018**, *98*, 083539. [CrossRef]
16. Yi, Z.; Gong, Y.; Sabir, M. Inflation with Gauss-Bonnet coupling. *Phys. Rev. D* **2018**, *98*, 083521. [CrossRef]
17. Odintsov, S.D.; Oikonomou, V.K. Viable Inflation in Scalar-Gauss-Bonnet Gravity and Reconstruction from Observational Indices. *Phys. Rev. D* **2018**, *98*, 044039. [CrossRef]
18. Yi, Z.; Gong, Y. Gauss–Bonnet Inflation and the String Swampland. *Universe* **2019**, *5*, 200. [CrossRef]
19. Fomin, I.V.; Chervon, S.V. Reconstruction of GR cosmological solutions in modified gravity theories. *Phys. Rev. D* **2019**, *100*, 023511. [CrossRef]
20. Odintsov, S.D.; Oikonomou, V.K.; Fronimos, F.P. Rectifying Einstein-Gauss-Bonnet Inflation in View of GW170817. *Nucl. Phys. B* **2020**, *958*, 115135. [CrossRef]
21. Odintsov, S.; Oikonomou, V. Swampland Implications of GW170817-compatible Einstein-Gauss-Bonnet Gravity. *Phys. Lett. B* **2020**, *805*, 135437. [CrossRef]
22. Fomin, I. Gauss–Bonnet term corrections in scalar field cosmology. *Eur. Phys. J. C* **2020**, *80*, 1145. [CrossRef]
23. Pozdeeva, E.O. Generalization of cosmological attractor approach to Einstein–Gauss–Bonnet gravity. *Eur. Phys. J. C* **2020**, *80*, 612. [CrossRef]
24. Pozdeeva, E.O.; Gangopadhyay, M.R.; Sami, M.; Toporensky, A.V.; Vernov, S.Y. Inflation with a quartic potential in the framework of Einstein-Gauss-Bonnet gravity. *Phys. Rev. D* **2020**, *102*, 043525. [CrossRef]
25. Nojiri, S.; Odintsov, S.D.; Sasaki, M. Gauss-Bonnet dark energy. *Phys. Rev. D* **2005**, *71*, 123509. [CrossRef]
26. Sami, M.; Toporensky, A.; Tretjakov, P.V.; Tsujikawa, S. The Fate of (phantom) dark energy universe with string curvature corrections. *Phys. Lett. B* **2005**, *619*, 193. [CrossRef]
27. Cognola, G.; Elizalde, E.; Nojiri, S.; Odintsov, S.D.; Zerbini, S. Dark energy in modified Gauss-Bonnet gravity: Late-time acceleration and the hierarchy problem. *Phys. Rev. D* **2006**, *73*, 084007. [CrossRef]

28. Elizalde, E.; Myrzakulov, R.; Obukhov, V.V.; Saez-Gomez, D. LambdaCDM epoch reconstruction from F(R,G) and modified Gauss-Bonnet gravities. *Class. Quant. Gravity* **2010**, *27*, 095007. [CrossRef]
29. Benetti, M.; da Costa, S.S.; Capozziello, S.; Alcaniz, J.S.; Laurentis, M.D. Observational constraints on Gauss-Bonnet cosmology. *Int. J. Mod. Phys. D* **2018**, *27*, 1850084. [CrossRef]
30. Odintsov, S.D.; Oikonomou, V.K.; Fronimos, F.P. Late-Time Cosmology of Scalar-Coupled $f(R,\mathcal{G})$ Gravity. *Class. Quant. Gravity* **2021**, *38*, 075009. [CrossRef]
31. Pozdeeva, E.O.; Sami, M.; Toporensky, A.V.; Vernov, S.Y. Stability analysis of de Sitter solutions in models with the Gauss-Bonnet term. *Phys. Rev. D* **2019**, *100*, 083527. [CrossRef]
32. Skugoreva, M.A.; Toporensky, A.V.; Vernov, S.Y. Global stability analysis for cosmological models with nonminimally coupled scalar fields. *Phys. Rev. D* **2014**, *90*, 064044. [CrossRef]
33. Pozdeeva, E.O.; Skugoreva, M.A.; Toporensky, A.V.; Vernov, S.Y. Possible evolution of a bouncing universe in cosmological models with nonminimally coupled scalar fields. *J. Cosmol. Astropart. Phys.* **2016**, *1612*, 006. [CrossRef]
34. Vernov, S.; Pozdeeva, E. De Sitter Solutions in Einstein–Gauss–Bonnet Gravity. *Universe* **2021**, *7*, 149. [CrossRef]
35. Pozdeeva, E.O.; Vernov, S.Y. Construction of inflationary scenarios with the Gauss–Bonnet term and nonminimal coupling. *Eur. Phys. J. C* **2021**, *81*, 633. [CrossRef]
36. Pozdeeva, E.O. Deviation from Slow-Roll Regime in the EGB Inflationary Models with $r \sim N_e^{-1}$. *Universe* **2021**, *7*, 181 [CrossRef]