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An Explicit Form of Ramp Function

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Abstract: In this paper, an analytical exact form of the ramp function is presented. This seminal function constitutes a fundamental concept of the digital signal processing theory and is also involved in many other areas of applied sciences and engineering. In particular, the ramp function is performed in a simple manner as the pointwise limit of a sequence of real and continuous functions with pointwise convergence. This limit is zero for strictly negative values of the real variable x, whereas it coincides with the independent variable x for strictly positive values of the variable x. Here, one may elucidate beforehand that the pointwise limit of a sequence of continuous functions can constitute a discontinuous function, on the condition that the convergence is not uniform. The novelty of this work, when compared to other research studies concerning analytical expressions of the ramp function, is that the proposed formula is not exhibited in terms of miscellaneous special functions, e.g., gamma function, biexponential function, or any other special functions, such as error function, hyperbolic function, orthogonal polynomials, etc. Hence, this formula may be much more practical, flexible, and useful in the computational procedures, which are inserted into digital signal processing techniques and other engineering practices.

Keywords: ramp function; analytical expression; absolute value

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1. Introduction

The ramp function, the notation of which is R(x), is a discontinuous, single-valued function of a real variable with a point discontinuity located at zero. For negative arguments, R(x) vanishes, whilst for positive arguments, R(x) is simply x [1]. In addition, its first derivative is the Heaviside step function, also known as the unit step function, whereas its second derivative is the Dirac delta distribution (or δ distribution), also known as the unit impulse [2]. Step, ramp, and parabolic functions are called singularity functions [2,3]. In fact, the ramp function has many applications in applied sciences/engineering and is mainly involved in digital signal processing and electrical engineering. Actually, it constitutes a signal, the amplitude of which varies linearly with respect to time and can be expressed by several definitions [4,5]. In digital signal processing, the unit ramp function is a discrete time signal that starts from zero and increases linearly. Here one may emphasize that the basic continuous-time (CT) and discrete-time (DT) signals include impulse, step, ramp, parabolic, rectangular pulse, triangular pulse, signum function, Sinc function, sinusoid (also known as sine wave, sinusoidal wave), and finally, real, along with complex exponentials [4–6]. The ramp function states that the signal will start from time zero and instantly will take a slant shape, and depending upon given time characteristics (i.e., either positive or negative, with it being positive here), the signal will follow the straight slant path either towards the right or the left, with it being towards the right here [6,7]. In this context, the ramp function constitutes a type of elementary function which exists only for the positive side and is zero for negative [8–10]. Moreover, the impulse function, the role of which is also very important in these fields, is obtained by differentiating the ramp function twice [9–11]. Apart from the previously mentioned examples highlighting the pivotal role of this function in digital signal processing and electrical engineering, it is important to note that the ramp function finds diverse applications, extending into areas such as finance and applied statistics (e.g., regression models) [2,4,12]. The currents through and voltages across these elements are obtained by solving integro-differential equations. Alternatively, the elements in the network are transformed from the time domain, and an algebraic equation is obtained, which is expressed in terms of input and output [11,12]. The commonly used inputs are impulse, step, ramp, sinusoids, exponentials, etc. In addition to the aforementioned applications above that demonstrate the central role of this function in digital signal processing and electrical engineering, one may also point out that the ramp function has many other applications in finance, as well as in applied statistics (e.g., regression models), etc. [2,4,12]. Actually, there are many explicit forms of this significant function that can be found in literature.

Especially in reference [4], a sophisticated and clear representation of this function was suggested through the following explicit form:

$$R(x) = \frac{x}{2} + \frac{x}{\pi}\left(arctan(x) + arctan\left(\frac{1}{x}\right)\right) \tag{1}$$

In ref. [13], the following exact form of this function was performed:

$$R(x) = \frac{x}{2} + i\,\frac{ln\,(x) - ln\,(-x)}{2\pi} \tag{2}$$

Nonetheless, a shortcoming of the above formula is that it cannot be defined for zero argument, i.e., at $x = 0$.

Furthermore, building upon the analysis presented in reference [14], which involved the analytical treatment of the Heaviside step function, a distinct closed-form expression for the ramp function can be derived, as indicated by the following formula:

$$R(x) = \frac{3x}{4} + \frac{x}{\pi}\cdot(arctan(x-1) + arctan(\frac{x-2}{x})) \tag{3}$$

Here, one may emphasize that according to the approach adopted in ref. [14], the singularity structure was left ambiguous. Evidently, the same problem will remain if the ramp function is derived by the aid of this formula, i.e., by multiplying the right-hand side of Equation (3) with the variable x.

Further, on the basis of ref. [15], the ramp function can be calculated as:

$$R(x) = \frac{x}{\pi}\cdot\left(arctan(x^n) + 2\,arctan(\frac{x^n}{x^{2n}+1}) + arctan(\frac{1}{x^n}) + 2\,arctan(\frac{x^{2n}-x^n+1}{x^{2n}+x^n+1})\right) \tag{4}$$

Meanwhile, there are many smooth analytical approximations to the ramp function, as can be seen in the literature [16–19]. One of the simplest approximations to this function is the following [16]:

$$R(x) = 1 + \frac{x}{2} + \frac{x}{\sqrt{x^2+\varepsilon^2}} \tag{5}$$

where $\varepsilon \in (0,1)$, such that $\varepsilon \ll 1$.

On the other hand, in ref. [20], an analytical form of the unit step function was proposed, and a qualitative study on the ramp and signum functions was carried out.

Concurrently, as it was signified beforehand, there are numerous applications of the ramp function in applied sciences and engineering, as can be observed in literature.

In ref. [21], a detailed study on neural networks operators by the aid of ramp functions was carried out, whilst an analogous valuable investigation took place in ref. [22], where an interpolation by neural network operators was activated by means of ramp functions.

Ref. [23] conducted a notable investigation into the utilization of fixed-point neuron models incorporating thresholds, emphasizing the significance of ramp and sigmoid activation functions. For a comprehensive exploration of approximate solutions for Volterra integral equations using an interpolation method centered on ramp functions, one can consult ref. [24].

In ref. [25], the role of smooth ramp functions on the activation of network interpolation operators was examined.

In ref. [26], quadratic programming by the aid of ramp functions and fast online Quadratic Programming–Model Predictive Control (QP-MPC) solutions was performed.

In ref. [27], a new and prominent implementation of the simplex method for solving linear programming problems was developed. In addition, its application for solving MPC problems on the basis of ramp functions was described.

In ref. [28], an implementation of the ramp function to a fundamental problem of fracture mechanics concerning multi-cracked simply supported beams was carried out, where the determination of the response of the beams was addressed under static loads and in the presence of multiple cracks, whilst in ref. [29], a substructure elimination method for evaluating the bending vibration of beams was performed. In this valuable work, a vibration analysis method was presented on the basis of the substructure elimination method for a general class of Bernoulli–Euler beams. Here, discontinuities were treated by the use of the Heaviside step function, whereas the non-smooth points were approached by means of the ramp function. In fact, referring to Euler–Bernoulli beams and Timoshenko beams, many remarkable investigations can be found in literature, where singularity functions, such as the unit step function and the ramp function, have been taken into consideration to carry out analytical treatments of discontinuity problems.

In refs. [30–32], the jump discontinuities on Euler–Bernoulli beams and Timoshenko beams were analytically treated by the aid of singularity functions, whereas in refs. [33,34], some basic concepts of the well–known Timoshenko beam theory were revisited and discussed in depth.

In ref. [35], a considerable study on the dynamics of viscoelastic discontinuous beams was accomplished. This investigation dealt with the dynamics of beams with an arbitrary number of Kelvin–Voigt viscoelastic rotational joints, translational supports, and attached, lumped masses.

In ref. [36], the Heaviside step function was implemented to approximate the discontinuities in Euler–Bernoulli discontinuous beams, where the analytical solution was finally carried out by means of uniform-beam Green's functions.

In ref. [37], an analytical treatment for Euler–Bernoulli vibrating discontinuous elastic beams was carried out. Heaviside step function and Dirac's delta distribution (also known as the unit impulse) were taken into account towards the analytical approach of the beam discontinuities.

In ref. [38], a remarkable study concerning the achievement of closed-form solutions for stochastic Euler–Bernoulli discontinuous beams was carried out, whilst in ref. [39], a valuable theoretical investigation on a general category of Euler–Bernoulli simply supported discontinuous beams was performed.

In ref. [40], a Euler–Bernoulli-like finite element method (FEM) for a general class of Timoshenko beams was presented and discussed, whereas in ref. [41], an exact stochastic solution for a general class of linear elastic beams subjected to delta-correlated loads was accomplished. In addition, in ref. [42], a considerable analytical study on the effects of axial load and thermal heating on the dynamic characteristics of axially moving Timoshenko beams was presented. On the other hand, there are many other engineering problems where the ramp function (along with other singularity functions) is involved and indeed plays a key role. For instance, the wavemaker problem is a fundamental and important issue in the study of marine and coastal engineering. In this context, in ref. [43], the transient waves were generated by a vertical, flexible wavemaker plate by means of a general ramp function. Moreover, in ref. [44], a numerical modeling framework based on complex analysis meshless (meshfree) methods, which can accurately and efficiently track arbitrary crack paths in two-dimensional linear elastic solids, was introduced and discussed.

In particular, the ramp function was applied in compatibility conditions in order to warrant that the deformations will leave the elastic continuum body in a compatible state. In ref. [45], a mathematical approach resulting in an implicit representation aiming at the analysis of piecewise affine discrete-time systems was carried out. In particular, a new framework was presented towards the stability analysis of discrete-time piecewise affine (PWA) systems. To this end, a novel implicit representation of PWA functions was introduced on the basis of ramp functions. Next, by exploiting some properties of ramp functions as a set of identities and inequalities, the authors obtained Lyapunov inequalities related to piecewise quadratic Lyapunov functions candidates. In ref. [46] the problem of assessing the stability of the origin of uncertain PWA systems was treated. In this framework, the authors extended the use of an implicit representation based on vector-valued ramp functions to continuous-time PWA systems. The main advantage of this method is that the robust stability analysis can be performed for the case where the uncertainties modify both the shape of the partition and the number of regions. In ref. [47], a mathematical treatment of the harmonic oscillator, which was considered to consist of a step function and a ramp function, was performed in a rigorous and straightforward manner. In ref. [48], a remarkable investigation on the use of multi-zone modeling for tunnel fires was carried out. In this approach, the fire and the ventilation conditions were specified by the aid of ramp functions. In ref. [49], an extended phase-field approach was performed to warrant the efficiency of the simulation of fatigue fracture processes. In this valuable work, the ramp function played a key role, since it participated in the mathematical derivations concerning the proposed interpolation method, which was adopted in order to avoid discontinuities. In ref. [50], a sound and effective finite element modeling (FEM) strategy, in order to determine the directivity of a thermoelastically generated laser ultrasound, was performed. Here, a smooth ramp function was applied to the free surface displacement in the time domain, to suppress the initial displacement, without introducing extra bulk waves by keeping the first derivatives of the displacements continuous.

Now, in the present study, which constitutes a theoretical investigation on this special function, the ramp function is exhibited as the pointwise limit of a sequence of real and continuous functions with pointwise convergence. This limit is proved to be zero for strictly negative values of the real variable x, whereas it is proved to be simply x for strictly positive values of x. To signify the novelty of the mathematical formula introduced in the current approach when compared to other analytical expressions of the ramp function existing in literature, let us remark that if this function is derived by the aid of the formula obtained in ref. [14], the singularity structure will be left ambiguous. This fact constitutes an advantage of the proposed formula in comparison to this approach. Also, another dominance of the formula proposed in the present work, when compared with formulae consisting of finite combinations of inverse trigonometric functions, is that the latter do not have unique definitions. In addition, one may elucidate that the formula performed here holds over the set $(-\infty, 0) \cup [0, +\infty)$, in opposition to the analytical representation of the ramp function obtained in ref. [13], which cannot be defined at $x = 0$. On the other hand, one may point out that the closed-form expression of the Heaviside step function obtained in ref. [20] cannot be measured, either in the Lebesgue sense or Riman's sense. Hence, it cannot be differentiated with respect to the variable x, and therefore, one cannot obtain Dirac's delta function on the basis of this formula. Obviously, the same conclusion holds if one tries to derive the ramp function by the aid of this formula, a fact that renders the proposed expression for the ramp function more flexible.

Further, one may also emphasize that the proposed exact formula is not expressed in terms of miscellaneous special functions, (elliptic integrals, etc.), a fact that may render this formula much more practical and helpful in the computational procedures which are inserted into digital signal processing techniques, along with other engineering practices.

2. Towards an Explicit Form of Ramp Function

2.1. Theorem

Let R, R^+, and N denote the sets of real numbers, positive real numbers, and positive integers, respectively. Further, let $x \in R$ and n ∈ N. The following single-valued function f: $R \to R^+$, with

$$f(x) = \lim_{n\to+\infty}\left(\frac{x\cdot n^{|x|+1} + ln\left(n^2\cdot(|x|+2n)\right)}{2n^{|x|+1}} + \frac{x\cdot(exp(n\cdot x)-1)}{2exp(n\cdot x)+2}\right) \tag{6}$$

coincides with the ramp function over the set $(-\infty, 0) \cup [0, +\infty)$.

2.2. Proof

In this subsection, we will give rigorous proof to the theorem formulated in Section 2.1. In particular, we will show that the values of the single-valued function f introduced by Equation (6) vanish for strictly negative arguments, whilst they coincide with the values of the real variable x for strictly positive arguments, as well as at $x = 0$. To this end, let us distinguish the following three cases concerning the independent variable x.

(i) $x \in (0, +\infty)$

In this context, one may deduce that

$$\lim_{n\to+\infty}(n\cdot x) = +\infty \Rightarrow \lim_{n\to+\infty} exp\ (n\cdot x) = +\infty \tag{7}$$

and therefore,

$$\lim_{n\to+\infty}\frac{1}{exp\ (n\cdot x)} = 0 \tag{8}$$

Next, to calculate the infinitesimal quantity $\lim_{n\to+\infty}\frac{x\cdot n^{|x|+1}+ln\left(n^2\cdot(|x|+2n)\right)}{2n^{|x|+1}}$, one may primarily observe that the above fraction, which the limiting operation is applied to, can be equivalently expanded as follows:

$$\begin{gathered}\frac{x\cdot n^{|x|+1}+ln\left(n^2\cdot(|x|+2n)\right)}{2n^{|x|+1}} = \frac{x}{2} + \frac{ln\ (n)}{n^{|x|+1}} + \frac{ln\ (|x|+2n)}{2n^{|x|+1}} \Leftrightarrow \\ \frac{x\cdot n^{|x|+1}+ln\left(n^2\cdot(|x|+2n)\right)}{2n^{|x|+1}} = \frac{x}{2} + \frac{ln\ (n)}{n^{|x|+1}} + \frac{ln\ (|x|+2n)}{2n^{|x|+1}} \Leftrightarrow \\ \frac{x\cdot n^{|x|+1}+ln\left(n^2\cdot(|x|+2n)\right)}{2n^{|x|+1}} = \frac{x}{2} + \frac{ln\ (n)}{n^{|x|+1}} + \frac{ln\ (n)}{2n^{|x|+1}} + \frac{ln\left(\frac{|x|}{n}+2\right)}{2n^{|x|+1}}\end{gathered} \tag{9}$$

Now, since the exponent $(|x|+1)$, which appears upon the denominator of the fractions appearing on both sides of Equation (9), is always a strictly positive quantity, i.e., $|x|+1 > 0\ \forall\ x \in (-\infty, 0) \cup [\,0, +\infty)$, one may also deduce that

$$\lim_{n\to+\infty}\frac{ln\ (n)}{n^{|x|+1}} = 0 \tag{10}$$

In addition, since the positive term $|x|$ does not vary with respect to the integer variable n, which is a natural number, as it was clarified beforehand, the quotient $\frac{|x|}{n}$ vanishes, letting n tend to infinity. In fact, it is known from calculus [51] that every sequence in the general form a(n) = $\frac{c}{n}$, where c denotes an arbitrary constant real number, is always a convergent sequence. In this framework, one may also infer:

$$\lim_{n\to+\infty}\frac{ln\left(\frac{|x|}{n}+2\right)}{2n^{|x|+1}} = 0 \tag{11}$$

Hence, one obtains

$$\lim_{n\to+\infty}\frac{x\cdot n^{|x|+1}+ln\left(n^2\cdot(|x|+2n)\right)}{2n^{|x|+1}}=\frac{x}{2} \tag{12}$$

Equation (6) can be combined with Equations (7), (8) and (12) respectively, to yield

$$f(x)=\frac{x}{2}+\frac{1}{2}\lim_{n\to+\infty}\frac{x-\frac{x}{exp\ (n\cdot x)}}{1+\frac{1}{exp\ (n\cdot x)}} \tag{13}$$

Moreover, since the real variable x does not vary with respect to the integer variable n, one may also deduce that

$$\lim_{n\to+\infty}\frac{x-\frac{x}{exp\ (n\cdot x)}}{1+\frac{1}{exp\ (n\cdot x)}}=x \tag{14}$$

Then, Equation (13) can be combined with Equation (14) to yield

$$\begin{gathered}f(x)=\tfrac{x}{2}+\tfrac{1}{2}\cdot\tfrac{x-0}{1+0}\ \Rightarrow\\ f(x)=x\end{gathered} \tag{15}$$

Thus, it was proved that for strictly positive arguments, i.e., when $x\in(0,+\infty)$, the function f in Equation (6) returns the value that was used as its argument, unchanged. Hence, $f(x)$ is simply x.

(ii) $x\in(-\infty,0)$

In this context, since the real variable x and the natural number n (which evidently is an integer variable) do not agree in sign, their product is strictly negative. Thus, one may deduce that

$$\lim_{n\to+\infty}(n\cdot x)=-\infty\ \Rightarrow\ \lim_{n\to+\infty}exp\ (n\cdot x)=0 \tag{16}$$

Here, one may also pinpoint that Equation (12), which was previously derived when we considered the variable x to be strictly positive, still holds. This significant observation is attributed to a fact that we will interpret and discuss just below. By focusing on the fraction $\frac{x\cdot n^{|x|+1}+ln\left(n^2\cdot(|x|+2n)\right)}{2n^{|x|+1}}$, which appears on Equation (12), one may pinpoint that the real variable x occurs in the denominator of this fraction only by its absolute value, i.e., $|x|$.

Thus, the sign of this real variable cannot influence the sign of the limit of the denominator in this aforementioned fraction, i.e., the term $2n^{|x|+1}$, letting the integer variable n tend to infinity.

In fact, the limit of the real quantity $2n^{|x|+1}$, letting n tend to infinity, is always equal to $+\infty$, i.e., it constitutes an infinitesimal quantity as well, regardless of the sign and the values of the real variable x, even if the variable x equals zero.

Further, Equation (6) can be combined with Equations (12) and (16), respectively, to yield

$$\begin{gathered}(x)=\tfrac{x}{2}+\tfrac{1}{2}\cdot\frac{\lim\limits_{n\to+\infty}(x\cdot exp(n\cdot x)-x)}{\lim\limits_{n\to+\infty}(exp(n\cdot x)+1)}\Rightarrow\\ f(x)=\tfrac{x}{2}+\tfrac{1}{2}\cdot\frac{x\cdot\lim\limits_{n\to+\infty}(exp(n\cdot x))-x}{\lim\limits_{n\to+\infty}(exp(n\cdot x))+1}\end{gathered} \tag{17}$$

At this point, we should elucidate that we have taken into account the fact that the real variable x does not vary with respect to the integer variable n. In this framework, the variable x was able to be pulled out of the limiting operation, letting n tend to infinity.

Thus, on the basis of Equation (17), it implies that

$$\begin{gathered}f(x)=\tfrac{x}{2}+\tfrac{1}{2}\cdot\tfrac{0\cdot x-x}{0+1}\ \Rightarrow\\ f(x)=\tfrac{x}{2}-\tfrac{x}{2}=0\end{gathered} \tag{18}$$

Hence, it was rigorously proved that the values of $f(x)$ introduced in Equation (6) vanish for strictly negative arguments, i.e., when the real variable $x \in (-\infty, 0)$. In addition, we have to emphasize that Equation (12), which was derived in the first case of the problem we studied, i.e., when $x \in (0, +\infty)$, is always valid over the set of real numbers $(-\infty, 0) \cup [\,0, +\infty)$, as we have previously shown. Moreover, we have also taken into consideration the fact that the real variable x and the integer variable n are always independent of each other. In this context, the real variable x can be pulled out of the limiting operations, letting n tend to infinity, since it can be roughly said that this variable practically behaves as a real constant during the limiting operation.

(iii) $x = 0$

In this ultimate case, one may deduce that

$$\lim_{n\to+\infty} exp\,(n\cdot x) = \lim_{n\to+\infty} exp\,(n\cdot 0) \tag{19}$$

and therefore,

$$\lim_{n\to+\infty} exp\,(n\cdot x) = \lim_{n\to+\infty} exp\,(0) = 1 \tag{20}$$

Moreover, one may notice that Equation (12) still holds.

Now, Equation (6) can be combined with Equations (20) and (12), respectively, to yield

$$\begin{gathered}(x) = \lim_{n\to+\infty}\left(\frac{ln\,\left(2n^3\right)}{2n} + \frac{0}{2+2}\right) \Rightarrow \\ f(x) = \lim_{n\to+\infty}\left(\frac{ln\,\left(2n^3\right)}{2n}\right) + \frac{0}{4}\end{gathered} \tag{21}$$

and therefore,

$$f(x) = 0 + \frac{0}{4} = 0 \tag{22}$$

Thus, it was proved that the value of $f(x)$ vanishes at $x = 0$ as the ramp function also does. After all, one may come to the conclusion that the single-valued real function introduced by Equation (6) is definitely identical to the ramp function over the set of real numbers.

3. Discussion

In Section 2, a mathematical analysis approach concerning the ramp function was carried out. Specifically, in Section 2.1, an explicit form of the ramp function was proposed as the limit of a sequence of real functions, letting n tend to infinity. Next, in Section 2.2, this limit was rigorously proved to be zero for strictly negative values of the real variable x, whereas it was proved to be simply x for strictly positive values of x. In fact, the proposed single-valued function coincides with the ramp function over the set $(-\infty, 0) \cup (0, +\infty)$. Moreover, one may also observe that the single-valued function f introduced by Equation (6) vanishes at $x = 0$. Indeed, the ramp function (by its definition) vanishes at $x = 0$ as well, since it is just x for positive arguments. In this framework, one may also conclude that the single-valued function introduced by Equation (6) coincides with the ramp function over the set of real numbers $(-\infty, 0) \cup [\,0, +\infty)$.

In addition, by focusing on the infinitesimal quantity $\lim_{n\to+\infty}\left(\frac{x\cdot(exp(n\cdot x)-1)}{2exp(n\cdot x)+2}\right)$, which can be equivalently written as: $\frac{1}{2}\lim_{n\to+\infty}\left(\frac{x\cdot(exp(n\cdot x)-1)}{exp(n\cdot x)+1}\right)$, one may observe that the following relationship holds:

$$\lim_{n\to+\infty}\left(\frac{x\cdot(exp(n\cdot x)-1)}{exp(n\cdot x)+1}\right) = |x| \tag{23}$$

Consequently, on the basis of Equation (23), it implies that the infinitesimal quantity $\lim_{n\to+\infty}\left(\frac{x\cdot(exp(n\cdot x)-1)}{exp(n\cdot x)+1}\right)$ coincides with the absolute value of the real variable x over the set

$(-\infty, 0) \cup (0, +\infty)$ and also at $x = 0$, since it is obvious that the numerator of the above fraction vanishes at x = 0, whereas its denominator is equal to 2.

Actually, the validity of Equation (23) $\forall x \in R$ is attributed to the fact that the value of the infinitesimal quantity $\lim_{n \to +\infty} exp\ (n \cdot x)$ depends on the sign of the real variable x, which is definitely independent of the integer variable n.

In this context, the infinitesimal quantity $\lim_{n \to +\infty} \left(\frac{x \cdot (exp(n \cdot x) - 1)}{exp(n \cdot x) + 1} \right)$ equals x for strictly positive arguments, whilst it equals $-x$ for strictly negative arguments and finally vanishes at $x = 0$.

4. Conclusions

The objective of this theoretical investigation was to introduce an analytical representation of the ramp function, which evidently is a very useful mathematical tool, and indeed, it participates in many areas of applied and engineering mathematics and physics. The novelty of this work, when compared to other analytical treatments of this significant function, is that the proposed exact mathematical formula is not exhibited in terms of any miscellaneous special functions or any other special functions, such as error function, hyperbolic function, orthogonal polynomials, etc. This fact may render the proposed formula more practical and helpful in the computational processes concerning digital signal processing techniques and other engineering practices.

Nevertheless, one may also observe that an advantage of the proposed closed-form expression of this special function is that it coincides with the ramp function over the set $(-\infty, 0) \cup [\,0, +\infty)$, since it was proved to vanish at $x = 0$, as the ramp function also does.

In closing, as a future work, one may also propose similar analytical representations to other singularity functions [52], e.g., Heaviside step function, signum function etc., by taking into consideration this theoretical approach.

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