



Proceeding Paper

# Introducing Quantum Mechanics in High Schools: A Proposal Based on Heisenberg's *Umdeutung*<sup>†</sup>

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**Abstract:** Teaching and learning QM at high school as well as the undergraduate level is a highly non-trivial task. Indeed, major changes are required in understanding the new physical reality, and students have to deal with counterintuitive concepts such as uncertainty and entanglement as well as advanced mathematical tools. In order to overcome these critical issues, a simple approach is presented here, which is based solely on two-vector and  $2 \times 2$  matrix algebra. Furthermore, it could also enable educational institutions to fill the gap between high school curricula and the current scientific and technological advances in physics by allowing students to gain some insight into topics such as qubits and quantum computers. The inspiration behind our proposal as well as its firm theoretical foundation are based on the famous *Umdeutung* (reinterpretation) paper by W. Heisenberg, which introduces QM in matrix form.

**Keywords:** quantum mechanics; physics teaching; two-level systems; history of physics



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## 1. Introduction

At high school as well as the undergraduate level, Quantum Mechanics (QM) is usually introduced through an overview of the main crucial experiments and theoretical attempts which took place at the beginning of the 20th century. Although retracing the historical path that led to the introduction of this conceptual and mathematical framework has undoubted advantages, there are also significant drawbacks, mainly in contexts such as a high school, where the students' lack of advanced mathematical tools have placed severe constraints on their understanding of quantum concepts.

On the other hand, QM implies major changes in understanding the world and physical reality. Introducing concepts such as probability, uncertainty, and superposition, and discussing issues such as non-locality and entanglement are highly non-trivial tasks. Students have to deal with a matter that is counterintuitive and in conflict with the classical view of the physical world [1]. In this respect, high school students' difficulties in accepting nondeterminism have recently been recognized [2] to induce a return to classical reasoning and a subsequent misunderstanding of the concept of quantum states. Last but not least, the introduction of wave functions and the Schrodinger equation, even for the simplest paradigmatic examples of the infinite square well and the harmonic oscillator, implies the solution of second-order ordinary differential equations, which are usually beyond high school students' standard knowledge of calculus.

All the above considerations have led us to think that a better strategy would be to focus our attention on two-level systems, which live in a finite dimensional Hilbert space. That allows us to introduce, from the very beginning, a simple  $2 \times 2$  matrix formulation of QM, where quantum states are identified with two vectors belonging to a finite vector space and where observables are  $2 \times 2$  matrices. In this way, students have

the possibility to become familiar with the unique conceptual issues of QM such as the superposition principle, non-locality, and entanglement without the need for an advanced mathematical background. That could also fill the gap between high school curricula and the current scientific and technological advances in physics by allowing students to have a glimpse at modern research topics. In fact, two-level systems, also known as qubits, are the basic building blocks of quantum information and computation. Furthermore, our proposal could be envisaged as a useful supplement to game-based and simulation teaching strategies [3,4].

In light of the above considerations, two-level systems seem to be the ideal candidates for the introduction of QM to advanced high school students. However, the other side of the coin is that two-level systems are more tightly linked with the notion of an operator acting on a state and that of eigenvalue equations. Thus, teachers have to justify the formalism; in particular, they should be able to explain why observables are operators (matrices) acting on states, and finally, why measured quantities are identified with the eigenvalues of such matrices. Our idea was then to look for a motivation in the history of physics. In this specific case, we found a firm basis for our proposal in Heisenberg's seminal paper from 1925 [5], the so-called *Umdeutung* paper, which was the first to recognize the role of matrices in quantum physics. The aim of Heisenberg's paper was to establish quantum theory by building only on observable quantities. In fact, he provided the calculational rules for computing transition frequencies between stationary states, without reference to the unobservable characteristics of such states. These rules were later identified by Born and Jordan [6,7] as matrix operations, with matrices being representations of operators based on eigenvectors of the Hamiltonian [8].

The advantages of Heisenberg's approach in undergraduate teaching have already been put forward by some authors [9–11], but the possibility of using it for advanced high-school teaching has not been explored yet. Our work aims to fill this gap. The stage for studying two-level systems may be set by following Heisenberg's line of reasoning. A mandatory prerequisite could be the basic historical introduction to quantum physics, which includes standard topics such as Planck's hypothesis, the photoelectric effect, and Bohr's model of the hydrogen atom [12]. A detailed analysis of Heisenberg's paper [5] is beyond the scope of this work but one of its inspiring points, which we assume as a starting point for our proposal, is Bohr's postulate that the frequencies of emitted radiation are proportional to the energy differences between two stationary states and not to the orbital frequencies of the electrons (as within classical physics). As a consequence, in QM we deal with physical quantities, which depend on two states rather than one. This naturally leads to the introduction of matrices.

The net result of our study is a novel teaching–learning sequence on QM, properly designed for advanced high school students and also very useful for in-service and pre-service teacher training.

## 2. Introducing Operators (Matrices): Basic Steps

In this section, we establish the theoretical basis for our proposal by recalling Bohr's postulate, subsequently describing Heisenberg's key ideas in a form suitable to high school students. We stress main logical steps and discuss a simple example: a harmonic oscillator. By further simplifying, we are led to a toy model with only two levels, a ground state and an excited one, described in terms of two vectors and  $2 \times 2$  matrices.

### 2.1. Bohr's Atomic Model

As is well known, one of Bohr's postulates [12] includes the hypothesis that an atom can be in one of a series of stationary states, each of which corresponds to a discrete value of energy. While an electron is in one of these states, its energy does not vary; however, it can radiate by going from one given state to another with lower energy, according to the fundamental relation:

$$\nu_{mn} = \frac{E_m - E_n}{h}, \quad E_m > E_n. \quad (1)$$

Thus, frequencies are guaranteed to obey the Rydberg–Ritz combination principle:

$$\nu_{mn} = \nu_{mp} + \nu_{pn} , \quad (2)$$

which is experimentally observed. This notation associates to each frequency two indices, one for the starting state of the electron and the other for the arriving state.

## 2.2. Heisenberg's Original Argument

Two main ideas led Heisenberg to matrix mechanics [5]. First, the recognition that at the atomic scale, classical mechanics is no longer valid. Second, that the correspondence principle must be valid; in fact, in Heisenberg's approach, each quantum equation has a corresponding classical formula.

The starting point is the consideration that in the quantum realm, only transitions between states are observable; hence, physical quantities should be associated with two states rather than one, and thus have two indices. Consider any dynamical quantity  $x(t)$ , which could be the position of a particle; consider then its Fourier representation (for simplicity we limit ourselves to the case in which  $x$  is periodic; in general, all the sums appearing in the following formulas are to be replaced with integrals):

$$x_n(t) = \sum_{j=-\infty}^{+\infty} a_j e^{ij\omega t} . \quad (3)$$

If  $x$  referred to a single quantum state, for instance a Bohr orbit, the Fourier coefficients  $a_j$  would depend on the corresponding quantum number, i.e.,  $a_j = a_j(n)$ , with  $\omega = \omega(n) = 2\pi\nu(n)$  being the corresponding angular frequency. However, single quantum states are not observable; only transition processes associated with two states are. Hence, Heisenberg replaces  $a_j(n) = a(n, n-j)$  and  $\omega(n) = \omega(n, n-j)$  with  $\omega(n, n-j) = 2\pi\nu_{n,n-j}$ , and the resulting expression is:

$$x(t) = \sum_{j=-\infty}^{+\infty} a(n, n-j) e^{ij\omega(n,n-j)t} . \quad (4)$$

This is the first “reinterpretation of kinematical relations” by Heisenberg and is suggested by the correspondence principle, according to which quantities related to quantum jumps between two states coincide with quantities related to single states in the limit of large quantum numbers. The next step is to represent products of dynamical quantities in the same way, which is required, for instance, to write down energies. Hence, by generalizing the convolution theorem for Fourier series and transforms, Heisenberg argues that the most natural assumption is to represent the square of  $x(t)$  as

$$x^2(t) = \sum_{k=-\infty}^{+\infty} b(n, n-k) e^{ik\omega(n,n-k)t} , \quad (5)$$

where, according to the combination principle,  $\omega(n, n-k) = \omega(n, n-j) + \omega(n-j, n-k)$ , and

$$b(n, n-k) = \sum_{j=-\infty}^{+\infty} a(n, n-j) a(n-j, n-k) . \quad (6)$$

This is Heisenberg's rule for multiplying transition amplitudes.

In his paper, Heisenberg proceeds by showing how to find transition amplitudes and frequencies from the dynamics of the system. In particular, he links the coefficients  $a(n, n-j)$  to Kramers' dispersion formula [13–15]—his main inspiration, together with Born's generalization to general systems [16]—which describes the interaction of an atom

with electromagnetic radiation, and to Planck's constant  $h$  through a reformulation of Sommerfeld's quantization condition. Thus he was able to formulate the following relation:

$$h = 4\pi m \sum_{j=0}^{\infty} \left\{ |a(n+j, n)|^2 \omega(n+j, n) - |a(n, n-j)|^2 \omega(n, n-j) \right\}, \quad (7)$$

which is nothing but the Thomas–Reiche–Kuhn sum rule [17,18]. He then applies his formalism to a simple system, the anharmonic oscillator, where he could determine the amplitudes and the frequencies by finding and solving some recursion relations which are satisfied by them. The quantities  $a(n, n-j)$  and  $\omega(n, n-j)$  were recognized by Born [6] to be elements of (infinite-dimensional) matrices, since Equation (6) is nothing but the row by column product of a matrix, with the elements  $a_{nm}$  with itself. Thus, physical quantities in the Heisenberg scheme as reformulated in [6] correspond to infinite matrices. Moreover, in the same paper [6], Born and Jordan recognize that the allowed energies for a quantum system are given by the diagonal elements of the matrix representing the Hamiltonian, namely, its eigenvalues.

### 2.3. A Simple Example: Harmonic Oscillator

At this point, the simple case of a harmonic oscillator [19] is briefly discussed in order to show Heisenberg's scheme at work, through a concrete example that could quite easily be digested by advanced high school students.

In general, the problem can be rephrased in the following way: given a conservative force  $F(x)$  that binds the electron in an atom, find the quantum mechanical properties, frequencies  $\omega_{nm}$ , and amplitudes  $a_{nm}$  associated with the transitions between stationary states. For a simple harmonic oscillator, the force is  $F(x) = -kx$ , so that the solution to the equation of motion  $F(x) = m\ddot{x}$  is:

$$x(t) = a \cos \omega_0 t, \quad (8)$$

where  $a$  is the fundamental amplitude and  $\omega_0 = \sqrt{k/m}$  is the frequency. From Sommerfeld's quantization condition, one easily obtains  $ma^2\omega_0\pi = nh$ , which gives the allowed values of the vibration amplitude:

$$a(n) = \sqrt{\frac{2\hbar n}{m\omega_0}}, \quad (9)$$

while frequency is independent on  $n$ , i.e.,  $\omega(n) = \omega_0$ . We now substitute amplitudes  $a(n)$  in the classical energy function  $E = \frac{1}{2}m\omega_0^2 a^2$  and obtain the quantum energy spectrum:

$$E_n = n\hbar\omega_0. \quad (10)$$

There is a single Fourier term (see Equation (8)), so that only transitions between adjacent states,  $n \rightarrow n-1$ , are allowed. Finally, the correspondence principle allows one to compute the radiation frequency and the transition amplitudes from the expressions for  $a(n)$  and  $\omega(n)$  obtained above:

$$\omega_{n,n-1} = \omega_0, \quad a_{n,n-1} = \sqrt{\frac{2\hbar n}{m\omega_0}}. \quad (11)$$

These quantities, as Born and Jordan pointed out [6], are identified with elements of matrices.

The simple procedure shown here provides a strong motivation for identifying operators that act on a Hilbert space of quantum states with matrices. At this stage, one is naturally led to introduce, for the sake of simplicity, a toy model—a system built of only

two levels, i.e., a ground state and an excited state, and whose observables are described by  $2 \times 2$  matrices acting on two-component vectors.

### 3. Results: Playing with Two-Level Systems

In the previous section, we laid the foundations for our teaching–learning sequence. The basic pillar of our proposal is the quantum two-state system, which may be introduced by making explicit reference to concrete physical examples (e.g., a single spin and a measurement apparatus or the polarization of a photon). Then, by taking the single-spin system, an identification has to be made between the corresponding space of states and a two-dimensional vector space. This allows one to choose the two basis vectors  $|u\rangle$  and  $|d\rangle$  as two-component column vectors and to construct a general state as the vector, which is a linear superposition of  $|u\rangle$  and  $|d\rangle$ . The single quantum spin is an example of a large class of simple systems called qubits. A qubit is the basic building block of quantum information and computation, in much the same way as the bit—a binary variable that constitutes the smallest piece of information—is the fundamental brick in classical information theory and current computer science.

The subsequent step deals with the introduction of physical observables, which are the object of measurement and are conveniently identified as  $2 \times 2$  matrices within the same vector space. Within the single-spin system, the matrix formed by spin components can be simply derived and identified with Pauli matrices. Subsequently, average values of observables can be easily computed as well as their eigenvalues and eigenvectors.

The representation of quantum states and observables as vectors and matrices of a two-dimensional vector space provides us with the simple machinery upon which peculiar quantum mechanical features can be built up, such as mixed and entangled states in composite systems.

It is now an easier task to illustrate the uncertainty principle by making reference to two different components of the spin, as it applies to many pairs of measurable quantities and not only to position and momentum. In this way, the deep meaning of a quantum measurement process and its differences with the classical case can be pointed out. The next step is to show how to combine single spins to get composite systems. This amounts to the introduction of non-locality issues, quantum correlations, and entanglement, allowing one to gain some insight into unique quantum features.

### 4. Discussion

Our teaching–learning sequence has been implemented within various training activities in QM, held in several high schools in southern Italy, and geared towards teachers in physics and mathematics as well as towards selected students attending the last year of a scientific high school. Preliminary results gathered through interviews as well as surveys filled in by participants both before and after the activities led us to the following considerations.

First, teachers and students have the possibility to become familiar with quantum issues such as entanglement and non-locality without an advanced mathematical background, which is, indeed, an advantage.

Second, it is possible to fill the gap between high school curricula and the current scientific and technological advances in physics (e.g., qubits, quantum computer, quantum teleportation). This could lead to an increase in the number of students who may choose to undertake scientific university programs.

Third, the proposal is also suitable for pre-service and in-service training programs for physics teachers. Indeed, teachers seem to be much more interested in learning basic principles and practical teaching strategies than in deepening their knowledge of formalism.

Finally, a promising strategy in physics education is the possibility of shaping teaching–learning proposals by relying on the historical path that led to a concept, as well as on the social and philosophical contexts in which the concept itself developed. In this way,

significant changes in teachers' and students' conceptions regarding the Nature of Science can be expected.

## 5. Conclusions

A novel strategy is presented here, which allows for the introduction of peculiar features of QM at the high school level without resorting to advanced mathematical tools. This non-trivial task has been accomplished by building upon vectors and matrices in a two-dimensional vector space. The inspiration behind our proposal, as well as its firm theoretical foundation, is the 1925 seminal paper by W. Heisenberg. This work provides a simple calculational method to deal with quantum mechanical states and observables, based on the identification of the physical quantities of interest with transition frequencies and amplitudes, and that indeed such frequencies and amplitudes indeed form matrices. Preliminary results gathered among both high school teachers and students are encouraging and offer useful insights for further improvements.

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## Abbreviations

QM Quantum Mechanics

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