symmetry

MDPI

Article

# Interior Multi-Peak Solutions for a Slightly Subcritical Nonlinear Neumann Equation

**Mohamed Ben Ayed [1,2], Khalil El Mehdi [1,3,*] and Fatimetou Mohamed Salem [3]**

[1] Department of Mathematics, College of Science, Qassim University, Buraydah 51452, Saudi Arabia; m.benayed@qu.edu.sa or mohamed.benayed@fss.rnu.tn
[2] Faculté des Sciences de Sfax, Université de Sfax, Route Soukra, Sfax 3052, Tunisia
[3] Faculté des Sciences et Techniques, Université de Nouakchott, Nouakchott BP 5026, Mauritania; fatimetou.salem@una.mr
* Correspondence: k.jiyid@qu.edu.sa

**Abstract:** In this paper, we consider the nonlinear Neumann problem $(\mathcal{P}_\varepsilon)$: $-\Delta u + V(x)u = K(x)u^{(n+2)/(n-2)-\varepsilon}$, $u > 0$ in $\Omega$, $\partial u/\partial \nu = 0$ on $\partial\Omega$, where $\Omega$ is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 4$, $\varepsilon$ is a small positive real, and $V$ and $K$ are non-constant smooth positive functions on $\overline{\Omega}$. First, we study the asymptotic behavior of solutions for $(\mathcal{P}_\varepsilon)$ which blow up at interior points as $\varepsilon$ moves towards zero. In particular, we give the precise location of blow-up points and blow-up rates. This description of the interior blow-up picture of solutions shows that, in contrast to a case where $K \equiv 1$, problem $(\mathcal{P}_\varepsilon)$ has no interior bubbling solutions with clustered bubbles. Second, we construct simple interior multi-peak solutions for $(\mathcal{P}_\varepsilon)$ which allow us to provide multiplicity results for $(\mathcal{P}_\varepsilon)$. The strategy of our proofs consists of testing the equation with vector fields which make it possible to obtain balancing conditions which are satisfied by the concentration parameters. Thanks to a careful analysis of these balancing conditions, we were able to obtain our results. Our results are proved without any assumptions of the symmetry or periodicity of the function $K$. Furthermore, no assumption of the symmetry of the domain is needed.

**Keywords:** partial differential equations; Schrödinger equation; Neumann elliptic problems; blow-up analysis; critical Sobolev exponent



**Citation:** Ben Ayed, M.; El Mehdi, K.; Mohamed Salem, F. Interior Multi-Peak Solutions for a Slightly Subcritical Nonlinear Neumann Equation. *Symmetry* **2024**, *16*, 291. https://doi.org/10.3390/sym16030291

Academic Editor: Calogero Vetro

Received: 2 February 2024
Revised: 21 February 2024
Accepted: 28 February 2024
Published: 2 March 2024



## 1. Introduction and Main Results

In this paper, we consider the following nonlinear Neumann equation:

$$(\mathcal{P}_{V,K,q}): \qquad \begin{cases} -\Delta u + V(x)u = K(x)u^q,\ u > 0 & \text{in} \quad \Omega \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 3$, $1 < q < \infty$ and $K$ and $V$ are smooth positive functions defined by $\overline{\Omega}$.

Such equations arise in various areas of applied sciences; for example, the Keller–Segel model in chemotaxis [1], the Gierer–Meinhardt model for biological pattern formation [2], and stationary waves for nonlinear Schrödinger equations, see, e.g., [3–5].

In the last few decades, equation $(\mathcal{P}_{V,K,q})$ has been widely studied. Most of the works have been carried out when the functions $K(x) \equiv 1$ and $V(x) = \mu > 0$. In this case, it is well known that the problem $(\mathcal{P}_{\mu,1,q})$ strongly depends on the constant $\mu$, the exponent $q$, and the dimension $n$. When $q$ is subcritical, that is, $1 < q < (n+2)/(n-2)$, the only solution to $(\mathcal{P}_{\mu,1,q})$ is the constant one for a small $\mu$ [6], whereas for a large $\mu$, non-constant solutions for $(\mathcal{P}_{\mu,1,q})$ exist and blow up at interior points or at boundary points, or at mixed points (some of them in the interior and others on the boundary), see the review in [7]. When $q$ is critical, that is, $q = (n+2)/(n-2)$, the problem $(\mathcal{P}_{\mu,1,q})$ becomes more difficult. On one hand, Zhu [8] proved that, if $\Omega$ is convex, $n = 3$ and $\mu$ is small, then $(\mathcal{P}_{\mu,1,q})$ has only constant solutions. On the other hand, if $n \in \{4,5,6\}$ and $\mu$ id small, $(\mathcal{P}_{\mu,1,q})$ admits non-constant solutions [9–11]. For a large $\mu$, $(\mathcal{P}_{\mu,1,q})$ also has solutions which blow up, as in the subcritical case, at boundary points as $\mu$ tends to infinity [12–15]. However, in contrast with the subcritical case, at least one blow-up point has to be on the boundary [16]. In [17,18], the authors considered the problem $(\mathcal{P}_{\mu,1,q})$ for a fixed $\mu$ when the exponent $q$ is close to the critical one; that is, $q = (n+2)/(n-2) \pm \varepsilon$ and $\varepsilon$ is a small positive parameter. They showed the existence of a single-boundary blow-up solution for $n \geq 4$. They also constructed single interior blowing-up solutions when $n = 3$. Recently, it has been proved that, unlike dimension three, problem $(\mathcal{P}_{\mu,1,q})$ has no solution, blowing up at only interior points when $n \geq 4$, $q = (n+2)/(n-2) + \varepsilon$ and $\varepsilon$ is small, positive, and real [19]. In light of the results mentioned above, we see that problem $(\mathcal{P}_{V,K,q})$ requires further study.

In [20], the authors considered problem $(\mathcal{P}_{V,1,\frac{n+2}{n-2}-\varepsilon})$; that is, a case in which the functions $K \equiv 1$ and $q = (n+2)/(n-2) - \varepsilon$ and $\varepsilon$ is small, positive, and real. They constructed simple interior bubbling solutions. They also showed the presence of interior bubbling solutions with clustered bubbles. Note that, in the results of [20], all concentration points in the interior bubbling solutions constructed (simple and clustered) converge to critical points of the function $V$ as $\varepsilon$ moves towards zero. The same phenomena appears in [21] when the author studied the location of the blow up of the ground states of $(\mathcal{P}_{V,1,\frac{n+2}{n-2}-\varepsilon})$ in the hall space. Indeed, he proved that, under some conditions of $V$, the ground-state solution concentrates at a global minimum of $V$. In view of these results, a natural question arises: what happens when the function $K$ is not constant? In particular, do interior bubbling solutions (simple and clustered) still exist? If this is the case, do the concentration points converge, as $\varepsilon$ moves towards zero, to critical points of $V$ or $K$? These questions motivate the present paper. We show that simple interior bubbling solutions still exist and, in contrast with problem $(\mathcal{P}_{V,1,\frac{n+2}{n-2}-\varepsilon})$ studied in [20], we prove that $(\mathcal{P}_{V,K,\frac{n+2}{n-2}-\varepsilon})$ has no interior bubbling solutions with clustered bubbles. In addition, we show that the presence of a non-constant function $K$, in equation $(\mathcal{P}_{V,K,\frac{n+2}{n-2}-\varepsilon})$, excludes the role played by the function $V$ in determining the locations of interior concentration points. Indeed, ignoring the presence of the function $V$, all the interior blow-up points converge to critical points of $K$ as $\varepsilon$ moves towards zero. To state our results, we need to define some notation. Throughout the remainder of this paper, we consider the following nonlinear Neumann problem:

$$(\mathcal{P}_\varepsilon): \qquad \begin{cases} -\Delta u + V(x)u = K(x)u^{p-\varepsilon},\ u > 0 \quad \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$ of class $C^1$, $n \geq 4$, $\varepsilon$ is a small positive parameter, $p+1 = \frac{2n}{n-2}$ is the critical Sobolev exponent for the embedding $H^1(\Omega) \hookrightarrow L^q(\Omega)$, and $K$ (resp., $V$) is a $C^3$ (resp. $C^2$) positive function defined by $\overline{\Omega}$.

Problem $(\mathcal{P}_\varepsilon)$ has a variational structure. Solutions to $(\mathcal{P}_\varepsilon)$ are the positive critical points of the functional

$$I_\varepsilon(u) = \frac{1}{2}\int_\Omega \Big(|\nabla u|^2 + V(x)u^2\Big) - \frac{1}{p+1-\varepsilon}\int_\Omega K(x)|u|^{p+1-\varepsilon} \tag{1}$$

defined by $H^1(\Omega)$ equipped with the norm $\|.\|$ and its corresponding inner product given by:

$$\|u\|^2 = \int_\Omega \Big(|\nabla u|^2 + u^2\Big)dx, \quad (u,v) = \int_\Omega (\nabla u \nabla v + uv)dx.$$

Note that all solutions $u_\varepsilon$ to $(\mathcal{P}_\varepsilon)$ satisfy $\|u_\varepsilon\| \geq C$ with a positive constant $C$ independent of $\varepsilon$. Thus, the concentration compactness principle [22,23] implies that, if $u_\varepsilon$ is an energy-bounded solution to $(\mathcal{P}_\varepsilon)$ which converges weakly to 0, then $u_\varepsilon$ has to blow up at a finite number $N$ of points of $\overline{\Omega}$. More precisely, $u_\varepsilon$ can be written as

$$u_\varepsilon = \sum_{i=1}^{N} K(a_{i,\varepsilon})^{(2-n)/4}\delta_{a_{i,\varepsilon},\lambda_{i,\varepsilon}} + v_\varepsilon \quad \text{where} \tag{2}$$

$$\|v_\varepsilon\|_{H^1} \to 0,\ \lambda_{i,\varepsilon} \to \infty,\ a_{i,\varepsilon} \to \overline{a}_i \in \overline{\Omega} \quad \forall i \quad \text{as} \quad \varepsilon \to 0, \tag{3}$$

$$\varepsilon_{ij} := \left(\frac{\lambda_{i,\varepsilon}}{\lambda_{j,\varepsilon}} + \frac{\lambda_{j,\varepsilon}}{\lambda_{i,\varepsilon}} + \lambda_{i,\varepsilon}\lambda_{j,\varepsilon}|a_{i,\varepsilon} - a_{j,\varepsilon}|^2\right)^{\frac{2-n}{2}} \to 0 \text{ as } \varepsilon \to 0 \quad \forall i \neq j \tag{4}$$

and $\delta_{a,\lambda}$ are the so-called bubbles defined by

$$\delta_{a,\lambda}(x) = c_0\frac{\lambda^{\frac{n-2}{2}}}{(1+\lambda^2 \mid x-a\mid^2)^{\frac{n-2}{2}}}, \qquad \lambda > 0, a,\ x \in \mathbb{R}^n,\ c_0 = (n(n-2))^{\frac{n-2}{4}} \tag{5}$$

and which are, see [24], the only solutions to the following problem

$$-\Delta u = u^{\frac{n+2}{n-2}},\ u > 0 \quad \text{in} \quad \mathbb{R}^n.$$

In this paper, our aim is to deal with the qualitative properties and existence of interior concentrating solutions for problem $(\mathcal{P}_\varepsilon)$. More precisely, we consider the case where

$$d(a_{i,\varepsilon}, \partial\Omega) \geq d_0 > 0 \quad \forall\, i \in \{1, \cdots, N\}. \tag{6}$$

We first start by studying the asymptotic behavior of solutions to $(\mathcal{P}_\varepsilon)$ which blow up at interior points as $\varepsilon$ moves towards zero. It should be noticed that the symmetry of the domain simplifies the choice of the blow up points and reduces the number of unknown variables. In this paper, our results are proved without any assumptions of the symmetry of domain or of the function $K$. We give a complete description of the interior blow-up pictures of solutions that weakly converge to zero. Namely, we prove:

**Theorem 1.** *Let $n \geq 4$, $K$ (resp., $V$) $: \overline{\Omega} \to \mathbb{R}$ be a $C^3$ (resp., $C^2$) positive function, and $y_1, \cdots y_q$ be the critical points of satisfying $K$; if $n \geq 5$, then we make following assumption*

$$-\overline{c}_2\frac{\Delta K(y_i)}{K(y_i)} + d_n V(y_i) \neq 0 \qquad \forall\, i \in \{1, \cdots, q\}, \tag{7}$$

*where*

$$\overline{c}_2 = \frac{n-2}{4n}c_0^{p+1}\int_{\mathbb{R}^n}\frac{|x|^2(|x|^2-1)}{(1+|x|^2)^{n+1}}\,dx \quad \text{and} \quad d_n = \frac{(n-2)c_0^2}{2}\int_{\mathbb{R}^n}\frac{(|x|^2-1)}{(1+|x|^2)^{n-1}}\,dx.$$

*Let $(u_\varepsilon)$ be a sequence of solutions of $(\mathcal{P}_\varepsilon)$ having the form (2) and satisfying (3), (4), and (6). In addition if the number $N$ of concentration points (defined in (2)) is bigger than or equal to 2, we assume that all the critical points $y_i'$s of $K$ are non-degenerate. Then, the following facts hold*

*(i) For any $i \in \{1, \cdots, N\}$, there exists $j_i \in \{1, \cdots, q\}$, such that the concentration point $a_{i,\varepsilon}$ converges to the critical point $y_{j_i}$ of $K$ as $\varepsilon \to 0$. In addition, if $n \geq 5$, we have*

$$-\overline{c}_2\frac{\Delta K(y_{j_i})}{K(y_{j_i})} + d_n V(y_{j_i}) > 0. \tag{8}$$

(ii) *For any $i \in \{1, \cdots, N\}$, we have*

$$\begin{cases} \frac{d_4}{\bar{c}_1} V(y_{j_i}) \frac{\ln \lambda_{i,\varepsilon}}{\lambda_{i,\varepsilon}^2} = \varepsilon(1 + o_\varepsilon(1)) & \text{if} \quad n = 4, \\ \left( -\frac{\bar{c}_2}{\bar{c}_1} \frac{\Delta K(y_{j_i})}{K(y_i)} + \frac{d_n}{\bar{c}_1} V(y_{j_i}) \right) \frac{1}{\lambda i, \varepsilon^2} = \varepsilon(1 + o_\varepsilon(1)) & \text{if} \quad n \geq 5, \end{cases} \tag{9}$$

*where $d_4 := 2\sqrt{2}\, meas(\mathbb{S}^3)$ and*

$$\bar{c}_1 = \frac{(n-2)^2}{4} c_0^{p+1} \int_{\mathbb{R}^n} \frac{(|x|^2 - 1)\ln(1 + |x|^2)}{(1 + |x|^2)^{n+1}}\, dx.$$

(iii) *If the number $N$ of concentration points satisfies $N \geq 2$, then $N \leq m$ and a positive constant $c$ exists independent of $\varepsilon$ such that the concentration points satisfy*

$$|a_{i,\varepsilon} - a_{k,\varepsilon}| \geq c \quad \forall\, i \neq k,$$

*where $m$ is defined by*

$$m = \begin{cases} \text{cardinal of } \{y \text{ is a non-degenerate critical point of } K\} & \text{if} \quad n = 4, \\ \text{cardinal of } \{y \text{ is a non-degenerate critical point of } K \text{ satisfying } (8)\} & \text{if} \quad n \geq 5. \end{cases}$$

**Remark 1.**

1. *When the number $N$ of the concentration points satisfies $N \geq 2$, the non-degeneracy assumption is used to show that two concentration points cannot converge to the same critical point of $K$. This shows that the presence of a Morse function $K$ in the equation $(\mathcal{P}_{V,K,\frac{n+2}{n-2}-\varepsilon})$ excludes the existence of interior bubbling solutions with clustered bubbles.*
2. *Theorem 1 also excludes the existence of solutions which resemble the form of a super-position of spikes centered at one point, as in the slightly super-critical problem [25].*
3. *It is easy to construct a function $K$ satisfying (7) and (8) for any positive function $V$. For example, assuming, without the loss of generality, $0 \in \Omega$ and taking a positive real $\gamma$ such that $B(0, 2\gamma) \subset \Omega$. Let $a := (\gamma, 0, \cdots, 0)$, we can take*

$$K(x) = R - \ln(\gamma^2 + 2|x - a|^2) - \ln(\gamma^2 + 2|x + a|^2)$$

*with $R$ chosen to be large so that $K > 0$ in $\Omega$. By easy computations, we can check that $\Delta K(x) < 0$ for any $x \in \Omega$, and $K$ has only three critical points which are $0$, $b := (\gamma/\sqrt{2}, 0, \cdots, 0)$, and $-b$. These critical points are non-degenerate. Clearly, (7) and (8) are satisfied for any positive function $V$.*

Our next result provides a kind of converse of Theorem 1. More precisely, our aim is to construct solutions to $(\mathcal{P}_\varepsilon)$ which blow up at multiple interior points as $\varepsilon$ moves towards zero.

**Theorem 2.** *Let $n \geq 4$, $K$ (resp., $V$) $: \overline{\Omega} \to \mathbb{R}$ be a $C^3$ (resp., $C^2$) positive function. Let $N \leq m$ (where $m$ is defined in Theorem 1) and $y_1, \cdots y_N$ be non-degenerate critical points of $K$. If $n \geq 5$, we further assume that they satisfy assumption (8). Then, there exists, for $\varepsilon$ small, a sequence of solutions to $(\mathcal{P}_\varepsilon)$ which decomposes as in (2) with the properties (3), (4), (6), and (9). In particular $(\mathcal{P}_\varepsilon)$ admits at least $2^m - 1$ solutions.*

To prove our results, we make a refined asymptotic analysis of the gradient of the functional $I_\varepsilon$ and we then test the equation using vector fields which make possible to obtain balancing conditions satisfied by the concentration parameters. Through a careful study of these balancing conditions, we obtain our results.

The rest of this paper is organized as follows: in Section 2, we make a precise estimate of the infinite dimensional part of $u_\varepsilon$. Section 3 is devoted to the expansion of the gradient of the functional $I_\varepsilon$. In Section 4, we study the asymptotic behavior of the solutions to $(\mathcal{P}_\varepsilon)$ which blow up at interior points as $\varepsilon$ moves towards zero. This allows us to provide proof for Theorem 1. In Section 5 we construct solutions of $(\mathcal{P}_\varepsilon)$ which blow up at multiple interior points as $\varepsilon$ moves towards zero which gives the proof for Theorem 2. Lastly, in Section 6, we present some future perspectives.

## 2. Estimate of the Infinite Dimensional Part

For $N \in \mathbb{N}^*$ , let $(u_\varepsilon)$ be a sequence with the form (2) with the properties (3), (4), and (6). It is well known that there is a unique way to choose $\lambda i, \varepsilon$, $a_{i,\varepsilon}$, and $v_\varepsilon$ such that

$$u_\varepsilon = \sum_{i=1}^{N} \alpha_{i,\varepsilon}\delta_{a_{i,\varepsilon},\lambda_{i,\varepsilon}} + v_\varepsilon \quad \text{with} \tag{10}$$

$$\begin{cases} \left|\alpha_{i,\varepsilon}^{4/n-2}K(a_{i,\varepsilon}) - 1\right| \to 0 \text{ as } \varepsilon \to 0 \quad \forall i \in \{1, \dots, N\}, \\ a_{i,\varepsilon} \to \bar{a}_i \in , \;\; \lambda_{i,\varepsilon} \to \infty \text{ as } \varepsilon \to 0 \quad \forall i \in \{1, \dots, N\}, \\ \varepsilon_{ij} \to 0 \text{ as } \varepsilon \to 0 \quad \forall i \neq j, \;\; \|v_\varepsilon\|_{H^1} \to 0, \; v_\varepsilon \in E_{a_\varepsilon,\lambda\varepsilon} \end{cases} \tag{11}$$

where $(a_\varepsilon, \lambda\varepsilon) \in \Omega^N \times (0,\infty)^N$, $E_{a_\varepsilon,\lambda\varepsilon}$ denotes

$$E_{a_\varepsilon,\lambda\varepsilon} = \Big\{ v \in H^1(\Omega) : \int_\Omega \nabla v \nabla \delta_{a_{i,\varepsilon},\lambda_{i,\varepsilon}} = \int_\Omega \nabla v \nabla \frac{\partial \delta_{a_{i,\varepsilon},\lambda_{i,\varepsilon}}}{\partial \lambda_{i,\varepsilon}} = \int_\Omega \nabla v \frac{\partial \delta_{a_{i,\varepsilon},\lambda_{i,\varepsilon}}}{\partial a_{i,\varepsilon}^j} = 0, \\ \forall 1 \leqslant i \leqslant N, \forall 1 \leqslant j \leqslant n \Big\}. \tag{12}$$

For the proof of this fact, see [26,27]. To simplify the notation, we will set, in the sequel, $a_i = a_{i,\varepsilon}$, $\lambda i = \lambda i, \varepsilon, \delta_i = \delta_{a_{i,\varepsilon},\lambda i,\varepsilon}, \alpha_i = \alpha_{i,\varepsilon}$, and $E_{a,\lambda} = E_{a_\varepsilon,\lambda\varepsilon}$. Throughout the sequel, we assume that $u_\varepsilon$ is written as in (10) and (11). To study the case of interior blowing-up solutions, we need to introduce the following set

$$\mathcal{O}(N, \mu_0) = \Big\{ (\alpha,\lambda, a, v) \in (\mathbb{R}_+)^N \times (\mathbb{R}_+)^N \times \Omega^N \times H^1(\Omega) : |\alpha_i^{4/n-2}K(a_i) - 1| < \mu_0, \\ \lambda_i > \mu_0^{-1}, \, \varepsilon \ln \lambda_i < \mu_0, \, d(a_i, \partial\Omega) > c, \varepsilon_{ij} < \mu_0, \, v \in E_{a,\lambda}, \, \|v\| < \mu_0 \Big\}, \tag{13}$$

where $\mu_0$ is positive, small, and real.

Next, we are going to deal with the $v$-part in (10). To this end, we perform an expansion of the associated functional $I_\varepsilon$ defined by (1) with respect to $v \in E_{a,\lambda}$ satisfying $|v| < \mu_0$, where $\mu_0$ is a positive small constant. Let $(\alpha, \lambda, a, 0) \in \mathcal{O}(N, \mu_0)$, taking $\bar{u} = \sum_{i=1}^N \alpha_i\delta_i$ and $v \in E_{a,\lambda}$ with $\|v\| < \mu_0$, we observe that

$$I_\varepsilon(\bar{u} + v) = \frac{1}{2}\int_\Omega |\nabla \bar{u}|^2 + \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{1}{2}\int_\Omega V(x)\bar{u}^2 + \frac{1}{2}\int_\Omega V(x)v^2 \\ + \int_\Omega V(x)\bar{u}\, v - \frac{1}{p+1-\varepsilon}\int_\Omega K(x)|\bar{u} + v|^{p+1-\varepsilon}.$$

But we have

$$\int_\Omega K(x)|\bar{u} + v|^{p+1-\varepsilon} = \int_\Omega K(x)\bar{u}^{p+1-\varepsilon} + (p+1-\varepsilon)\int_\Omega K(x)\bar{u}^{p-\varepsilon}v \\ + \frac{(p+1-\varepsilon)(p-\varepsilon)}{2}\int_\Omega K(x)\bar{u}^{p-1-\varepsilon}v^2 + R_\varepsilon(v),$$

where

$$R_\varepsilon(v) = O\Big(\int_\Omega |v|^{p+1-\varepsilon}\Big) + (\text{if } n \leqslant 5)\, O\Big(\int_\Omega \bar{u}^{p-2-\varepsilon}|v|^3\Big) = O\Big(\|v\|^{\min(3,p+1-\varepsilon)}\Big) \tag{14}$$

which implies that

$$I_\varepsilon(\bar{u}+v) = I_\varepsilon(\bar{u}) + \langle l_\varepsilon, v\rangle + \frac{1}{2}Q_\varepsilon(v) + R_\varepsilon(v) \quad \text{where}$$

$$\langle l_\varepsilon, v\rangle = \int_\Omega V(x)\bar{u}v - \int_\Omega K(x)\bar{u}^{p-\varepsilon}v, \tag{15}$$

$$Q_\varepsilon(v) = \int_\Omega |\nabla v|^2 + \int_\Omega V(x)v^2(x) - (p-\varepsilon)\int_\Omega K(x)\bar{u}^{p-1-\varepsilon}v^2. \tag{16}$$

Notice that the derivatives of $R_\varepsilon$ satisfy

$$R'_\varepsilon(v) = O\Big(\|v\|^{\min(2,p-\varepsilon)}\Big) \quad \text{and} \quad R''_\varepsilon(v) = O\Big(\|v\|^{\min(1,p-1-\varepsilon)}\Big). \tag{17}$$

Next, we are going to prove the uniform coercivity of the quadratic form $Q_\varepsilon$.

**Proposition 1.** *Let $n \geq 3$ and $(\alpha,\lambda,a,0) \in \mathcal{O}(N,\mu_0)$. Then, there exists $\varepsilon_0 > 0$ and $C > 0$ such that, for $\varepsilon \in (0,\varepsilon_0)$, the quadratic form $Q_\varepsilon$, defined by (16), satisfies*

$$Q_\varepsilon(v) \geq C\|v\|^2 \;\; \forall v \in \; E_{a,\lambda}.$$

**Proof.** On one hand, since $\varepsilon \ln \lambda_i$ is small and $\Omega$ is bounded, Taylor's expansion implies that

$$\delta_i^{-\varepsilon} = c_0^{-\varepsilon}\lambda_i^{-\varepsilon(n-2)/2}\Big(1 + O\Big(\varepsilon\ln\Big(1+\lambda_i^2|x-a_i|^2\Big)\Big)\Big) = 1 + o(1). \tag{18}$$

On the other hand, letting $v \in E_{a,\lambda}$, we have

$$\varepsilon\int_\Omega K\bar{u}^{p-1-\varepsilon}v^2 \leqslant c\varepsilon\sum_{i=1}^N\int_\Omega \delta_i^{p-1-\varepsilon}v^2 \leqslant c\varepsilon\|v\|^2, \tag{19}$$

$$\int_\Omega K\bar{u}^{p-1-\varepsilon}v^2 = \sum_{i=1}^N\int_\Omega K(\alpha_i\delta_i)^{p-1-\varepsilon}v^2 + \begin{cases}\sum_{j\neq i} O\Big(\int_\Omega (\delta_i\delta_j)^{(p-1)/2}v^2\Big) & \text{if} \quad n \geq 4,\\ \sum_{j\neq i} O\big(\int_\Omega \delta_i^3\delta_j v^2\big) & \text{if} \quad n = 3.\end{cases} \tag{20}$$

But, using estimate $E_2$ of [26] and Holder's inequality, we obtain

$$\int_\Omega (\delta_i\delta_j)^{(p-1)/2}v^2 \leqslant c\Big(\varepsilon_{ij}^{n/(n-2)}\ln\varepsilon_{ij}^{-1}\Big)^{2/n}\|v\|^2 = c\varepsilon_{ij}^{2/(n-2)}(\ln\varepsilon_{ij}^{-1})^{2/n}\|v\|^2. \tag{21}$$

Thus, combining (18)–(21), we obtain

$$Q_\varepsilon(v) = \int_\Omega |\nabla v|^2 + \int_\Omega Vv^2 - p\sum_{i=1}^N \alpha_i^{p-1}\int_\Omega K\delta_i^{p-1}v^2 + o(\|v\|^2). \tag{22}$$

But, we have

$$\alpha_i^{p-1}\int_\Omega K\delta_i^{p-1}v^2 = \alpha_i^{p-1}K(a_i)\int_\Omega \delta_i^{p-1}v^2 + O\Big(\int_\Omega |K(x)-K(a_i)|\delta_i^{p-1}v^2\Big)$$

and we notice that

$$\int_\Omega |K(x)-K(a_i)|\delta_i^{p-1}v^2 \leq c\int_\Omega |x-a_i|\delta_i^{p-1}v^2 = o(\|v\|^2).$$

Using the fact that $\alpha_i^{p-1}K(a_i) = 1 + o(1)$, we obtain

$$Q_\varepsilon(v) = \int_\Omega |\nabla v|^2 + \int_\Omega Vv^2 - p\sum_{i=1}^N\int_\Omega \delta_i^{p-1}v^2 + o\Big(\|v\|^2\Big). \tag{23}$$

But, according to the proof of Proposition 1 in [20], a positive constant $c$ exists such that

$$\int_\Omega |\nabla v|^2 + \int_\Omega V v^2 - p \sum_{i=1}^N \int_\Omega \delta_i^{p-1} v^2 \geq c\|v\|^2$$

which gives the desired result. The proof of the proposition is thereby complete. □

Next, we are going to give the estimate of the infinite dimensional variable $v_\varepsilon$. Our result reads as follows.

**Proposition 2.** *Let $n \geq 3$ and $(\alpha, \lambda, a, 0) \in \mathcal{O}(N, \mu_0)$. Then, if $\mu_0 > 0$ is small enough, a unique $\bar{v}_\varepsilon \in E_{a,\lambda}$ exists which minimizes $I_\varepsilon\left(\sum_{i=1}^N \alpha_i \delta_{a_i,\lambda_i} + v\right)$ with respect to $v \in E_{a,\lambda}$, and $\|v\|$ is small. In particular, we have*

$$\left\langle I'_\varepsilon\Big(\sum_{i=1}^N \alpha_i \delta_{a_i,\lambda_i} + \bar{v}_\varepsilon\Big), w\right\rangle = 0 \quad \forall\, w \in E_{a,\lambda}. \tag{24}$$

*In addition, $\bar{v}_\varepsilon$ satisfies the following estimate*

$$\|\bar{v}_\varepsilon\| \leq c R_{\varepsilon,a,\lambda}, \quad \textit{where} \tag{25}$$

$$R_{\varepsilon,a,\lambda} = \varepsilon + \sum_{i=1}^N \frac{|\nabla K(a_i)|}{\lambda_i} + \begin{cases} \sum_{j\neq i} \varepsilon_{ij} + \sum \lambda_i^{(2-n)/2} & \textit{if } n \leq 5, \\ \sum_{j\neq i} \varepsilon_{ij} (\ln \varepsilon_{ij}^{-1})^{2/3} + \lambda_i^{-2} (\ln \lambda_i)^{2/3} & \textit{if } n = 6, \\ \sum_{j\neq i} \varepsilon_{ij}^{(n+2)/(2(n-2))} (\ln \varepsilon_{ij}^{-1})^{(n+2)/(2n)} + \lambda_i^{-2} & \textit{if } n \geq 7. \end{cases}$$

**Proof.** Using estimate (17), Proposition 1, and the implicit function theorem, we derive that, for a $\varepsilon$ small, $\bar{v}_\varepsilon \in E_{a,\lambda}$ exists, such that $\|\bar{v}_\varepsilon\| = O(\|l_\varepsilon\|)$, where $l_\varepsilon$ is defined by (15). Thus, we need to estimate $l_\varepsilon$. To this end, letting $v \in E_{a,\lambda}$, we observe that

$$\int_\Omega V \bar{u} |v| \leqslant c \sum_{i=1}^N \alpha_i \int_\Omega \delta_i |v| \leqslant c \sum_{i=1}^N \|v\| R(\lambda_i) \quad \text{where}$$

$$R(\lambda_i) = \begin{cases} \lambda_i^{-2} \ln^{2/3}(\lambda_i) & \text{if } n = 6, \\ \lambda_i^{-\min(2, \frac{n-2}{2})} & \text{if } n \neq 6. \end{cases} \tag{26}$$

We also observe that

$$\int_\Omega K \bar{u}^{p-\varepsilon} v = \sum_{i=1}^N \alpha_i^{p-\varepsilon} \int_\Omega K \delta_i^{p-\varepsilon} v + \begin{cases} \sum_{j\neq i} O\left(\int_\Omega (\delta_i \delta_j)^{\frac{p-\varepsilon}{2}} |v|\right) & \text{if} \quad n \geq 6, \\ \sum_{j\neq i} O\left(\int_\Omega \delta_i^{p-1-\varepsilon} \delta_j |v|\right) & \text{if} \quad n \leq 5. \end{cases} \tag{27}$$

But, using (18) and estimate $E_2$ of [26], we obtain

$$\int_\Omega (\delta_i \delta_j)^{\frac{p-\varepsilon}{2}} |v| \leqslant c \int_\Omega (\delta_i \delta_j)^{\frac{p}{2}} |v| \leqslant c \left(\int_\Omega (\delta_i \delta_j)^{\frac{n}{n-2}}\right)^{\frac{n+2}{2n}} \|v\| \leqslant c \|v\| \varepsilon_{ij}^{\frac{n+2}{2(n-2)}} \ln^{\frac{n+2}{2n}} \varepsilon_{ij}^{-1}. \tag{28}$$

For $n \leq 5$, we have $1 < \frac{2n}{n+2} < \frac{8n}{n^2-4}$. Thus, it follows from (18) and Lemma 6.6 of [19] that

$$\int_\Omega \delta_i^{p-1-\varepsilon} \delta_j |v| = O\left(\int_\Omega \delta_i^{p-1} \delta_j |v|\right) = O\left(\|v\| \left(\int_\Omega \delta_j^{\frac{2n}{n+2}} \delta_i^{\frac{8n}{n^2-4}}\right)^{\frac{n+2}{2n}}\right) = O(\|v\|\, \varepsilon_{ij}). \tag{29}$$

For the other term in right hand side of (27), using estimate (18), we obtain

$$\int_\Omega K\delta_i^{p-\varepsilon}v = c_0^{-\varepsilon}\lambda\varepsilon^{-\frac{\varepsilon(n-2)}{2}}\int_\Omega K\delta_i^p v + O\Big(\varepsilon\int_\Omega \delta_i^p|v|\ln\Big(1+\lambda_i^2|x-a_i|^2\Big)\Big)$$
$$= c_0^{-\varepsilon}\lambda\varepsilon^{-\frac{\varepsilon(n-2)}{2}}K(a_i)\int_\Omega \delta_i^p v + O\Big(|\nabla K(a_i)|\int_\Omega |x-a_i|\delta_i^p|v|\Big)$$
$$+ O\Big(\int_\Omega |x-a_i|^2\delta_i^p|v| + \varepsilon\,\|v\|\Big). \tag{30}$$

Observe that

$$\int_\Omega |x-a_i|\delta_i^p|v| \le c\|v\|\Big(\int_\Omega |x-a_i|^{\frac{2n}{n+2}}\delta_i^{p+1}\Big)^{(n+2)/(2n)} \leqslant \frac{c\|v\|}{\lambda_i}, \tag{31}$$
$$\int_\Omega |x-a_i|^2\delta_i^p|v| \leqslant c\|v\|\Big(\int_\Omega |x-a_i|^{\frac{4n}{n+2}}\delta_i^{p+1}\Big)^{(n+2)/(2n)} \leqslant \frac{c\|v\|}{\lambda_i^2}. \tag{32}$$

Now, using (6), the fact that $\varepsilon\ln\lambda_i$ is small, and $v_\varepsilon \in E_{a,\lambda}$, we obtain

$$c_0^{-\varepsilon}\lambda\varepsilon^{-\frac{\varepsilon(n-2)}{2}}K(a_i)\int_\Omega \delta_i^p v = O\Big(\int_{\partial\Omega}|\frac{\partial\delta_i}{\partial\nu}v_\varepsilon|\Big)$$
$$= O\Bigg(\frac{1}{\lambda_i^{(n-2)/2}}\int_{\partial\Omega}|v|\Bigg) = O\Big(\frac{\|v\|}{\lambda_i^{(n-2)/2}}\Big). \tag{33}$$

It follows from (30)–(33) that

$$\int_\Omega K\delta_i^{p-\varepsilon}v = O\Big(\frac{|\nabla K(a_i)|}{\lambda_i}\|v\| + \frac{\|v\|}{\lambda i^{\min(2,(n-2)/2)}} + \varepsilon\,\|v\|\Big). \tag{34}$$

Combining (26)–(29) and (34), our proposition follows. □

## 3. Expansion of the Gradient of the Associated Functional

In this section, we are going to perform the expansion of the gradient of the associated Euler–Lagrange functional $I_\varepsilon$ in $\mathcal{O}(N,\mu_0)$. Notice that, for $u,h\in H^1(\Omega)$, we have

$$\langle I'_\varepsilon(u),h\rangle = \int_\Omega \nabla u.\nabla h + \int_\Omega V\,uh - \int_\Omega K|u|^{p-1-\varepsilon}uh. \tag{35}$$

Let $(\alpha,\lambda,a,0)\in\mathcal{O}(N,\mu_0)$. In (35), we will take $u=\sum_{i=1}^N\alpha_i\delta_i + \bar{v}_\varepsilon := \bar{u}+\bar{v}_\varepsilon$ and $h=f_i\in\{\delta_i,\lambda_i\partial\delta_i/\partial\lambda_i,\lambda_i^{-1}\partial\delta_i/\partial a_i\}$ with $1\le i\le N$. Thus, we need to estimate each term in (35). We start by dealing with the last integral in the right hand side of (35). Namely, we prove the proposition below.

**Proposition 3.** *Let $n\ge 3$ and $(\alpha,\lambda,a)$ be such that $(\alpha,\lambda,a,0)\in\mathcal{O}(N,\mu_0)$. Let us denote that $\bar{u}=\sum_{i=1}^N\alpha_i\delta_i$ and $u_\varepsilon=\bar{u}+\bar{v}_\varepsilon$, where $\bar{v}_\varepsilon$ in defined in Proposition 2. Then, for $f_i\in\{\delta_i,\lambda_i\partial\delta_i/\partial\lambda_i,\lambda_i^{-1}\partial\delta_i/\partial a_i\}$ with $1\le i\le N$, the following fact holds*

$$\int_\Omega K|u_\varepsilon|^{p-1-\varepsilon}u_\varepsilon f_i = \int_\Omega K(\alpha_i\delta_i)^{p-\varepsilon}f_i + (p-\varepsilon)\sum_{k\neq i}\int_\Omega K(\alpha_i\delta_i)^{p-1-\varepsilon}(\alpha_k\delta_k)f_i$$
$$+\sum_{k\neq i}\int_\Omega K(\alpha_k\delta_k)^{p-\varepsilon}f_i + (p-\varepsilon)\int_\Omega K(\alpha_i\delta_i)^{p-1-\varepsilon}\bar{v}_\varepsilon f_i$$
$$+O\Big(\|\bar{v}_\varepsilon\|^2 + \sum_{k\neq r}\varepsilon_{kr}^{\frac{n}{n-2}}\ln\varepsilon_{kr}^{-1}\Big) + (\textit{if } n=3)\ O\Big(\sum_{k\neq r}\varepsilon_{kr}^2\ln^{2/3}\varepsilon_{kr}^{-1}\Big). \tag{36}$$

**Proof.** We have

$$\int_\Omega K|u_\varepsilon|^{p-1-\varepsilon}u_\varepsilon f_i = \int_\Omega K\overline{u}^{p-\varepsilon}f_i + (p-\varepsilon)\int_\Omega K\overline{u}^{p-1-\varepsilon}\overline{v}_\varepsilon f_i + O\Big(\int_{[\overline{u}\le|\overline{v}_\varepsilon|]}|\overline{v}_\varepsilon|^{p-\varepsilon}|f_i| + \int_{[|\overline{v}_\varepsilon|\le\overline{u}]}\overline{u}^{p-2-\varepsilon}|\overline{v}_\varepsilon|^2|f_i|\Big). \tag{37}$$

Observe that

$$\int_{[\overline{u}\le|\overline{v}_\varepsilon|]}|\overline{v}_\varepsilon|^{p-\varepsilon}|f_i| + \int_{[|\overline{v}_\varepsilon|\le\overline{u}]}\overline{u}^{p-2-\varepsilon}|\overline{v}_\varepsilon|^2|f_i| \le \int_{[\overline{u}\le|\overline{v}_\varepsilon|]}|\overline{v}_\varepsilon|^{p+1-\varepsilon} + \int_\Omega \overline{u}^{p-1-\varepsilon}|\overline{v}_\varepsilon|^2 \le c\|\overline{v}_\varepsilon\|^2.$$

Thus

$$\int_\Omega K|u_\varepsilon|^{p-1-\varepsilon}u_\varepsilon f_i = \int_\Omega K\bar{u}^{p-\varepsilon}f_i + (p-\varepsilon)\int_\Omega K\bar{u}^{p-1-\varepsilon}\overline{v}_\varepsilon f_i + O\Big(\|\overline{v}_\varepsilon\|^2\Big). \tag{38}$$

To deal with the second integral in the right hand side of (38), we write

$$\int_\Omega K\bar{u}^{p-1-\varepsilon}\overline{v}_\varepsilon f_i = \int_{\Omega_i}\cdots + \int_{\Omega\setminus\Omega_i}\cdots, \quad \text{where} \tag{39}$$

$$\Omega_i = \{x\in\Omega : \sum_{k\ne i}\alpha_k\delta_k(x) \le \frac{1}{2}\alpha_i\delta_i(x)\}.$$

For $n\ge 6$, we have $p-1\le 1$. Using (28), we obtain

$$\int_{\Omega\setminus\Omega_i} K|u_\varepsilon|^{p-1-\varepsilon}|\overline{v}_\varepsilon||f_i| \le c\sum_{k\ne i}\int_{\Omega\setminus\Omega_i}(\delta_k\delta_i)^{\frac{p-\varepsilon}{2}}|\overline{v}_\varepsilon| \le c\sum_{k\ne i}\|\overline{v}_\varepsilon\|\varepsilon_{ik}^{\frac{n+2}{2(n-2)}}(\ln\varepsilon_{ik}^{-1})^{\frac{n+2}{2n}}. \tag{40}$$

and for $n\le 5$, using (29), we obtain

$$\int_{\Omega\setminus\Omega_i} K|u_\varepsilon|^{p-1-\varepsilon}|\overline{v}_\varepsilon||f_i| \le c\sum_{k\ne i}\int_{\Omega\setminus\Omega_i}(\delta_k)^{p-1-\varepsilon}\delta_i|\overline{v}_\varepsilon| \le c\|\overline{v}_\varepsilon\|\sum_{k\ne i}\varepsilon_{ik} \le c\|\overline{v}_\varepsilon\|^2 + c\sum_{k\ne i}\varepsilon_{ik}^2. \tag{41}$$

For the first integral in the right hand side of (39), we write

$$\begin{aligned}\int_{\Omega_i} K\bar{u}^{p-1-\varepsilon}\overline{v}_\varepsilon f_i &= \int_{\Omega_i} K(\alpha_i\delta_i)^{p-1-\varepsilon}\overline{v}_\varepsilon f_i + O\Big(\int_{\Omega_i}(\alpha_i\delta_i)^{p-2-\varepsilon}\Big(\sum_{k\ne i}(\alpha_k\delta_k)\Big)|\overline{v}_\varepsilon||f_i|\Big)\\ &= \int_\Omega K(\alpha_i\delta_i)^{p-1-\varepsilon}\overline{v}_\varepsilon f_i - \int_{\Omega\setminus\Omega_i} K(\alpha_i\delta_i)^{p-1-\varepsilon}\overline{v}_\varepsilon f_i\\ &+ (\text{if } n\ge 6)O\Big(\sum_{k\ne i}\int_{\Omega_i}(\delta_i\delta_k)^{\frac{p}{2}}|\overline{v}_\varepsilon|\Big) + (\text{if } n\le 5)O\Big(\sum_{k\ne i}\int_{\Omega_i}\delta_i^{p-1}\delta_k|\overline{v}_\varepsilon|\Big).\end{aligned} \tag{42}$$

Using Estimate $E_2$ from [26], (39), (40), and (42), we obtain

$$\int_\Omega K\bar{u}^{p-1-\varepsilon}\overline{v}_\varepsilon f_i = \alpha_i^{p-1-\varepsilon}\int_\Omega K\delta_i^{p-1-\varepsilon}\overline{v}_\varepsilon f_i + O\Big(\|\overline{v}_\varepsilon\|^2\Big) + O\Big(\sum_{k\ne i}\Big(\varepsilon_{ik}^{\frac{n}{n-2}}\ln\varepsilon_{ik}^{-1}\Big)^{\min(2\frac{n-2}{n};\frac{n+2}{n})}\Big). \tag{43}$$

Now, we are going to estimate the first integral on the right hand side of (38). To this end, letting $\Omega_i$ be defined by (39), we write

$$\int_\Omega K\bar{u}^{p-\varepsilon}f_i = \int_{\Omega_i} K\bar{u}_\varepsilon^{p-\varepsilon}f_i + \int_{\Omega\setminus\Omega_i} K\bar{u}_\varepsilon^{p-\varepsilon}f_i := I_1 + I_2. \tag{44}$$

For the first integral $(I_1)$ in (44), using (18) and the fact that $|f_i| \le c\delta_i$, it holds that

$$\begin{aligned}
I_1 &= \int_{\Omega_i} K(\alpha_i\delta_i)^{p-\varepsilon}f_i + (p-\varepsilon)\int_{\Omega_i} K(\alpha_i\delta_i)^{p-1-\varepsilon}\Big(\sum_{k\neq i}\alpha_k\delta_k\Big)f_i \\
&+ O\Big(\int_{\Omega_i}(\alpha_i\delta_i)^{p-2}\Big(\sum_{k\neq i}\alpha_k\delta_k\Big)^2|f_i|\Big) \\
&= \int_\Omega K(\alpha_i\delta_i)^{p-\varepsilon}f_i + (p-\varepsilon)\sum_{k\neq i}\int_\Omega K(\alpha_i\delta_i)^{p-1-\varepsilon}\alpha_k\delta_k f_i \\
&+ (\text{ if } n\ge 4)\sum_{k\neq i} O\Big(\int_\Omega(\delta_i\delta_k)^{\frac{n}{n-2}}\Big) + (\text{ if } n=3)\, O\Big(\sum_{k\neq i}\int_\Omega \delta_i^4\delta_k^2\Big).
\end{aligned}$$

Thus, using Estimate $E_2$ from [26], we obtain

$$\begin{aligned}
I_1 &= \alpha_i^{p-\varepsilon}\int_\Omega K\delta_i^{p-\varepsilon}f_i + (p-\varepsilon)\sum_{k\neq i}\int_\Omega K(\alpha_i\delta_i)^{p-1-\varepsilon}(\alpha_k\delta_k)f_i \\
&+ +(\text{ if } n\ge 4)\, O\Big(\varepsilon_{ik}^{\frac{n}{n-2}}\ln\varepsilon_{ik}^{-1}\Big) + +(\text{ if } n=3)\, O\Big(\varepsilon_{ik}^2(\ln\varepsilon_{ik}^{-1})^{\frac{2}{3}}\Big).
\end{aligned} \tag{45}$$

For the second integral on the right hand side of (44), using (18), the fact that $|f_i| \le c\delta_i$, and Estimate $E_2$ from [26], we have

$$\begin{aligned}
\int_{\Omega\setminus\Omega_i} K\bar{u}^{p-\varepsilon}f_i &= \int_{\Omega\setminus\Omega_i} K\Big(\sum_{k\neq i}\alpha_k\delta_k\Big)^{p-\varepsilon}f_i + O\Big(\int_{\Omega\setminus\Omega_i}\Big(\sum_{k\neq i}\alpha_k\delta_k\Big)^{p-1}(\alpha_i\delta_i)|f_i|\Big) \\
&= \sum_{k\neq i}\int_{\Omega\setminus\Omega_i} K(\alpha_k\delta_k)^{p-\varepsilon}f_i + O\Big(\sum_{k,r\neq i,k\neq r}\int_{\Omega\setminus\Omega_i}\inf_{k\neq r,k\neq i}(\delta_k,\delta_r)\sup_{k\neq r,k,r\neq i}(\delta_k,\delta_r)^{p-1}\delta_i\Big) \\
&+ O\Big(\sum_{k\neq i}\int_{\Omega\setminus\Omega_i}(\delta_k\delta_i)^{\frac{n}{n-2}}\Big) + (\text{if } n=3)\ O\Big(\sum_{k\neq i}\int_{\Omega\setminus\Omega_i}\delta_k^4\delta_i^2\Big) \\
&= \sum_{k\neq i}\int_\Omega K(\alpha_k\delta_k)^{p-\varepsilon}f_i + O\Big(\sum_{k\neq r}\int_\Omega(\delta_k\delta_r)^{\frac{n}{n-2}}\Big) + (\text{if } n=3)\ O\Big(\sum_{k\neq i}\varepsilon_{ik}^2\ln^{2/3}\varepsilon_{ik}^{-1}\Big) \\
&= \sum_{k\neq i}\int_\Omega K(\alpha_k\delta_k)^{p-\varepsilon}f_i + O\Big(\sum_{k\neq r}\varepsilon_{kr}^{\frac{n}{n-2}}\ln\varepsilon_{kr}^{-1}\Big) + (\text{if } n=3)\ O\Big(\sum_{k\neq i}\varepsilon_{ik}^2\ln^{2/3}\varepsilon_{ik}^{-1}\Big).
\end{aligned} \tag{46}$$

Combining (38), (43), (44), (45), and (46), we obtain the desired result. □

Next, we deal with the linear term in Proposition 3 with respect to $\bar{v}_\varepsilon$. Namely, we prove the following Lemma.

**Lemma 1.** *Let $n \ge 3$ and $(\alpha,\lambda,a)$ such that $(\alpha,\lambda,a,0)\in\mathcal{O}(N,\mu_0)$. Then, for $i\in\{1,\dots,N\}$, the following fact holds:*

$$\Big|\int_\Omega K\delta_i^{p-1-\varepsilon}\bar{v}_\varepsilon f_i\Big| \le c\|\bar{v}_\varepsilon\|\left(\varepsilon + \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2}\right) + \begin{cases} c\|\bar{v}_\varepsilon\|/\lambda_i^{\frac{n-2}{2}} & \text{if } f_i\in\{\delta_i,\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}\}, \\ c\|\bar{v}_\varepsilon\|/\lambda_i^{\frac{n}{2}} & \text{if } f_i = \frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial a_i}, \end{cases}$$

*where $\bar{v}_\varepsilon$ is defined in Proposition 2.*

**Proof.** Using (18), the fact that $|f_i| \le c\delta_i$, (31), and (32), we obtain

$$
\begin{aligned}
&\int_\Omega K\delta_i^{p-1-\varepsilon}\bar{v}_\varepsilon f_i \\
&\quad = K(a_i)\int_\Omega \delta_i^{p-1-\varepsilon}\bar{v}_\varepsilon f_i + O\Big(|\nabla K(a_i)|\int_\Omega |x-a_i|\delta_i^p|\bar{v}_\varepsilon| + \int_\Omega |x-a_i|^2\delta_i^p|\bar{v}_\varepsilon|\Big) \\
&\quad = c_0^{-\varepsilon}\lambda_i^{\frac{-\varepsilon(n-2)}{2}}K(a_i)\int_\Omega \delta_i^{p-1}\bar{v}_\varepsilon f_i + O\Big(\frac{|\nabla K(a_i)|}{\lambda_i}\|\bar{v}_\varepsilon\| + \frac{\|\bar{v}_\varepsilon\|}{\lambda_i^2}\Big) \\
&\quad + O\Big(\varepsilon\int_\Omega \delta_i^p|\bar{v}_\varepsilon|\ln(1+\lambda_i^2|x-a_i|^2)\Big) \\
&\quad = c_0^{-\varepsilon}\lambda_i^{\frac{-\varepsilon(n-2)}{2}}K(a_i)\int_\Omega \delta_i^{p-1}\bar{v}_\varepsilon f_i + O\Big(\|\bar{v}_\varepsilon\|\Big(\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} + \varepsilon\Big)\Big). \qquad (47)
\end{aligned}
$$

If $f_i = \delta_i$, since $\bar{v}_\varepsilon \in E_{a,\lambda}$ we have

$$\int_\Omega \delta_i^{p-1}\bar{v}_\varepsilon f_i = \int_\Omega \delta_i^p\bar{v}_\varepsilon = -\int_\Omega \Delta\delta_i\bar{v}_\varepsilon = -\int_{\partial\Omega}\frac{\partial\delta_i}{\partial\nu}\bar{v}_\varepsilon.$$

But, since $a_i \to \bar{a}_i \in \Omega$, we have $|\partial\delta_i/\partial\nu| \le c\lambda_i^{(2-n)/2}$ uniformly on $\partial\Omega$. Therefore we obtain

$$\int_\Omega \delta_i^{p-1}\bar{v}_\varepsilon f_i = O\Big(\frac{\|\bar{v}_\varepsilon\|}{\lambda_i^{(n-2)/2}}\Big). \qquad (48)$$

If $f_i = \lambda_i\frac{\partial\delta_i}{\partial\lambda_i}$, using again the fact that $\bar{v}_\varepsilon \in E_{a,\lambda}$ and $a_i \to \bar{a}_i \in \Omega$, we obtain

$$\int_\Omega \delta_i^{p-1}\bar{v}_\varepsilon f_i = \lambda_i\frac{1}{p}\int_\Omega -\Delta\Big(\frac{\partial\delta_i}{\partial\lambda_i}\Big)\bar{v}_\varepsilon = -\frac{\lambda_i}{p}\int_{\partial\Omega}\frac{\partial}{\partial\nu}\Big(\frac{\partial\delta_i}{\partial\lambda_i}\Big)\bar{v}_\varepsilon = O\Big(\frac{\|\bar{v}_\varepsilon\|}{\lambda_i^{(n-2)/2}}\Big). \qquad (49)$$

Lastly, if $f_i = \frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial a_i}$, we obtain, in the same way,

$$\int_\Omega \delta_i^{p-1}\bar{v}_\varepsilon f_i = \frac{1}{\lambda_i}\frac{1}{p}\int_\Omega -\Delta\Big(\frac{\partial\delta_i}{\partial a_i}\Big)\bar{v}_\varepsilon = -\frac{1}{p\lambda_i}\int_{\partial\Omega}\frac{\partial}{\partial\nu}\Big(\frac{\partial\delta_i}{\partial a_i}\Big)\bar{v}_\varepsilon = O\Big(\frac{\|\bar{v}_\varepsilon\|}{\lambda_i^{n/2}}\Big). \qquad (50)$$

Clearly, our lemma follows from (47)–(50). □

Next, we are going to make the statement in Proposition 3 more precise.
We start with the case where $f_i = \delta_i$.

**Proposition 4.** *Let $n \ge 3$ and $(\alpha,\lambda,a)$ such that $(\alpha,\lambda,a,0) \in \mathcal{O}(N,\mu_0)$. Let us denote that $u_\varepsilon = \sum_{i=1}^N \alpha_i\delta_i + \bar{v}_\varepsilon := \bar{u} + \bar{v}_\varepsilon$, where $\bar{v}_\varepsilon$ is defined in Proposition 2. Then, for $1 \le i \le N$, we have*

$$
\begin{aligned}
\int_\Omega K|u_\varepsilon|^{p-1-\varepsilon}u_\varepsilon\delta_i &= \alpha_i^{p-\varepsilon}\lambda_i^{\frac{-\varepsilon(n-2)}{2}}K(a_i)S_n \\
&\quad + O\Big(\varepsilon + \frac{1}{\lambda_i^2} + \|\bar{v}_\varepsilon\|^2 + \sum_{k\ne i}\varepsilon_{ik}\Big) + (\text{if } n=3)\ O\Big(\frac{1}{\lambda_i}\Big).
\end{aligned}
$$

**Proof.** Using (38) and (43) with $f_i = \delta_i$, we obtain

$$\int_\Omega K|u_\varepsilon|^{p-1-\varepsilon}u_\varepsilon\delta_i = \int_\Omega K\bar{u}^{p-\varepsilon}\delta_i + (p-\varepsilon)\alpha_i^{p-1-\varepsilon}\int_\Omega K\delta_i^{p-\varepsilon}\bar{v}_\varepsilon + O\Big(\|\bar{v}_\varepsilon\|^2 + \sum_{k\ne i}\varepsilon_{ik}\Big). \qquad (51)$$

The second integral on the right hand side of (51) is estimated in Lemma 1. For the first one, we write

$$\int_\Omega K\bar{u}^{p-\varepsilon}\delta_i = \alpha_i^{p-\varepsilon}\int_\Omega K\delta_i^{p+1-\varepsilon} + \sum_{k\neq i} O\Big(\int_\Omega \delta_i^p\delta_k + \int_\Omega \delta_i\delta_k^p\Big)$$
$$= \alpha_i^{p-\varepsilon}\int_\Omega K\delta_i^{p+1-\varepsilon} + O\Big(\sum_{k\neq i}\varepsilon_{ik}\Big).$$

Now, observe that, for $r$ positive small, using (18) we obtain

$$\int_\Omega K\delta_i^{p+1-\varepsilon} = K(a_i)\int_{B(a_i,r)}\delta_i^{p+1-\varepsilon} + O\bigg(\int_{B(a_i,r)}|x-a_i|^2\delta_i^{p+1}\bigg) + \int_{\Omega\setminus B(a_i,r)} K\delta_i^{p+1-\varepsilon}$$
$$= K(a_i)\int_{\mathbb{R}^n}\frac{c_0^{p+1-\varepsilon}\lambda_i^{\frac{-\varepsilon(n-2)}{2}}dx}{(1+|x|^2)^{n-\frac{\varepsilon(n-2)}{2}}} + O\bigg(\frac{1}{\lambda_i^2}+\frac{1}{\lambda_i^n}\bigg)$$
$$= K(a_i)\lambda_i^{\frac{-\varepsilon(n-2)}{2}}S_n + O\bigg(\varepsilon+\frac{1}{\lambda_i^2}\bigg).$$

Combining the previous estimates with Lemma 1, we easily derive our proposition. □

Next, we take $f_i = \lambda_i\frac{\partial\delta_i}{\partial\lambda_i}$ in Proposition 3 and our aim is to prove the following result.

**Proposition 5.** *Let $n \geq 4$ and $(\alpha,\lambda,a)$ such that $(\alpha,\lambda,a,0)\in\mathcal{O}(N,\mu_0)$. Let us denote that $u_\varepsilon = \sum_{i=1}^N \alpha_i\delta_i + \bar{v}_\varepsilon := \bar{u} + \bar{v}_\varepsilon$, where $\bar{v}_\varepsilon$ is defined in Proposition 2. Then, for $i \leq N$, we have*

$$\int_\Omega K|u_\varepsilon|^{p-1-\varepsilon}u_\varepsilon\lambda_i\frac{\partial\delta_i}{\partial\lambda_i} = \alpha_i^{p-\varepsilon}\lambda_i^{\frac{-\varepsilon(n-2)}{2}}\bigg(-\bar{c}_1\varepsilon K(a_i) - \bar{c}_2\frac{\Delta K(a_i)}{\lambda_i^2}\bigg) + O\Big(\|\bar{v}_\varepsilon\|^2\Big) + O(R)$$
$$+ c_1\sum_{k\neq i}\lambda i\frac{\partial\varepsilon_{ik}}{\partial\lambda_i}\alpha_k\bigg(\alpha_i^{p-1-\varepsilon}K(a_i)\lambda_i^{\frac{-\varepsilon(n-2)}{2}} + \alpha_k^{p-1-\varepsilon}K(a_k)\lambda_k^{\frac{-\varepsilon(n-2)}{2}}\bigg),$$

*where $\bar{c}_1$, $\bar{c}_2$ are defined in Theorem 1 and*

$$R = \varepsilon^2 + \frac{1}{\lambda_i^{n-2}} + \sum_{k\neq r}\varepsilon_{kr}^{\frac{n}{n-2}}\ln\varepsilon_{kr}^{-1} + \sum_{k=1}^N\Big(\frac{|\nabla K(a_k)|^2}{\lambda_k^2} + \frac{(\ln\lambda_k)^2}{\lambda_k^4}\Big),$$
$$c_1 = \int_{\mathbb{R}^n}\frac{c_0^{\frac{2n}{n-2}}}{(1+|x|^2)^{\frac{n+2}{2}}}.$$

**Proof.** Applying Proposition 3 with $f_i = \lambda_i(\partial\delta_i)/\partial\lambda_i$, we need to estimate the integrals involved in (36). For a small positive $r$, since $K$ is a $C^3$-function on $\overline{\Omega}$, we have

$$\int_\Omega K\delta_i^{p-\varepsilon}\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}$$
$$= K(a_i)\int_{B(a_i,r)}\delta_i^{p-\varepsilon}\lambda_i\frac{\partial\delta_i}{\partial\lambda_i} + \frac{1}{2}\int_{B(a_i,r)}D^2K(a_i)(x-a_i,x-a_i)\delta_i^{p-\varepsilon}\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}$$
$$+ O\bigg(\int_{B(a_i,r)}|x-a_i|^4\delta_i^{p+1}\bigg) + O\bigg(\frac{1}{\lambda_i^n}\bigg). \tag{52}$$

Next, we recall the following estimate which is extracted from [20] (see estimate (91) of [20])

$$\int_{B(a_i,r)}\delta_i^{p-\varepsilon}\lambda_i\frac{\partial\delta_i}{\partial\lambda_i} = -\varepsilon\bar{c}_1\lambda_i^{\frac{-\varepsilon(n-2)}{2}} + O\Big(\frac{\ln\lambda_i}{\lambda_i^n}+\varepsilon^2\Big). \tag{53}$$

Combining estimates (52) and (53), we obtain

$$\int_\Omega K\delta_i^{p-\varepsilon}\lambda_i\frac{\partial\delta_i}{\partial\lambda_i} = -\frac{\bar{c}_1\varepsilon K(a_i)}{\lambda_i^{\frac{\varepsilon(n-2)}{2}}} + \frac{\Delta K(a_i)}{2n}\int_{B(a_i,r)}|x-a_i|^2\delta_i^{p-\varepsilon}\lambda_i\frac{\partial\delta_i}{\partial\lambda_i} + O\Big(\varepsilon^2 + \frac{(\ln\lambda_i)^{\sigma_n}}{\lambda_i^4}\Big). \tag{54}$$

But, using (18), we have

$$\begin{aligned}\frac{1}{2n}\int_{B(a_i,r)}|x-a_i|^2\delta_i^{p-\varepsilon}\lambda_i\frac{\partial\delta_i}{\partial\lambda_i} &= c_0^{-\varepsilon}\lambda_i^{\frac{-\varepsilon(n-2)}{2}}\frac{1}{\lambda_i^2}\frac{(n-2)}{4n}\int_{B(0,\lambda_i r)}\frac{|x|^2(1-|x|^2)dx}{(1+|x|^2)^{n+1}} + O\Big(\frac{\varepsilon}{\lambda_i^2}\Big)\\ &= -\bar{c}_2\lambda_i^{\frac{-\varepsilon(n-2)}{2}}\frac{1}{\lambda_i^2} + O\Big(\frac{1}{\lambda_i^n} + \frac{\varepsilon}{\lambda_i^2}\Big).\end{aligned} \tag{55}$$

Thus, combining (54) and (55), we obtain

$$\int_\Omega K\delta_i^{p-\varepsilon}\lambda_i\frac{\partial\delta_i}{\partial\lambda_i} = -\bar{c}_1\,\varepsilon\lambda_i^{\frac{-\varepsilon(n-2)}{2}}K(a_i) - \bar{c}_2\lambda_i^{\frac{-\varepsilon(n-2)}{2}}\frac{\Delta K(a_i)}{\lambda_i^2} + O\Big(\varepsilon^2 + \frac{(\ln\lambda_i)^{\sigma_n}}{\lambda_i^4}\Big). \tag{56}$$

Next, we are going to estimate the second integral on the right hand side of (36). To this end, using (18), we obtain

$$\begin{aligned}p\int_\Omega K\delta_i^{p-1-\varepsilon}\delta_k\lambda_i\frac{\partial\delta_i}{\partial\lambda_i} &= pK(a_i)\int_\Omega\delta_i^{p-1-\varepsilon}\delta_k\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}\\ &+ O\Big(|\nabla K(a_i)|\int_\Omega|x-a_i|\delta_i^p\delta_k\Big) + O\Big(\int_\Omega|x-a_i|^2\delta_i^p\delta_k\Big)\\ &= pK(a_i)c_0^{-\varepsilon}\lambda_i^{\frac{-\varepsilon(n-2)}{2}}\int_{\mathbb{R}^n}\delta_i^{p-1}\delta_k\lambda_i\frac{\partial\delta_i}{\partial\lambda_i} + O\Big(\int_{\mathbb{R}^n\setminus\Omega}\delta_i^p\delta_k\Big)\\ &+ O\Big(\varepsilon\int_\Omega\delta_i^p\delta_k\ln\big(1+\lambda_i^2|x-a_i|^2\big)\Big)\\ &+ O\Big(|\nabla K(a_i)|\Big(\int_\Omega|x-a_i|^{\frac{n}{2}}\delta_i^{\frac{2n}{n-2}}\Big)^{\frac{2}{n}}\Big(\int_\Omega(\delta_i\delta_k)^{\frac{n}{n-2}}\Big)^{\frac{n-2}{n}}\Big)\\ &+ O\Big(\Big(\int_\Omega|x-a_i|^n\delta_i^{p+1}\Big)^{2/n}\Big\}\Big(\int_\Omega(\delta_i\delta_k)^{\frac{n}{n-2}}\Big)^{\frac{n-2}{n}}\Big).\end{aligned} \tag{57}$$

But, using estimate F16 of [26], we have

$$p\int_{\mathbb{R}^n}\delta_i^{p-1}\delta_k\lambda_i\frac{\partial\delta_i}{\partial\lambda_i} = \lambda_i\int_{\mathbb{R}^n}\delta_k^p\frac{\partial\delta_i}{\partial\lambda_i} = c_1\lambda_i\frac{\partial\varepsilon_{ik}}{\partial\lambda_i} + O\Big(\varepsilon_{ik}^{\frac{n}{n-2}}\ln\varepsilon_{ik}^{-1}\Big). \tag{58}$$

We also have

$$\begin{aligned}\int_{\mathbb{R}^n\setminus\Omega}\delta_i^p\delta_k = \int_{\mathbb{R}^n\setminus\Omega}\delta_i^{p-1}(\delta_i\delta_k) &\le \Big(\int_{\mathbb{R}^n\setminus\Omega}\delta_i^{p+1}\Big)^{\frac{2}{n}}\Big(\int_\Omega(\delta_i\delta_k)^{\frac{n}{n-2}}\Big)^{\frac{n-2}{n}}\\ &\le \frac{c}{\lambda_i^2}\varepsilon_{ik}\ln^{\frac{n-2}{n}}\varepsilon_{ik}^{-1},\end{aligned} \tag{59}$$

$$\begin{aligned}\varepsilon\int_{\mathbb{R}^n}\delta_i^p\delta_k\ln\big(1+\lambda_i^2|x-a_i|^2\big) &\le \varepsilon\int_{\mathbb{R}^n}(\delta_i\delta_k)\delta_i^{p-1}\ln\big(1+\lambda_i^2|x-a_i|^2\big)\\ &\le \varepsilon\varepsilon_{ik}\ln^{\frac{n-2}{n}}\varepsilon_{ik}^{-1}.\end{aligned} \tag{60}$$

And (57)–(60) imply that

$$p\int_\Omega K\delta_i^{p-1-\varepsilon}\delta_k\lambda_i\frac{\partial\delta_i}{\partial\lambda_i} = c_0^{-\varepsilon}\lambda_i^{\frac{-\varepsilon(n-2)}{2}}K(a_i)c_1\lambda_i\frac{\partial\varepsilon_{ik}}{\partial\lambda_i} + O\Big(\varepsilon_{ik}^{\frac{n}{n-2}}\ln\varepsilon_{ik}^{-1}\Big)$$
$$+ O\Big(\Big(\varepsilon + \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{(\ln\lambda_i)^{2/n}}{\lambda_i^2}\Big)\varepsilon_{ik}(\ln\varepsilon_{ik}^{-1})^{(n-2)/n}\Big). \qquad (61)$$

Now, in the same way, we consider the third integral in the right hand side of (36) and we write

$$\int_\Omega K\delta_k^{p-\varepsilon}\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}$$
$$= K(a_k)\int_\Omega \delta_k^{p-\varepsilon}\lambda_i\frac{\partial\delta_i}{\partial\lambda_i} + O\Big(|\nabla K(a_k)|\int_\Omega |x-a_k|\delta_k^p\delta_i\Big) + O\Big(\int_\Omega |x-a_k|^2\delta_k^p\delta_i\Big)$$
$$= K(a_k)c_0^{-\varepsilon}\lambda_k^{\frac{-\varepsilon(n-2)}{2}}c_1\lambda_i\frac{\partial\varepsilon_{ik}}{\partial\lambda_i} + O\Big(\varepsilon_{ik}^{n/(n-2)}\ln\varepsilon_{ik}^{-1}\Big)$$
$$+ O\Big(\Big(\varepsilon + \frac{|\nabla K(a_k)|}{\lambda_k} + \frac{(\ln\lambda_k)^{2/n}}{\lambda_k^2}\Big)\varepsilon_{ik}^{n/(n-2)}(\ln\varepsilon_{ik}^{-1})^{(n-2)/n}\Big). \qquad (62)$$

Combining (56), (61), (62), Lemma 1, and Proposition 3, we obtain the desired result. □

Now, taking $f_i = \frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial a_i}$ in Proposition 3, we are going to prove the following crucial result.

**Proposition 6.** *Let $n \geq 3$ and $(\alpha, \lambda, a)$ be such that $(\alpha, \lambda, a, 0) \in \mathcal{O}(N, \mu_0)$. Let us denote that $u_\varepsilon = \sum_{i=1}^N \alpha_i\delta_i + \bar{v}_\varepsilon$, where $\bar{v}_\varepsilon$ is defined in Proposition 2. Then, for $1 \leq i \leq N$, the following fact holds*

$$\int_\Omega K|u_\varepsilon|^{p-1-\varepsilon}u_\varepsilon\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial a_i} = \alpha_i^{p-\varepsilon}\lambda_i^{\frac{-\varepsilon(n-2)}{2}}\bar{c}_4\frac{\nabla K(a_i)}{\lambda_i} + c_1\sum_{k\neq i}\alpha_k\frac{1}{\lambda_i}\frac{\partial\varepsilon_{ik}}{\partial a_i}\Big(\alpha_i^{p-1-\varepsilon}K(a_i)\lambda_i^{\frac{-\varepsilon(n-2)}{2}}$$
$$+ \alpha_k^{p-1-\varepsilon}K(a_k)\lambda_k^{\frac{-\varepsilon(n-2)}{2}}\Big) + O\Big(\|\bar{v}_\varepsilon\|^2\Big) + O(R_i),$$

*where*

$$R_i = \varepsilon^2 + \frac{1}{\lambda_i^3} + \sum_k\frac{|\nabla K(a_k)|^2}{\lambda_k^2} + \sum_{k\neq i}\lambda_k|a_i - a_k|\varepsilon_{ik}^{\frac{n+1}{n-2}} + \sum_{k\neq r}\varepsilon_{kr}^{\frac{n}{n-2}}\ln\varepsilon_{kr}^{-1} + \sum_k\frac{(\ln\lambda_k)^{4/n}}{\lambda_k^4}$$
$$+ (\textit{if } n=3)\, O\Big(\sum_{k\neq r}\varepsilon_{kr}^2\ln^{2/3}\varepsilon_{kr}^{-1}\Big) \qquad \textit{and} \qquad \bar{c}_4 = \frac{n-2}{n}c_0^{p+1}\int_{\mathbb{R}^n}\frac{|x|^2dx}{(1+|x|^2)^{n+1}}.$$

**Proof.** Taking $r$ positive small and denoting that $B_i := B(a_i, r)$, we write

$$\int_\Omega K\delta_i^{p-\varepsilon}\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial a_i} = \int_{B_i} K\delta_i^{p-\varepsilon}\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial a_i} + O\Big(\int_{\mathbb{R}^n\setminus B_i}\delta_i^p\frac{1}{\lambda_i}\Big|\frac{\partial\delta_i}{\partial a_i}\Big|\Big)$$
$$= \int_{B_i}\nabla K(a_i)(x-a_i)\delta_i^{p-\varepsilon}\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial a_i} + O\Big(\int_{B_i}|x-a_i|^3\delta_i^{p+1} + \frac{1}{\lambda_i^{n+1}}\Big). \qquad (63)$$

But using (18), we have

$$\int_{B_i}\nabla K(a_i)(x-a_i)\delta_i^{p-\varepsilon}\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial a_i}$$
$$= \frac{c_0^{-\varepsilon}}{\lambda_i^{\frac{\varepsilon(n-2)}{2}}}\int_{B_i}\nabla K(a_i)(x-a_i)\delta_i^p\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial a_i} + O\Big(\varepsilon\frac{|\nabla K(a_i)|}{\lambda_i}\Big). \qquad (64)$$

Notice that, for $1 \le j \le n$, we have

$$\int_{B_i} \nabla K(a_i)(x-a_i)\delta_i^p \frac{1}{\lambda_i}\frac{\partial \delta_i}{\partial (a_i)_j} = \sum_{k=1}^{n} \frac{\partial K}{\partial x_k}(a_i)\int_{B_i}(x-a_i)_k \delta_i^p \frac{1}{\lambda_i}\frac{\partial \delta_i}{\partial (a_i)_j}$$
$$= \frac{\partial K}{\partial x_j}(a_i)\int_{B_i}(x-a_i)_j \delta_i^p \frac{1}{\lambda_i}\frac{\partial \delta_i}{\partial (a_i)_j}$$
$$= (n-2)c_0^{p+1}\frac{\partial K}{\partial x_j}(a_i)\int_{B_i}\frac{\lambda_i^{n+1}(x-a_i)_j^2}{(1+\lambda_i^2|x-a_i|^2)^{n+1}}dx.$$

We notice that the last integral is independent of the index $j$. Then we obtain

$$\int_{B_i} \nabla K(a_i)(x-a_i)\delta_i^p \frac{1}{\lambda_i}\frac{\partial \delta_i}{\partial (a_i)_j} = (n-2)c_0^{p+1}\frac{\partial K}{\partial x_j}(a_i)\frac{1}{n}\int_{B_i}\frac{\lambda_i^{n+1}|x-a_i|^2}{(1+\lambda_i^2|x-a_i|^2)^{n+1}}dx$$
$$= \frac{(n-2)}{n}c_0^{p+1}\frac{\partial K}{\partial x_j}(a_i).\frac{1}{\lambda_i}\int_{B(0,\lambda_i r)}\frac{|x|^2}{(1+|x|^2)^{n+1}}dx$$
$$= \frac{\partial K}{\partial x_j}(a_i)\frac{\bar{c}_4}{\lambda_i} + O\left(\frac{1}{\lambda_i^{n+1}}\right). \tag{65}$$

Clearly, (63), (64), and (65) imply that

$$\int_{\Omega} K\delta_i^{p-\varepsilon}\frac{1}{\lambda_i}\frac{\partial \delta_i}{\partial a_i} = c_0^{-\varepsilon}\lambda_i^{\frac{-\varepsilon(n-2)}{2}}\bar{c}_4\frac{\nabla K(a_i)}{\lambda_i} + O\left(\frac{\varepsilon|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^3}\right). \tag{66}$$

For the second integral on the right hand side of (36), following the proof of (57), we write

$$p\int_{\Omega} K\delta_i^{p-1-\varepsilon}\delta_k \frac{1}{\lambda_i}\frac{\partial \delta_i}{\partial a_i}$$
$$= pK(a_i)\int_{\Omega}\delta_i^{p-1-\varepsilon}\delta_k\frac{1}{\lambda_i}\frac{\partial \delta_i}{\partial a_i} + O\left(|\nabla K(a_i)|\int_{\Omega}|x-a_i|\delta_i^p\delta_k\right) + O\left(\int_{\Omega}|x-a_i|^2\delta_i^p\delta_k\right)$$
$$= pK(a_i)c_0^{-\varepsilon}\lambda_i^{\frac{-\varepsilon(n-2)}{2}}\int_{\mathbb{R}^n}\delta_i^{p-1}\delta_k\frac{1}{\lambda_i}\frac{\partial \delta_i}{\partial a_i}$$
$$+ O\Big(\Big(\frac{(\ln\lambda_i)^{\frac{2}{n}}}{\lambda_i^2} + \varepsilon + \frac{|\nabla K(a_i)|}{\lambda_i}\Big)\varepsilon_{ik}(\ln\varepsilon_{ik}^{-1})^{\frac{(n-2)}{n}}\Big). \tag{67}$$

But, using estimate F11 of [26], we have

$$p\int_{\mathbb{R}^n}\delta_i^{p-1}\delta_k\frac{1}{\lambda_i}\frac{\partial \delta_i}{\partial a_i} = \frac{1}{\lambda_i}\int_{\mathbb{R}^n}\delta_k^p\frac{\partial \delta_i}{\partial a_i} = c_1\frac{1}{\lambda_i}\frac{\partial \varepsilon_{ik}}{\partial a_i} + O\left(\lambda_k|a_i-a_k|\varepsilon_{ik}^{\frac{n+1}{n-2}}\right). \tag{68}$$

Combining (67) and (68), we obtain

$$p\int_{\Omega} K\delta_i^{p-1-\varepsilon}\delta_k\frac{1}{\lambda_i}\frac{\partial \delta_i}{\partial a_i} = c_0^{-\varepsilon}\lambda_i^{\frac{-\varepsilon(n-2)}{2}}K(a_i)c_1\frac{1}{\lambda_i}\frac{\partial \varepsilon_{ik}}{\partial a_i} + O\left(\lambda_k|a_i-a_k|\varepsilon_{ik}^{\frac{n+1}{n-2}}\right)$$
$$+ O\left(\left(\varepsilon + \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{(\ln\lambda_i)^{2/n}}{\lambda_i^2}\right)\varepsilon_{ik}(\ln\varepsilon_{ik}^{-1})^{(n-2)/n}\right). \tag{69}$$

Lastly, in the same way, we deal with the third integral on the right hand side of (36), and we write

$$
\begin{aligned}
&\int_\Omega K\delta_k^{p-\varepsilon}\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial a_i}\\
&= K(a_k)\int_\Omega \delta_k^{p-\varepsilon}\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial a_i} + O\Big(|\nabla K(a_k)|\int_\Omega \delta_k^{p-1}(\delta_k\delta_i)|x-a_k|\Big) + O\Big(\int_{B(a_k,r)}|x-a_k|^2\delta_k^p\delta_i\Big)\\
&= K(a_k)c_0^{-\varepsilon}\lambda_k^{\frac{-\varepsilon(n-2)}{2}}\Big(c_1\frac{1}{\lambda_i}\frac{\partial\varepsilon_{ik}}{\partial a_i} + O\Big(\lambda_k|a_i-a_k|\varepsilon_{ik}^{\frac{n+1}{n-2}}\Big)\Big)\\
&\quad + O\Big(\Big(\varepsilon + \frac{|\nabla K(a_k)|}{\lambda_k} + \frac{(\ln\lambda_k)^{2/n}}{\lambda_k^2}\Big)\varepsilon_{ik}(\ln\varepsilon_{ik}^{-1})^{(n-2)/n}\Big). \qquad (70)
\end{aligned}
$$

Combining (66), (69), (70), Lemma 1, and Proposition 3, we obtain the desired result. □

Now, we are ready to give the expansions of the gradient of the associated Euler–Lagrange functional $I_\varepsilon$ in the set $\mathcal{O}(N,\mu_0)$. Namely we prove the following crucial result

**Proposition 7.** *Let $(\alpha,\lambda,a)$ be such that $(\alpha,\lambda,a,0)\in\mathcal{O}(N,\mu_0)$ and Let $u_\varepsilon = \sum_{i=1}^N \alpha_i\delta_{a_i,\lambda_i} + \bar{v}_\varepsilon$, where $\bar{v}_\varepsilon$ is defined in Proposition 2. Then, for $1\le i\le N$, the following facts hold:*

*(i)* *For $n\ge 3$, we have*

$$\langle I'_\varepsilon(u_\varepsilon),\delta_i\rangle = \alpha_i S_n\Big(1-\alpha_i^{p-1-\varepsilon}\lambda_i^{-\frac{\varepsilon(n-2)}{2}}K(a_i)\Big) + O(R_{1i}(\varepsilon,a,\lambda)),$$

*where $S_n$ is defined in (83).*

*(ii)* *For $n\ge 4$, we have*

$$
\begin{aligned}
\Big\langle I'_\varepsilon(u_\varepsilon),\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}\Big\rangle &= -d_n\alpha_i\, V(a_i)\frac{\ln^{\sigma_n}\lambda_i}{\lambda_i^2} + \alpha_i^{p-\varepsilon}\lambda_i^{-\frac{\varepsilon(n-2)}{2}}\Big(\bar{c}_2\frac{\Delta K(a_i)}{\lambda_i^2} + \bar{c}_1\varepsilon K(a_i)\Big),\\
&\quad + c_1\sum_{k\ne i}\alpha_k\lambda_i\frac{\partial\varepsilon_{ik}}{\partial\lambda_i}\Big[1-\sum_{j=i,k}\alpha_j^{p-1-\varepsilon}K(a_j)\lambda_j^{-\frac{\varepsilon(n-2)}{2}}\Big] + O\Big(R_{2i}(\varepsilon,a,\lambda)\Big),
\end{aligned}
$$

*where $d_4 = 2\sqrt{2}\, meas(\mathbb{S}^3)$, $c_1$ is defined in Proposition 5, and where $\bar{c}_1$, $\bar{c}_2$, $d_n$ for $n\ge 5$ are defined in Theorem 1.*

*(iii)* *For $n\ge 3$, we have*

$$
\begin{aligned}
\Big\langle I'_\varepsilon(u_\varepsilon),\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial a_i}\Big\rangle &= -\alpha_i^{p-\varepsilon}\lambda_i^{-\frac{\varepsilon(n-2)}{2}}\,\bar{c}_4\frac{\nabla K(a_i)}{\lambda_i}\\
&\quad + c_1\sum_{k\ne i}\alpha_k\frac{1}{\lambda_i}\frac{\partial\varepsilon_{ik}}{\partial a_i}\Big[1-\sum_{j=i,k}\alpha_i^{p-1-\varepsilon}K(a_i)\lambda_i^{-\frac{\varepsilon(n-2)}{2}}\Big] + O\Big(R_{3i}(\varepsilon,a,\lambda)\Big),
\end{aligned}
$$

*where $\bar{c}_4$ is defined in Proposition 6 and where*

$$
\begin{aligned}
R_{1i}(\varepsilon,a,\lambda) &= \varepsilon + \|\bar{v}_\varepsilon\|^2 + \sum_{j\ne i}\varepsilon_{ij} + \sum_j\frac{1}{\lambda_j^{n-2}} + \frac{\ln^{\sigma_n}\lambda_i}{\lambda_i^2} \quad \text{with} \quad \sigma_n := \begin{cases} 0 & \text{if } n\ne 4,\\ 1 & \text{if } n=4,\end{cases}\\
R_{2i}(\varepsilon,a,\lambda) &= \varepsilon^2 + \sum_{k\ne r}\varepsilon_{kr}^{\frac{n}{n-2}}\ln\varepsilon_{kr}^{-1} + \sum_{k=1}^N\frac{|\nabla K(a_k)|^2}{\lambda_k^2} + \sum_{k=1}^N\frac{(\ln\lambda_k)^{4/n}}{\lambda_k^4} + \sum_{k=1}^N\frac{1}{\lambda_k^{n-2}}\\
&\quad + \sum_{k\ne i}\varepsilon_{ik}|a_i-a_k|^2|\ln^{\sigma_n}|a_i-a_k|| + \sum_{k\ne i}\varepsilon_{ik}\frac{\ln^{\sigma_n}(\min(\lambda_i,\lambda_k))}{\min(\lambda_i,\lambda_k)^2} + \|\bar{v}_\varepsilon\|^2,\\
R_{3i}(\varepsilon,a,\lambda) &= \varepsilon^2 + \frac{\ln^{\sigma_n}\lambda_i}{\lambda_i^3} + \sum_k\frac{|\nabla K(a_k)|^2}{\lambda_k^2} + \sum_{k\ne i}\lambda_k|a_i-a_k|\varepsilon_{ik}^{\frac{n+1}{n-2}} + \sum_{k\ne r}\varepsilon_{kr}^{\frac{n}{n-2}}\ln\varepsilon_{kr}^{-1}\\
&\quad + \sum_k\frac{(\ln\lambda_k)^{4/n}}{\lambda_k^4} + \|\bar{v}_\varepsilon\|^2 + \frac{1}{\lambda_i}\sum_{k\ne i}\varepsilon_{ik} + \sum_k\frac{1}{\lambda_k^{n-1}}.
\end{aligned}
$$

**Proof.** Claim $(i)$ follows from estimates (50)–(54) and (56) from [20] and Proposition 4.

Next, we are going to prove estimate (ii). First , we know that (see Estimate (51) of [19])

$$\int_{\Omega} \nabla u_{\varepsilon} \nabla \left( \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \right) = \sum_{k \neq i} c_1 \alpha_k \lambda_i \frac{\partial \varepsilon_{ik}}{\partial \lambda_i} + O\left( \sum_{k \neq i} \varepsilon_{ik}^{\frac{n}{n-2}} \ln \varepsilon_{ik}^{-1} + \sum_k \frac{1}{\lambda_k^{n-2}} \right). \tag{71}$$

Second, taking $r$ positive small and using estimates (52),(53) and Lemma 6.6 of [19], we obtain

$$\begin{aligned}
\int_{\Omega} V u_{\varepsilon} \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} &= \alpha_i V(a_i) \int_{B(a_i,r)} \delta_i \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} + O\Big( \int_{B(a_i,r)} |x - a_i|^2 \delta_i^2 \Big) + O\Big( \int_{\Omega \setminus B(a_i,r)} \delta_i^2 \Big) \\
&\quad + \sum_{j \neq i} \int_{\Omega} \alpha_j V \delta_j \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} + O\Big( \int_{\Omega} |\overline{v}_{\varepsilon}| \delta_i \Big) \\
&= -d_n V(a_i) \alpha_i \frac{\ln^{\sigma_n} \lambda_i}{\lambda_i^2} + O\Big( \sum_{k \neq i} \varepsilon_{ik} |a_i - a_k|^2 | \ln^{\sigma_n} |a_i - a_k| \Big) \\
&\quad + O\Big( \frac{1}{\lambda_i^{\min(n-2;4)}} \Big) + O\left( \sum_{k \neq i} \varepsilon_{ik} \ln^{\sigma_n} \frac{(\min(\lambda_i, \lambda_k))}{\min(\lambda_i, \lambda_k)^2} \right) + O\Big( \|\overline{v}_{\varepsilon}\|^2 \Big) \\
&\quad + (\text{if } n = 6) O\left( \frac{\ln^{\frac{4}{3}} \lambda_i}{\lambda_i^4} \right)
\end{aligned} \tag{72}$$

where we have used (26).

Combining (71), (72) and Proposition 5, we easily obtain Claim $(ii)$.

To prove Claim $(iii)$, we first use Proposition 3.4 of [19] to derive

$$\int_{\Omega} \nabla u_{\varepsilon} \cdot \nabla \left( \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial a_i} \right) = c_1 \frac{1}{\lambda_i} \sum_{k \neq i} \alpha_k \frac{\partial \varepsilon_{ik}}{\partial a_i} + O\Big( \sum_k \frac{1}{\lambda_k^{n-1}} + \sum_{k \neq i} \lambda_k |a_i - a_k| \varepsilon_{ik}^{\frac{n+1}{n-2}} \Big). \tag{73}$$

Second, taking $r$ positive small, we write

$$\begin{aligned}
\int_{\Omega} V u_{\varepsilon} \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial a_i} &= O\Big( \int_{B(a_i,r)} \delta_i \frac{1}{\lambda_i} \Big| \frac{\partial \delta_i}{\partial a_i} \Big| |x - a_i| + \int_{\Omega \setminus B(a_i,r)} \delta_i \frac{1}{\lambda_i} \Big| \frac{\partial \delta_i}{\partial a_i} \Big| \Big) \\
&\quad + O\Big( \sum_{k \neq i} \int_{\Omega} V \delta_k \frac{1}{\lambda_i} \Big| \frac{\partial \delta_i}{\partial a_i} \Big| + \int_{\Omega} |\overline{v}_{\varepsilon}| \frac{1}{\lambda_i} \Big| \frac{\partial \delta_i}{\partial a_i} \Big| \Big).
\end{aligned} \tag{74}$$

But, using Lemma 6.3 of [19], we have

$$\int_{B(a_i,r)} \delta_i \frac{1}{\lambda_i} \Big| \frac{\partial \delta_i}{\partial a_i} \Big| |x - a_i| \leq \frac{c}{\lambda_i} \int_{B(a_i,r)} \delta_i^2 \leq c \begin{cases} \lambda_i^{-3} & \text{if } n \geq 5, \\ \lambda_i^{-3} \ln \lambda_i & \text{if } n = 4, \\ \lambda_i^{-2} & \text{if } n = 3. \end{cases} \tag{75}$$

$$\int_{\Omega \setminus B(a_i,r)} \delta_i \frac{1}{\lambda_i} \Big| \frac{\partial \delta_i}{\partial a_i} \Big| \leq \frac{c}{\lambda_i^{n-1}}, \tag{76}$$

$$\begin{aligned}
\int_{\Omega} |\overline{v}_{\varepsilon}| \frac{1}{\lambda_i} \Big| \frac{\partial \delta_i}{\partial a_i} \Big| &\leq c \|\overline{v}_{\varepsilon}\| \left( \int_{\Omega} \left( \frac{\delta_i}{\lambda_i |x - a_i|} \right)^{\frac{2n}{n+2}} \right)^{\frac{n+2}{2n}} \\
&\leq c \|\overline{v}_{\varepsilon}\| \begin{cases} \lambda_i^{-3/2} & \text{if } n = 3, \\ \lambda_i^{-2} \ln^{3/4} \lambda_i & \text{if } n = 4, \\ \lambda_i^{-2} & \text{if } n \geq 5. \end{cases}
\end{aligned} \tag{77}$$

Furthermore, for $k \neq i$, it holds that

$$\int_\Omega V\delta_k \frac{1}{\lambda_i}\Big|\frac{\partial \delta_i}{\partial a_i}\Big| \leq c\int_\Omega \frac{\delta_i\delta_k}{\lambda_i|x-a_i|}$$
$$\leq \frac{c}{\lambda_i}\Big(\int_\Omega (\delta_i\delta_k)^{\frac{n-1}{n-2}}\Big)^{\frac{n-2}{n-1}}\Big(\int_\Omega \frac{dx}{|x-a_i|^{n-1}}\Big)^{\frac{1}{n-1}} \leq \frac{c}{\lambda_i}\varepsilon_{ik}. \quad (78)$$

Combining estimates (74)–(78), we obtain

$$\int_\Omega V u_\varepsilon \frac{1}{\lambda_i}\frac{\partial \delta_i}{\partial a_i} = O\Big(\|\overline{v}_\varepsilon\|^2 + \frac{1}{\lambda_i}\sum_{k\neq i}\varepsilon_{ik} + \frac{(\ln\lambda_i)^{\sigma_n}}{\lambda_i^{\min(3;n-1)}}\Big). \quad (79)$$

Combining estimates (73), (79) and Proposition 6, we easily derive estimate (iii). This completes the proof of our proposition. □

## 4. Asymptotic Behavior of Interior Bubbling Solutions

Our aim in this section is to study the asymptotic behavior of solutions to $(\mathcal{P}_\varepsilon)$ which blow up at interior points as $\varepsilon$ moves towards zero. We begin by proving the following crucial fact:

**Lemma 2.** *Let $n \geq 3$ and $(u_\varepsilon)$ be a sequence of solutions of $(\mathcal{P}_\varepsilon)$. Then, for all $i \in \{1,\ldots,N\}$, the following fact holds:*

$$\varepsilon \ln\lambda_i \longrightarrow 0 \quad \textit{as } \varepsilon \longrightarrow 0.$$

**Proof.** Multiplying $(\mathcal{P}_\varepsilon)$ by $\delta_i$ and integrating over $\Omega$, we obtain

$$-\sum_{j=1}^N \alpha_j \int_\Omega \Delta\delta_j \cdot \delta_i - \int_\Omega \Delta v_\varepsilon \delta_i + \sum_{j=1}^N \alpha_j \int_\Omega V(x)\delta_j(x)\delta_i + \int_\Omega V(x)v_\varepsilon(x)\delta_i$$
$$= \int_\Omega K(x)\Big(\sum_{j=1}^N \alpha_j\delta_j + v_\varepsilon\Big)^{p-\varepsilon}\delta_i. \quad (80)$$

First, using Lemma 6.6 of [19] and Appendix B of [28], we obtain

$$-\int_\Omega \Delta\delta_j\delta_i = \int_\Omega \delta_j^p\delta_i = O(\varepsilon_{ij}) = o(1) \quad \forall j \neq i, \quad (81)$$
$$-\int_\Omega \Delta\delta_i\cdot\delta_i = \int_\Omega \delta_i^{p+1} = S_n + O\Big(\frac{1}{\lambda_i^n}\Big), \quad (82)$$

where

$$S_n = c_0^{p+1}\int_{\mathbb{R}^n}\frac{dx}{(1+|x|^2)^n}. \quad (83)$$

Second, since $v_\varepsilon \in E_{a,\lambda}$ and $\partial u_\varepsilon/\partial\nu = 0$, we observe that

$$-\int_\Omega \Delta v_\varepsilon\delta_i = \int_\Omega \nabla v_\varepsilon\nabla\delta_i - \int_{\partial\Omega}\frac{\partial v_\varepsilon}{\partial\nu}\delta_i = \sum_{j=1}^N \alpha_j\int_{\partial\Omega}\frac{\partial\delta_j}{\partial\nu}\delta_i = O\Big(\sum_{k=1}^n \frac{1}{\lambda_k^{n-2}}\Big) = o(1), \quad (84)$$
$$\int_\Omega V(x)\delta_i^2 + \int_\Omega V(x)|v_\varepsilon(x)|\delta_i(x) = O\Big(\frac{\ln^{\sigma_n}\lambda_i}{\lambda_i^{\min(2,n-2)}} + \|v_\varepsilon\|\Big(\int \delta_i^{\frac{2n}{n+2}}\Big)^{\frac{n+2}{2n}}\Big) = o(1) \quad (85)$$

where $\sigma_n = 0$ if $n \neq 4$ and $\sigma_n = 1$ if $n = 4$.

Third, using Lemma 6.6 of [19], we get, for $j \neq i$,

$$\int_\Omega V(x)\delta_j\delta_i = O\Big(\int_\Omega \delta_j\delta_i\Big) = O(\varepsilon_{ij}) = o(1). \quad (86)$$

Next, we are going to estimate the right hand side of (80). To this end, we write

$$\int_\Omega K\Big(\sum_{j=1}^N \alpha_j\delta_j + v_\varepsilon\Big)^{p-\varepsilon}\delta_i = \int_\Omega K(\alpha_i\delta_i)^{p-\varepsilon}\delta_i + O\Big(\sum_{j\neq i}\int_\Omega \Big(\delta_j^{p-\varepsilon}\delta_i + \delta_i^{p-\varepsilon}\delta_j\Big)\Big) + O\Big(\int_\Omega \Big(|v_\varepsilon|^{p-\varepsilon}\delta_i + \delta_i^{p-\varepsilon}|v_\varepsilon|\Big)\Big). \tag{87}$$

But, using Estimate $E_2$ of [26], we have

$$\begin{aligned}\int_\Omega \Big(\delta_j^{p-\varepsilon}\delta_i + \delta_j\delta_i^{p-\varepsilon}\Big) &= \int_\Omega \Big((\delta_i\delta_j)\delta_j^{p-1-\varepsilon} + (\delta_i\delta_j)\delta_i^{p-1-\varepsilon}\Big)\\ &\leq \Big(\int_\Omega (\delta_i\delta_j)^{\frac{n}{n-2}}\Big)^{\frac{n-2}{n}}\Big(\int_\Omega \delta_j^{(p-1-\varepsilon)\frac{n}{2}}\Big)^{\frac{2}{n}} + \int_\Omega (\delta_i\delta_j)^{\frac{n}{n-2}}\Big)^{\frac{n-2}{n}}\Big(\int_\Omega \delta_i^{(p-1-\varepsilon)\frac{n}{2}}\Big)^{\frac{2}{n}}\Big)\\ &= O\Big(\varepsilon_{ij}(\ln\varepsilon_{ij}^{-1})^{(n-2)/n}\Big) = o(1).\end{aligned} \tag{88}$$

We also have

$$\int_\Omega \Big(|v_\varepsilon|^{p-\varepsilon}\delta_i + \delta_i^{p-\varepsilon}|v_\varepsilon|\Big) \leq c\|v_\varepsilon\|^{p-\varepsilon}\Big(\int_\Omega \delta_i^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{2n}} + \|v_\varepsilon\|\Big(\int_\Omega \delta_i^{(p-\varepsilon)\frac{2n}{n+2}}\Big)^{\frac{n+2}{2n}} = o(1). \tag{89}$$

Concerning the first integral in the right hand side of (87), we write

$$\begin{aligned}\int_\Omega K\delta_i^{p+1-\varepsilon} &= K(a_i)\int_\Omega \delta_i^{p+1-\varepsilon} + O\Big(\int_\Omega |x-a_i|\delta_i^{p+1-\varepsilon}\Big)\\ &= K(a_i)\int_{\mathbb{R}^n} \frac{c_0^{p+1-\varepsilon}\lambda_i^{n-\varepsilon(n-2)/2}}{(1+\lambda_i^2|x-a_i|^2)^{n-\varepsilon(n-2)/2}} + O\Big(\frac{1}{\lambda_i}\int_{\mathbb{R}^n} \frac{|x|\lambda_i^{-\varepsilon(n-2)/2}}{(1+|x|^2)^{n-\varepsilon(n-2)/2}}\Big)\\ &\quad + O\Big(\int_{\mathbb{R}^n\setminus\Omega} \frac{\lambda_i^{n-\varepsilon(n-2)/2}}{(1+\lambda_i^2|x-a_i|^2)^{n-\varepsilon(n-2)/2}}\Big).\end{aligned}$$

But, we observe that

$$\int_{\mathbb{R}^n} \frac{c_0^{p+1-\varepsilon}\lambda_i^{n-\varepsilon(n-2)/2}}{(1+\lambda_i^2|x-a_i|^2)^{n-\varepsilon(n-2))/2}} = \int_{\mathbb{R}^n} \frac{c_0^{p+1}c_0^{-\varepsilon}\lambda_i^{-\varepsilon(n-2)/2}}{(1+|x|^2)^{n-\varepsilon(n-2)/2}} = \lambda_i^{-\varepsilon(n-2)/2}(S_n + O(\varepsilon)),$$

$$\int_{\mathbb{R}^n} \frac{|x|\lambda_i^{-\varepsilon(n-2)/2}}{(1+|x|^2)^{n-\varepsilon(n-2)/2}} \leq c\int_{\mathbb{R}^n} \frac{|x|\lambda_i^{-\varepsilon(n-2)/2}}{(1+|x|^2)^n} = O\Big(\lambda_i^{-\varepsilon(n-2)/2}\Big),$$

$$\int_{\mathbb{R}^n\setminus\Omega} \frac{\lambda_i^{n-\varepsilon(n-2)/2}}{\Big(1+\lambda_i^2|x-a_i|^2\Big)^{n-\varepsilon(n-2)/2}} = O\Big(\frac{1}{\lambda_i^{n-\varepsilon(n-2)/2}}\Big) = o(1).$$

The above estimates imply that

$$\alpha_i^{p-\varepsilon}\int_\Omega K\delta_i^{p+1-\varepsilon} = \alpha_i^{p-\varepsilon}K(a_i)\lambda_i^{-\frac{\varepsilon(n-2)}{2}}(S_n + O(\varepsilon)) + o(1). \tag{90}$$

Combining estimates (80)–(90) and using the fact that $\alpha_i^{p-1}K(a_i) = 1 + o(1)$, we obtain

$$S_n + o(1) = \lambda_i^{-\varepsilon(n-2)/2}(S_n + o(1))$$

which implies that $\lambda_i^{-\varepsilon(n-2)/2} = 1 + o(1)$. The proof of the Lemma is thereby complete. □

Next, we consider $(u_\varepsilon)$ a sequence of solutions to $(\mathcal{P}_\varepsilon)$ which have the form (2) and satisfy (3), (4), and (6). We know that $u_\varepsilon$ can be written in the form (10) where $\alpha_i$, $\lambda_i$, $a_i$, and $v_\varepsilon$ satisfy (11). Using Lemma 2, we see that $(\alpha, \lambda, a, v_\varepsilon) \in \mathcal{O}(N, \mu_0)$. Since $u_\varepsilon$ is a solution to $(\mathcal{P}_\varepsilon)$, we see that (24) is satisfied with $v_\varepsilon$. Thus, through its uniqueness, we obtain $v_\varepsilon = \bar{v}_\varepsilon$, where $\bar{v}_\varepsilon$ is defined in Proposition 2. Therefore $v_\varepsilon$ satisfies Estimate (25). We start by proving Theorem 1 in the case of a single interior blow-up point, that is $N = 1$. In this case estimate (25) becomes

$$\|v_\varepsilon\| \leq c\Big(\varepsilon + \frac{|\nabla K(a)|}{\lambda}\Big) + c\begin{cases} \lambda^{-\min(2,(n-2)/2)} & \text{if } n \neq 6, \\ \lambda^{-2}\ln^{2/3}\lambda & \text{if } n = 6. \end{cases} \tag{91}$$

Combining (91) and Proposition 7, we obtain

$$\alpha^{p-1-\varepsilon}\lambda^{-\varepsilon\frac{(n-2)}{2}}K(a) = 1 + O(\varepsilon) + \begin{cases} \lambda^{-2}\ln\lambda & \text{if } n = 4, \\ \lambda^{-2} & \text{if } n \geq 5 \end{cases} \tag{92}$$

$$-d_4 V(a)\frac{\ln\lambda}{\lambda^2} + \alpha^{p-1-\varepsilon}\lambda^{-\varepsilon\frac{(n-2)}{2}}\bar{c}_1\,\varepsilon\, K(a) = O\Big(\varepsilon^2 + \frac{1}{\lambda^2}\Big) \quad (\text{for} \quad n = 4) \tag{93}$$

$$\Big(-d_n V(a) + \alpha^{p-1-\varepsilon}\lambda^{-\varepsilon\frac{(n-2)}{2}}\bar{c}_2\Delta K(a)\Big)\frac{1}{\lambda^2} + \alpha^{p-1-\varepsilon}\lambda^{-\varepsilon\frac{(n-2)}{2}}\bar{c}_1\varepsilon K(a)$$
$$= O\Big(\varepsilon^2 + \frac{|\nabla K(a)|^2}{\lambda^2} + \frac{1}{\lambda^{n-2}} + \frac{\ln^2\lambda}{\lambda^4}\Big) \quad (\text{for} \quad n \geq 5) \tag{94}$$

$$\frac{\nabla K(a)}{\lambda} = O\Big(\varepsilon^2 + \frac{(\ln\lambda)^{\sigma_n}}{\lambda^3} + \frac{1}{\lambda^{\min(4,n-2)}}\Big) \quad (\text{for} \quad n \geq 4). \tag{95}$$

Putting (92) and (95) in (93) and (94), we obtain

$$-d_4 V(a)\frac{\ln\lambda}{\lambda^2} + \bar{c}_1\,\varepsilon = O\Big(\varepsilon^2 + \frac{1}{\lambda^2}\Big) \quad (\text{for} \quad n = 4), \tag{96}$$

$$\Big(-d_n V(a) + \bar{c}_2\frac{\Delta K(a)}{K(a)}\Big)\frac{1}{\lambda^2} + \bar{c}_1\varepsilon = O\Big(\varepsilon^2 + \frac{1}{\lambda^{n-2}} + \frac{\ln^2\lambda}{\lambda^4}\Big) \quad (\text{for} \quad n \geq 5). \tag{97}$$

Using (96) and (97), we obtain

$$\begin{cases} \varepsilon \leq c\lambda^{-2}\ln\lambda & \text{if} \quad n = 4, \\ \varepsilon \leq c\lambda^{-2} & \text{if} \quad n \geq 5. \end{cases} \tag{98}$$

Putting (98) in (95), we derive that

$$\begin{cases} |\nabla K(a)| \leq c\lambda^{-1} & \text{if} \quad n = 4, \\ |\nabla K(a)| \leq c\lambda^{-2} & \text{if} \quad n \geq 5. \end{cases}$$

This implies that the concentration point $a$ converges to a critical point $y$ of $K$. Using this information, we see that (96) and (97) show that (8) and (9) are satisfied. This completes the proof of Theorem 1 in the case of $N = 1$.

Next, we are going to prove Theorem 1 in the case of multiple interior blow-up points; that is, $N \geq 2$. Without loss of generality, we can assume that

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N.$$

First, using the estimate of $\|v_\varepsilon\|$ given by Proposition 2 and the fact that $u_\varepsilon$ is a solution to $(\mathcal{P}_\varepsilon)$, the Claims of Proposition 7 become

$$(E_{\alpha_i}) \quad \alpha_i^{p-1-\varepsilon}\lambda_i^{-\varepsilon\frac{(n-2)}{2}}K(a_i) = 1 + O\Big(\varepsilon + \sum_{j\neq i}\varepsilon_{ij} + \sum_k \frac{\ln^{\sigma_n}\lambda_k}{\lambda_k^2}\Big)$$

$$(E_{\lambda_i}) \quad \alpha_i\Big(-d_nV(a_i)(\ln\lambda_i)^{\sigma_n} + \bar{c}_2\frac{\Delta K(a_i)}{K(a_i)}\Big)\frac{1}{\lambda_i^2} + \alpha_i\bar{c}_1\varepsilon - c_1\sum_{k\neq i}\alpha_k\lambda_i\frac{\partial\varepsilon_{ik}}{\partial\lambda_i}$$
$$= o\Big(\varepsilon + \sum_{k\neq r}\varepsilon_{kr} + \sum_k\frac{(\ln\lambda_k)^{\sigma_n}}{\lambda_k^2}\Big) + O\Big(\sum_k\frac{|\nabla K(a_k)|^2}{\lambda_k^2}\Big)$$

$$(E_{a_i}) \quad \Big|\frac{\nabla K(a_i)}{\lambda_i}\Big| \leq c\Big(\varepsilon^2 + \sum_{k\neq r}\varepsilon_{kr} + \frac{(\ln\lambda_i)^{\sigma_n}}{\lambda_i^3} + \sum_k\frac{1}{\lambda_k^{n-2}} + \sum_k\frac{|\nabla K(a_k)|^2}{\lambda_k^2} + \sum_k\frac{\ln^2\lambda_k}{\lambda_k^4}\Big),$$

where $d_4 = \text{meas}(\mathbb{S}^3)c_0^2$, $d_n$ is defined in Theorem 1 for $n \geq 5$ and where we have used $(E_{\alpha_i})$ in $(E_{\lambda_i})$ and $(E_{a_i})$.

Summing $(E_{a_i})$, we obtain

$$\sum_{i=1}^N\Big|\frac{\nabla K(a_i)}{\lambda_i}\Big| \leq c\Big(\varepsilon^2 + \sum_{k\neq r}\varepsilon_{kr} + \sum_i\frac{(\ln\lambda_i)^{\sigma_n}}{\lambda_i^3} + \sum_i\frac{1}{\lambda_i^{n-2}}\Big). \tag{99}$$

Putting (99) in $(E_{\lambda_i})$, we obtain

$$-\alpha_i^2 d_4V(a_i)\frac{\ln\lambda_i}{\lambda_i^2} + \alpha_i^2\bar{c}_1\varepsilon - c_1\sum_{k\neq i}\alpha_i\alpha_k\lambda_i\frac{\partial\varepsilon_{ik}}{\partial\lambda_i}$$
$$= o\Big(\varepsilon + \sum_{k\neq r}\varepsilon_{kr} + \sum_k\frac{\ln\lambda_k}{\lambda_k^2}\Big) \qquad \text{if } n = 4, \tag{100}$$

$$\Big(\bar{c}_2\frac{\Delta K(a_i)}{K(a_i)} - d_nV(a_i)\Big)\frac{\alpha_i^2}{\lambda_i^2} + \alpha_i^2\bar{c}_1\varepsilon - c_1\sum_{k\neq i}\alpha_i\alpha_k\lambda_i\frac{\partial\varepsilon_{ik}}{\partial\lambda_i}$$
$$= o\Big(\varepsilon + \sum_{k\neq r}\varepsilon_{kr} + \sum_k\frac{1}{\lambda_k^2}\Big)\text{if } n \geq 5. \tag{101}$$

Now, for the sake of clarity, we will split the rest of the proof into three claims.

**Claim 1.** *For $n \geq 4$, we have*

$$\varepsilon + \sum_{k\neq r}\varepsilon_{kr} \leq \frac{c}{\lambda_1^2}(\ln\lambda_1)^{\sigma_n}.$$

*To prove Claim 1, we first notice that*

$$-\lambda_k\frac{\partial\varepsilon_{ik}}{\partial\lambda_k} \geq c\,\varepsilon_{ik} \quad \textit{and} \quad -\lambda_i\frac{\partial\varepsilon_{ik}}{\partial\lambda_i} - 2\lambda_k\frac{\partial\varepsilon_{ik}}{\partial\lambda_k} \geq c\,\varepsilon_{ik} \quad \textit{for} \quad \lambda_k \geq \lambda_i. \tag{102}$$

*Thus, multiplying (100) and (101) by $2^i$ and summing over $i \in \{1, \cdots, N\}$, we obtain*

$$O\Big(\frac{(\ln\lambda_1)^{\sigma_n}}{\lambda_1^2}\Big) + \bar{c}_1\varepsilon\sum_i\alpha_i^2 2^i - c_1\sum_i\sum_{k\neq i}\alpha_i\alpha_k 2^i\lambda_i\frac{\partial\varepsilon_{ik}}{\partial\lambda_i} = o\Big(\varepsilon + \sum_{k\neq r}\varepsilon_{kr} + \frac{(\ln\lambda 1)^{\sigma_n}}{\lambda_1^2}\Big). \tag{103}$$

*Clearly, the combination of (102) and (103) completes the proof of Claim 1.*

*To proceed further, we introduce the following set*

$$\mathcal{D} = \{1, \cdots, N\} \setminus \{j \leq N : \frac{\lambda_1}{\lambda_j} \to 0\}. \tag{104}$$

*Then, our second claim reads:*

**Claim 2.** *For each $j \in \mathcal{D}$, there exists $i_j$ such that the concentration point $a_j$ converges to a critical point $y_{i_j}$ of K. In addition, we have*

$$\lambda_j |a_j - y_{i_j}| \leq c \ln^{\sigma_n} \lambda_j \quad \textit{and} \quad i_j \neq i_k \quad \forall\, j, k \in \mathcal{D} \quad \textit{with} \quad j \neq k.$$

*To prove Claim 2, letting $j \in \mathcal{D}$, we put Claim 1 and estimate (99) in $(E_{a_j})$. This leads to*

$$\frac{|\nabla K(a_j)|}{\lambda_j} \leq c \frac{(\ln \lambda_1)^{\sigma_n}}{\lambda_1^2} \leq c \frac{(\ln \lambda_j)^{\sigma_n}}{\lambda_j^2}, \tag{105}$$

*which implies that $|\nabla K(a_j)|$ tends to 0. Hence, there exists $i_j$ such that the concentration point $a_j$ converges to a critical point $y_{i_j}$ of K. Furthermore, since $y_{i_j}$ is assumed to be non-degenerate, Estimate (105) implies that*

$$\lambda_j |a_j - y_{i_j}| \leq \lambda_j |\nabla K(a_j)| \leq c\, (\ln \lambda_j)^{\sigma_n}.$$

*To complete the proof of Claim 2, arguing by contradiction, we assume that $j, k \in \mathcal{D}$ exist with $j \neq k$ satisfying $y_{i_j} = y_{i_k}$. Since $j, k \in \mathcal{D}$, we obtain*

$$\lambda_j \lambda_k |a_j - a_k|^2 \leq c\, (\ln^{\sigma_n} \lambda_1)^2.$$

*This implies that*

$$\varepsilon_{jk} \geq \Big(c + c(\ln^{\sigma_n} \lambda_1)^2\Big)^{(2-n)/n} \geq c(\ln^{\sigma_n} \lambda_1)^{2-n} >> \frac{\ln^{\sigma_n} \lambda_1}{\lambda_1^2}$$

*which gives a contradiction to Claim 1. The proof of Claim 2 is thereby complete.*

Next, we state and prove the third claim.

**Claim 3.** *The set $\mathcal{D}$ is equal to $\{1, \cdots, N\}$, that is, all the rate $\lambda_j'$s of the concentration are of the same order.*

To prove Claim 3, arguing by contradiction, we assume that $\mathcal{D} \neq \{1, \cdots, N\}$. Let $j = \max \mathcal{D}$. Multiplying $(E_{\lambda_i})$ by $2^i \alpha_i$ and summing over $i \geq j+1$, we obtain

$$O\Big(\frac{(\ln \lambda j + 1)^{\sigma_n}}{\lambda j + 1^2}\Big) + \bar{c}_1 \varepsilon \sum_{i \geq j+1} \alpha_i^2 2^i - c_1 \sum_{i \geq j+1} \sum_{l \neq i} \alpha_i \alpha_l 2^i \lambda_i \frac{\partial \varepsilon_{il}}{\partial \lambda_i} = o\Big(\varepsilon + \sum_{k \neq r} \varepsilon_{kr} + \frac{(\ln \lambda_1)^{\sigma_n}}{\lambda_1^2}\Big).$$

Thus, using (102) and Claim 1, we obtain

$$\sum_{i \geq j+1} \sum_{l \neq i} \varepsilon_{il} + \varepsilon = o\Big(\frac{(\ln \lambda_1)^{\sigma_n}}{\lambda_1^2}\Big). \tag{106}$$

Now, using (106) and Claim 2, we obtain

$$\begin{cases} \varepsilon_{1k} = o\Big(\frac{(\ln \lambda_1)^{\sigma_n}}{\lambda_1^2}\Big) & \text{if } k \geq j+1, \\ \varepsilon_{1k} \leq \frac{c}{(\lambda_1 \lambda_k |a_1 - a_k|^2)^{(n-2)/2}} \leq \frac{c}{\lambda_1^{n-2}} = o\Big(\frac{(\ln \lambda_1)^{\sigma_n}}{\lambda_1^2}\Big) & \text{if } k \leq j. \end{cases} \tag{107}$$

Writing $(E_{\lambda_1})$ and using (107), Claim 1 and (106), we obtain, for $n = 4$,

$$-d_4 V(a_1) \alpha_1 \frac{\ln \lambda_1}{\lambda_1^2} = o\Big(\frac{\ln \lambda_1}{\lambda_1^2}\Big),$$

which gives a contradiction. Therefore our claim follows for $n = 4$. In the same way for $n \geq 5$, using (106) and (107), (101), with $i = 1$, implies that

$$\alpha_1\Big( - d_n V(a_1) + \bar{c}_2 \frac{\Delta K(a_1)}{K(a_1)}\Big)\frac{1}{\lambda_1^2} = o\Big(\frac{1}{\lambda_1^2}\Big). \tag{108}$$

In addition, using Claim 2, the concentration point $a_1$ converges to a critical point $y_{i_1}$ of $K$. Thus estimate (108) becomes

$$\Big( - d_n V(y_{i_1}) + \bar{c}_2 \frac{\Delta K(y_{i_1})}{K(y_{i_1})}\Big) = o(1)$$

which gives a contradiction. This implies that our claim also follows for $n \geq 5$.

To complete the proof of Theorem 1, it remains to be shown that Estimate (9) holds. Combining Claims 2 and 3, we see that

$$\varepsilon_{ij} \leq \frac{c}{(\lambda_i\lambda_j)^{(n-2)/2}} = o\Big(\frac{\ln^{\sigma_n}\lambda_1}{\lambda_1^2}\Big) \quad \forall \quad i \neq j.$$

Thus, for $n = 4$, using (100), for each $i$, we obtain

$$-d_4 V(y_{k_i})\frac{\ln\lambda_i}{\lambda_i^2} + \bar{c}_1\,\varepsilon = o\Big(\frac{\ln\lambda_i}{\lambda_i^2}\Big)$$

which implies estimate (9) for $n = 4$.

For $n \geq 5$, using again (101), we obtain

$$\Big( - d_n V(y_{k_i}) + \bar{c}_2 \frac{\Delta K(y_{k_i})}{K(y_{k_i})}\Big)\frac{1}{\lambda_i^2} + \bar{c}_1\,\varepsilon = o\Big(\frac{1}{\lambda_i^2}\Big)$$

which implies estimate (9) for $n \geq 5$. This completes the proof of Theorem 1.

## 5. Construction of Interior Bubbling Solutions

The goal of this section is to prove Theorem 2; that is, we are going to construct solutions to $(\mathcal{P}_\varepsilon)$ which blow up at $N$ interior point(s) as $\varepsilon$ goes to zero, with $N \geq 1$. As the proof of the theorem is simpler in the case of one concentration interior point, we will focus on the case of multiple interior blow-up points. The proof of the theorem for one blow-up point is easily deduced from our proof by eliminating the terms which involve more than one point. We will follow [20] (see also [29]). Let $y_1, \cdots, y_N$ be non-degenerate critical points of $K$ satisfying (8) if $n \geq 5$. Inspired by Theorem 1, we introduce the following set which depends on the kind of the blow-up points we want to obtain.

$$\begin{aligned}\mathbb{M}(N,\varepsilon) = \{(\alpha,\lambda,a,v) \in (\mathbb{R}_+)^N \times (\mathbb{R}_+)^N \times \Omega^N \times H^1(\Omega) : |\alpha_i^{\frac{4}{n-2}} K(a_i) - 1| < c\,\varepsilon\,\ln^2\varepsilon,\\ \frac{1}{c} < \frac{\lambda_i^2\varepsilon}{\ln^{\sigma_n}\lambda_i} < c, |a_i - y_i| < c\,\varepsilon^{1/5} \quad \forall 1 \leq i \leq N, v \in E_{a,\lambda} \text{ and } \|v\| < c\sqrt{\varepsilon}\},\end{aligned} \tag{109}$$

where $c$ is a positive constant, $E_{a,\lambda}$ is defined by (12), $\sigma_n = 0$ if $n \geq 5$ and $\sigma_n = 1$ if $n = 4$.

Notice that such a condition imposed on the parameter $\lambda_i$ in $\mathbb{M}(N,\varepsilon)$ implies that

$$\frac{1}{\lambda_i} = O(f(\varepsilon)) \quad \text{with} \quad f(\varepsilon) = \begin{cases} \sqrt{\varepsilon} & \text{if} \quad n \geq 5, \\ \sqrt{\varepsilon/|\ln\varepsilon|} & \text{if} \quad n = 4. \end{cases} \tag{110}$$

We also introduce the following function

$$g_\varepsilon : \mathbb{M}(N,\varepsilon) \to \mathbb{R}, \quad (\alpha,\lambda,a,v) \mapsto g_\varepsilon(\alpha,\lambda,a,v) = I_\varepsilon\Big(\sum_{i=1}^N \alpha_i\delta_{a_i,\lambda_i} + v\Big).$$

Since the variable $v \in E_{a,\lambda}$, the Euler–Lagrange multiplier theorem implies that the following proposition holds.

**Proposition 8.** *$(\alpha, \lambda, a, v) \in \mathbb{M}(N, \varepsilon)$ is a critical point of $g_\varepsilon$ if, and only if, $u = \sum_{i=1}^N \alpha_i \delta_i + v$ is a critical point of $I_\varepsilon$; that is, if, and only if, ts $(A, B, C) \in \mathbb{R}^N \times \mathbb{R}^N \times (\mathbb{R}^n)^N$ exists such that the following system holds*

$$\frac{\partial g_\varepsilon}{\partial \alpha_i}(\alpha, \lambda, a, v) = 0 \; \forall i \in \{1, ..., N\}, \tag{111}$$

$$\frac{\partial g_\varepsilon}{\partial \lambda_i}(\alpha, \lambda, a, v) = B_i \int_\Omega \nabla v \lambda_i \frac{\partial^2 \delta_i}{\partial \lambda_i^2} + \sum_{j=1}^n C_{ij} \int_\Omega \nabla v \frac{1}{\lambda_i} \frac{\partial^2 \delta_i}{\partial \lambda_i \partial a_i^j} \; \forall i \in \{1, ..., N\}, \tag{112}$$

$$\frac{\partial g_\varepsilon}{\partial a_i}(\alpha, \lambda, a, v) = B_i \int_\Omega \nabla v \lambda_i \frac{\partial^2 \delta_i}{\partial \lambda_i \partial a_i} + \sum_{j=1}^n C_{ij} \int_\Omega \nabla v \frac{1}{\lambda_i} \frac{\partial^2 \delta_i}{\partial a_i^j \partial a_i} \; \forall i \in \{1, ..., N\}, \tag{113}$$

$$\frac{\partial g_\varepsilon}{\partial v}(\alpha, \lambda, a, v) = \sum_{k=1}^N \Big( A_k \frac{\partial \varphi_k}{\partial v} + B_k \frac{\partial \psi_k}{\partial v} + \sum_{j=1}^n C_{kj} \frac{\partial \xi_{kj}}{\partial v} \Big), \tag{114}$$

*where*

$$\varphi_k(\alpha, \lambda, a, v) = \int_\Omega \nabla v \nabla \delta_k, \qquad \psi_k(\alpha, \lambda, a, v) = \lambda_k \int_\Omega \nabla v \nabla \frac{\partial \delta_k}{\partial \lambda_k},$$
$$\xi_{kj}(\alpha, \lambda, a, v) = \frac{1}{\lambda_k} \int_\Omega \nabla v \nabla \frac{\partial \delta_k}{\partial a_k^j}.$$

The proof of Theorem 2 will be carried out through a careful analysis of the previous system on $\mathbb{M}(N, \varepsilon)$. Observe that $\overline{v}_\varepsilon$, defined in Proposition 2, satisfies equation (114). In the sequel, we will write $v_\varepsilon$ instead of $\overline{v}_\varepsilon$. Taking $(\alpha, \lambda, a, 0) \in \mathbb{M}(N, \varepsilon)$, we see that $u_\varepsilon = \sum_{i=1}^N \alpha_i \delta_i + v_\varepsilon$ is a critical point of $I_\varepsilon$ if and only if $(\alpha, \lambda, a)$ satisfies the following system for each $1 \le i \le N$

$$(E_{\alpha_i}) \quad \langle I'_\varepsilon(u), \delta_i \rangle = 0 \tag{115}$$

$$(E_{\lambda_i}) \quad \Big\langle I'_\varepsilon(u), \alpha_i \frac{\partial \delta_i}{\partial \lambda i} \Big\rangle = B_i \int_\Omega \nabla v \lambda_i \frac{\partial^2 \delta_i}{\partial \lambda_i^2} + \sum_{j=1}^n C_{ij} \int_\Omega \nabla v \frac{1}{\lambda_i} \frac{\partial^2 \delta_i}{\partial \lambda_i \partial a_i^j} \tag{116}$$

$$(E_{a_i}) \quad \Big\langle I'_\varepsilon(u), \alpha_i \frac{\partial \delta_i}{\partial a_i} \Big\rangle = B_i \int_\Omega \nabla v \lambda_i \frac{\partial^2 \delta_i}{\partial \lambda_i \partial a_i} + \sum_{j=1}^n C_{ij} \int_\Omega \nabla v \frac{1}{\lambda_i} \frac{\partial^2 \delta_i}{\partial a_i^j \partial a_i}. \tag{117}$$

Notice that, since $(\alpha, \lambda, a, 0) \in \mathbb{M}(N, \varepsilon)$, we have

$$|a_i - a_j| \ge c > 0 \quad \text{and} \quad \varepsilon_{ij} = \begin{cases} O(\varepsilon^{(n-2)/2}) & \text{if} \quad n \ge 5, \\ \varepsilon / |\ln \varepsilon| & \text{if} \quad n = 4. \end{cases}$$

The following result is a direct consequence of Proposition 7.

**Lemma 3.** *For a small ε, the following statements hold:*

$$\|v_\varepsilon\| \le c \begin{cases} \varepsilon^{7/10} & \textit{if} \quad n \ge 5, \\ \sqrt{\varepsilon / |\ln \varepsilon|} & \textit{if} \quad n = 4, \end{cases} \quad R_{1i} = \le c \begin{cases} \varepsilon & \textit{if} \quad n \ge 5, \\ \varepsilon / |\ln \varepsilon| & \textit{if} \quad n = 4, \end{cases},$$
$$R_{2i}; \; R_{3i} \; \le c \begin{cases} \varepsilon^{7/5} & \textit{if} \quad n \ge 5, \\ \varepsilon / |\ln \varepsilon| & \textit{if} \quad n = 4, \end{cases}$$

*where $R_{1i}$, $R_{2i}$, and $R_{3i}$ are defined in Propositions 7.*

Next, our aim is to estimate the numbers $A_i's$, $B_i's$, and $C_{ij}'s$ which appear in Equations (115)–(117).

**Lemma 4.** *Let $(\alpha,\lambda,a,0)\in\mathbb{M}(N,\varepsilon)$. Then, for a small $\varepsilon$, the following estimates hold:*

$$A_i=O(\varepsilon\ln^2\varepsilon),\quad B_i=O(\varepsilon)\quad\textit{and}\quad C_{ij}=O(\phi(\varepsilon))\quad\forall\,1\le i\le N,\,\forall\,1\le j\le n,$$

*where*

$$\phi(\varepsilon)=\begin{cases}\varepsilon^{7/10} & \textit{if}\quad n\ge 5,\\ \varepsilon^{7/10}/\sqrt{|\ln\varepsilon|} & \textit{if}\quad n=4.\end{cases}$$

**Proof.** Applying $\frac{\partial g_\varepsilon}{\partial v}$ (see (114)) to the functions $\delta_i$, $\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}$ and $\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial a_i^j}$, we obtain the following quasi-diagonal system

$$cA_i+O\Big((f(\varepsilon))^{n-2}\Big(\sum_{k=1}^{N}(|A_k|+|B_k|+\sum_{j=1}^{n}|C_{kj}|)\Big)\Big)=\Big\langle\frac{\partial g_\varepsilon}{\partial v},\delta_i\Big\rangle,$$

$$c'B_i+O\Big((f(\varepsilon))^{n-2}\Big(\sum_{k=1}^{N}(|A_k|+|B_k|+\sum_{j=1}^{n}|C_{kj}|)\Big)\Big)=\Big\langle\frac{\partial g_\varepsilon}{\partial v},\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}\Big\rangle,$$

$$c''C_{il}+O\Big((f(\varepsilon))^{n-2}\Big(\sum_{k=1}^{N}(|A_k|+|B_k|+\sum_{j=1}^{n}|C_{kj}|)\Big)\Big)=\Big\langle\frac{\partial g_\varepsilon}{\partial v},\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial a_i^l}\Big\rangle,$$

where

$$c=\int_{\mathbb{R}^n}|\nabla\delta_i|^2,\quad c'=\int_{\mathbb{R}^n}\Big|\lambda_i\nabla\frac{\partial\delta_i}{\partial\lambda_i}\Big|^2\quad\text{and}\quad c''=\int_{\mathbb{R}^n}\Big|\frac{1}{\lambda_i}\nabla\frac{\partial\delta_i}{\partial a_i^j}\Big|^2.\tag{118}$$

Combining Proposition 7, Lemma 3 and the fact that $(\alpha,\lambda,a,0)\in\mathbb{M}(N,\varepsilon)$, we derive that for all $i\in\{1,\cdots,N\}$ we have

$$\Big\langle\frac{\partial g_\varepsilon}{\partial v},\delta_i\Big\rangle=O(\varepsilon\ln^2\varepsilon),\quad\Big\langle\frac{\partial g_\varepsilon}{\partial v},\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}\Big\rangle=O(\varepsilon)\quad\text{and}\quad\Big\langle\frac{\partial g_\varepsilon}{\partial v},\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial a_i}\Big\rangle=O(\phi(\varepsilon)).$$

Thus, we obtain

$$M(A,B,C)^T=(O(\varepsilon\ln^2\varepsilon),O(\varepsilon),O(\phi(\varepsilon))^T,$$

where $M$ is the matrix defined by $m_{ij}=O((f(\varepsilon))^{n-2})\ \forall\,i\neq j$ and

$$m_{ii}=\begin{cases}c+O((f(\varepsilon))^{n-2}) & \text{for}\quad 1\le i\le N\\ c'+O((f(\varepsilon))^{n-2}) & \text{for}\quad N+1\le i\le 2N\\ c''+O((f(\varepsilon))^{n-2}) & \text{for}\quad 2N+1\le i\le N(n+2),\end{cases}$$

where $c$, $c'$, and $c''$ are defined in (118).

Hence Lemma 4 follows. □

Next, we are going to study equations $(E_{\alpha_i})$, $(E_{\lambda_i})$, $(E_{a_i})$. To obtain an easy system to solve, we perform the following change of variables

$$\beta_i=1-\alpha_i^{p-1}K(a_i),\quad\frac{\ln^{\sigma_n}\lambda_i}{\lambda_i^2}=\frac{\overline{c}_1}{\chi(y_i)}\,\varepsilon(1+\wedge_i),\quad z_i=a_i-y_i\quad 1\le i\le N,\tag{119}$$

where

$$\chi(y_i)=\begin{cases}d_4\,V(y_i) & \text{if}\quad n=4,\\ -\overline{c}_2\frac{\Delta K(y_i)}{K(y_i)}+d_n\,V(y_i) & \text{if}\quad n\ge 5.\end{cases}$$

Using this change of variables, we rewrite our system in the following simple form:

**Lemma 5.** *For $\varepsilon$ small, the system (115)–(117) is equivalent to the following system*

$$(\mathcal{S}) \quad \begin{cases} \beta_i = O(\varepsilon|\ln\varepsilon|) & \forall 1 \le i \le N \\ \wedge_i = O(\varepsilon^{2/5} + |z_i|^2) & \forall 1 \le i \le N \\ D^2K(y_i)(z_i,.) = O\Big(\varepsilon^{9/10} + |z_i|^2\Big) & \forall 1 \le i \le N. \end{cases} \quad (\textit{for} \quad n \ge 5),$$

$$(\mathcal{S}) \quad \begin{cases} \beta_i = O(\varepsilon|\ln\varepsilon|) & \forall 1 \le i \le N \\ \wedge_i = O(1/|\ln\varepsilon| + |z_i|^2) & \forall 1 \le i \le N \\ D^2K(y_i)(z_i,.) = O\Big(\sqrt{\varepsilon/|\ln\varepsilon|} + |z_i|^2\Big) & \forall 1 \le i \le N. \end{cases} \quad (\textit{for} \quad n = 4).$$

**Proof.** Using the fact that

$$\alpha_i^{p-1-\varepsilon} = \alpha_i^{p-1} + O(\varepsilon) \quad \text{and} \quad \lambda_i^{-\varepsilon\frac{n-2}{2}} = 1 + O(\varepsilon \ln \lambda_i),$$

we see that equation (115) is equivalent to

$$(E'_{\alpha_i}) \quad \beta_i = O(\varepsilon|\ln\varepsilon|) \quad \forall 1 \le i \le N.$$

For the second equation (116), we start by the case where $n \ge 5$. Using Proposition 7 and Lemmas 3 and 4, we obtain

$$\overline{c}_1\, \varepsilon - \Big( -\overline{c}_2 \frac{\Delta K(a_i)}{K(a_i)} + d_n\, V(a_i) \Big) \frac{1}{\lambda_i^2} = O\Big(\varepsilon^{7/5}\Big).$$

Writing

$$V(a_i) = K(y_i) + O(|z_i|); \quad \frac{\Delta K(a_i)}{K(a_i)} = \frac{\Delta K(y_i)}{K(y_i)} + O(|z_i|),$$

we obtain

$$\wedge_i = O(\varepsilon^{2/5} + |z_i|). \tag{120}$$

Now, Using again Proposition 7 and Lemmas 3 and 4, we obtain

$$\nabla K(a_i) = O(\lambda_i \varepsilon^{7/5}) = O(\varepsilon^{9/10})$$

which implies the third equation in the system $(\mathcal{S})$. Using the fact that $y_i$ is a non-degenerate critical point of $K$, we deduce that

$$|z_i| \le c\Big(\varepsilon^{9/10} + |z_i|^2\Big).$$

Putting the last inequality in (120), we obtain the second equation in the system $(\mathcal{S})$ which completes the proof of the lemma for $n \ge 5$. In the same way, we prove the lemma in the case where $n = 4$. $\square$

To complete the proof of Theorem 2, we rewrite the system $(\mathcal{S})$ in the following form

$$\begin{cases} \beta_i = U_{1,i}(\varepsilon, \beta, \wedge, z) & \forall 1 \le i \le N \\ \wedge_i = U_{2,i}(\varepsilon, \beta, \wedge, z) & \forall 1 \le i \le N \\ D^2K(y_i)(z_i,.) = U_{3,i}(\varepsilon, \beta, \wedge, z) & \forall 1 \le i \le N, \end{cases}$$

and we define the following linear map

$$L : \mathbb{R}^N \times \mathbb{R}^N \times (\mathbb{R}^n)^N \to \mathbb{R}^N \times \mathbb{R}^N \times (\mathbb{R}^n)^N$$
$$(\beta, \wedge, z) \mapsto (\beta, \wedge, D^2K(y_1)(z_1,.), \cdots, D^2K(y_N)(z_N,.)),$$

where $\beta = (\beta_1, \cdots, \beta_N)$, $\wedge = (\wedge_1, \cdots, \wedge_N)$ and $z = (z_1, \cdots, z_N)$.

We see that $L$ is invertible. Thus, applying Brouwer's fixed point theorem, we deduce that the system $(\mathcal{S})$ has at least one solution $(\beta_\varepsilon, \wedge_\varepsilon, z_\varepsilon)$ for a small $\varepsilon$ (for more details, see [20]). As in [20], we prove that the constructed function $u_\varepsilon = \sum_{i=1}^{N} \alpha_i \delta_i + v_\varepsilon$ is positive. Lastly, as a straightforward consequence of this construction, we see that $(\mathcal{P}_\varepsilon)$ admits at least $2^m - 1$ solutions provided that $\varepsilon$ is small, where $m$ is defined in Theorem 1. This completes the proof of Theorem 2.

## 6. Conclusions

By using a careful asymptotic analysis of the gradient of the associated Euler–Lagrange functional in the neighborhood of the so-called bubbles, we were able to give a complete description of the interior blow-up picture of solutions of $(\mathcal{P}_\varepsilon)$ that weakly converge to zero. We also constructed interior multi-peak solutions which lead to a multiplicity result for the problem. Notice that, when the number of concentration points is bigger than or equal to two, a non-degeneracy assumption for the critical points of $K$ was needed to prove that the concentration points are uniformly separated. However, some questions remain open:

(i) What happens if a degenerate critical point of $K$ exists, particularly when $K$ satisfies some flatness assumption?

(ii) Do solutions exist which involve some interior concentration points which converge to the boundary?

**Author Contributions:** M.B.A., K.E.M. and F.M.S.: conceptualization, methodology, investigation, writing original draft, writing—review and editing. All authors contributed equally to the writing of this paper. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Data Availability Statement:** Data are contained within the article.

**Acknowledgments:** Researchers would like to thank the Deanship of Scientific Research, Qassim University for funding publication of this project.

**Conflicts of Interest:** The authors declare no conflicts of interest.

## References

1. Keller, E.F.; Segel, L.A. Initiation of slime mold aggregation viewed as an instability. *J. Theor. Biol.* **1970**, *26*, 399–415. [CrossRef] [PubMed]
2. Gierer, A.; Meinhardt, H. A theory of biological pattern formation. *Kybernetik* **1972**, *12*, 30–39. [CrossRef] [PubMed]
3. Choi, D.; Niu, Q. Bose Einstein condensation in an optical lattice. *Phys. Rev. Lett.* **1999**, *82*, 2022–2025. [CrossRef]
4. Erdös, L.; Schlein, B.; Yau, H. Derivation of the Gross-Pitaevskii equation for the dynamics of Bose-Einstein condensate. *Ann. Math.* **2010**, *172*, 291–370. [CrossRef]
5. Gross, E. Structure of a quantized vortex in boson systems. *Nuovo C.* **1961**, *20*, 454–466. [CrossRef]
6. Lin, C.S.; Ni, W.N.; Takagi, I. Large amplitude stationary solutions to a chemotaxis system. *J. Differ. Equ.* **1988**, *72*, 1–27. [CrossRef]
7. Ni, W.-M. Qualitative properties of solutions to elliptic problems. In *Handbook of Differential Equations: Stationary Partial Differential Equations*; North-Holland: Amsterdam, The Netherlands, 2004; pp. 157–233.
8. Zhu, M. Uniqueness results through a priori estimates, I. A three dimensional Neumann problem. *J. Differ. Equ.* **1999**, *154*, 284–317. [CrossRef]
9. Yadava, S.L. Existence and nonexistence of positive radial solutions of Neumann problems with critical Sobolev exponents. *Arch. Rat. Mech. Anal.* **1991**,*115*, 275–296.
10. Rey, O.; Wei, J. Arbitrary number of positive solutions for elliptic problem with critical nonlinearity. *J. Eur. Math. Soc.* **2005**, *7*, 449–476.
11. Wei, J.; Xu, B.; Yang, W. On Lin-Ni's Conjecture in dimensions four and six. *Sci. China Math.* **2019**, *49*, 281–306. [CrossRef]
12. Adimurthi, A.; Mancini, G. The Neumann problem for elliptic equations with critical nonlinearity. In *A Tribute in Honour of G. Prodi*; Scuola Normale Superiore: Pisa, Italy, 1991; pp. 9–25.
13. Adimurthi, A.; Mancini, G. Geometry and topology of the boundary in the critical Neumann problem. *J. Reine Angew. Math.* **1994**, *456*, 1–18.
14. Wang, X.J. Neumann problem of semilinear elliptic equations involving critical Sobolev exponents. *J. Differ. Equ.* **1991**, *93*, 283–310. [CrossRef]

15. Wang, Z.Q. Construction of multi-peaked solution for a nonlinear Neumann problem with critical exponent. *J. Nonlinear Anal. TMA* **1996**, *27*, 1281–1306. [CrossRef]
16. Rey, O. The question of interior blow-up points for an elliptic Neumann problem: The critical case. *J. Math. Pures Appl.* **2002**, *81*, 655–696. [CrossRef]
17. Rey, O.; Wei, J. Blow-up solutions for an elliptic Neumann problem with sub- or supercritical nonlinearity, I: $N = 3$. *J. Funct. Anal.* **2004**, *212*, 472–499. [CrossRef]
18. Rey, O.; Wei, J. Blow-up solutions for an elliptic Neumann problem with sub- or supercritical nonlinearity, II: $N \geq 4$. *Ann. Inst. Henri Poincaré C Anal. Lin.* **2005**, *22*, 459–484. [CrossRef]
19. Ben Ayed, M.; El Mehdi, K. Non-Existence of Interior Bubbling solutions for Slightly Supercritical Elliptic Problems. *Bound. Value Probl.* **2023** *2023*, 90. [CrossRef]
20. El Mehdi, K.; Mohamed Salem, F. Interior Bubbling Solutions for an Elliptic Equation with Slightly Subcritical Nonlinearity. *Mathematics* **2023**, *11*, 1471. [CrossRef]
21. Wang, X. On location of blow up of ground states of semilinear elliptic equations in $\mathbb{R}^n$ involving critical Sovolev exponents. *J. Differ. Equ.* **1996**, *127*, 148–173. [CrossRef]
22. Lions, P.L. The concentration-compactness principle in the calculus of variations. The limit case. Part I. *Rev. Mat. Iberoam.* **1985**, *1*, 145–201. [CrossRef]
23. Struwe, M. A global compactness result for elliptic boundary value problems involving limiting nonlinearities. *Math. Z.* **1984**, *187*, 511–517. [CrossRef]
24. Caffarelli, L.; Gidas, B.; Spruck, J. Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth. *Comm. Pure Appl. Math.* **1989**, *42*, 271–297. [CrossRef]
25. Del Pino, M.; Musso, M.; Pistoia, A. Super-critical boundary bubbling in a semilinear Neumann problem. *Ann. Inst. Henri Poincaré C Anal. Lin.* **2005**, *22*, 45–82. [CrossRef]
26. Bahri, A. Critical points at infinity in some variational problems. In *Research Notes in Mathematics*; Longman-Pitman: London, UK, 1989; Volume 182.
27. Rey, O. Boundary effect for an elliptic Neumann problem with critical nonlinearity. *Comm. Partial. Differ. Equ.* **1997**, *22*, 1055–1139. [CrossRef]
28. Rey, O. The role of the Green's function in a nonlinear elliptic problem involving the critical Sobolev exponent. *J. Funct. Anal.* **1990**, *89*, 1–52. [CrossRef]
29. Bahri, A.; Li, Y.; Rey, O. On a variational problem with lack of compactness: The topological effect of the critical points at infinity. *Calc. Var. Partial. Diff. Equ.* **1995**, *3*, 67–94. [CrossRef]