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Article

Generalized Hamilton's Principle for Inelastic Bodies Within Non-Equilibrium Thermodynamics

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Abstract: Within the thermodynamic framework with internal variables, the classical Hamilton's principle for elastic bodies is extended to inelastic bodies composed of materials whose free energy densities are point functions of internal variables, or the so-termed Green-inelastic bodies, subject to finite deformation and non-conservative external forces. Yet this general result holds true even without the Green-inelasticity presumption under a more general interpretation of the infinitesimal internal rearrangement. Three special cases are discussed following the generalized form: (a) the Green-elastic bodies whose free energy can be identified with the strain energy; (b) the Green-inelastic bodies composed of materials compliant with the additive decomposition of strain; and (c) the Green-inelastic bodies undergoing isothermal relaxation processes where the thermodynamic forces conjugate to internal variables, or the so-termed internal forces prove to be potential forces. This paper can be viewed as an extension of Yang *et al.* [1].

Keywords: Hamilton's principle; inelastic body; internal variable; internal force

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1. Introduction

Hamilton's principle, in connection with its generalized forms, is one of the most effective approaches in studying various non-linear systems subject to dynamic actions. It usually takes a brief and neat expression with clear and definite physical meanings pertaining to work and energy.

The efforts to extend and apply Hamilton's principle to continuum cast back to one and half century ago, as indicated by Truesdell and Toupin [2] and Biot [3], and it continues to attract a lot of interest. More frequently, it is introduced as a starting point in the studies of elastic media or structures, e.g., Toupin [4] on elastic dielectrics, Mindlin [5] on linear elasticity with micro-structure, Batra [6] on thermo-elastic fluids and solids, Simo *et al.* [7] on the Hamiltonian structure of non-linear elasticity, Grinfeld and Norris [8] on fluid-permeable elastic continua and Ma *et al.* [9] on microstructure-dependent Timoshenko beams.

Additionally, efforts have been made trying to establish generalized forms of Hamilton's principle for the inelastic continuum [1,10–14]. These studies share the common background of non-equilibrium thermodynamics where the vast phenomena of inelasticity are abstracted in a general way known as the internal variables. As remarked by Rice [15], one of the initiators of such concept, this theory is general enough to encompass relevant material behaviors while so simple that the essential content is kept clear. Stolz [10,11] proposed the functional approach in nonlinear dynamics where the notions of Lagrange equations and, for the case of conservative systems, Hamilton's principle, were established for inelastic continuum and shockwaves, as reviewed by Germain [12]. But in these earlier studies, the internal variables were not treated as generalized coordinates like the displacements and temperature and some preconditions in thermodynamics were not sufficiently stressed. Recent representative studies include Sievers and Anthony [13] and Anthony [14], mainly on complex fluid and heat conduction and Yang *et al.* [1] on Green-inelastic solids.

This paper can be viewed as an extension of Yang *et al.* [1] in four aspects: (a) the most general form of the Hamilton's principle for Green-inelastic bodies is given and the classical form for those undergoing relaxation processes, as focused by Yang [1], is handled as a special case; (b) it is expounded that the general form holds true even for non-Green-inelasticity under a more general interpretation of the infinitesimal internal rearrangement; (c) Green-inelastic bodies compliant with the additive decomposition of strain is investigated as a special case; and (d) the entire problem is discussed under lager deformation situations.

The paper is arranged as follows: Sections 2 and 3 introduce the essential backgrounds of large deformation continuum mechanics and thermodynamics with internal variables, respectively. In Section 4, the generalized Hamilton's principle for inelastic bodies is derived from the principle of virtual work, and in Section 5 three significant special cases are investigated.

2. Prerequisites in Large Deformation Continuum Mechanics

In this section, we simply formulate the prerequisites of our primary concerns in large (or finite) deformation continuum mechanics following the monograph of Truesdell and Toupin [2], where the integrated background is available.

2.1. Work Conjugate Strain and Stress in Lagrange Formulism

Consider a body with volume V_0 and mass density ρ_0 in the reference configuration and volume V and mass density ρ in the displaced or current configuration. To describe finite deformation, let $\boldsymbol{X} = X_I E_I$ denote a material point where X_I and E_I are Lagrangian coordinates and basis, and let

$\boldsymbol{x} = x_i e_i$ denote a point in the current configuration where x_i and e_i are Eulerian coordinates and basis.

The motion of the body is represented by the functional relation $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t)$. When $\boldsymbol{\chi}$ is continuously differentiable with respect to ***X***, the second-rank tensor $\boldsymbol{F}(\boldsymbol{X}, t)$ is defined, namely the deformation gradient:

$$\boldsymbol{F} = \partial \boldsymbol{\chi} / \partial \boldsymbol{X} \tag{1}$$

The inverse of $\boldsymbol{F}(\boldsymbol{X}, t)$ exists if the material points in the neighborhood of ***X*** are in one-to-one correspondence with their current positions:

$$\boldsymbol{F}^{-1}(\boldsymbol{X}, t) = \partial \boldsymbol{X} / \partial \boldsymbol{x} \big|_{x=\boldsymbol{\chi}(X,t)} \tag{2}$$

The Lagrange strain tensor, denoted ***E***, objective and symmetric, measuring finite deformation from an arbitrary reference configuration, can be derived:

$$\boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{F}^{\mathrm{T}} \cdot \boldsymbol{F} - \boldsymbol{I}\right) \tag{3}$$

where the superscript T denotes transposition and ***I*** is the second-rank unit tensor.

In the Eulerian description, the deformation power is defined as:

$$P_{\mathrm{d}} = \int_V \boldsymbol{T} : \boldsymbol{D} \mathrm{d}V \tag{4}$$

where ***T*** is the real (or Cauchy) stress and ***D*** the deformation-rate tensor given by $\boldsymbol{D} = \left(\boldsymbol{L} + \boldsymbol{L}^{\mathrm{T}}\right)/2$, in which ***L*** is an Eulerian tensor denoting velocity gradient, $\boldsymbol{L} = \partial \boldsymbol{v} / \partial \boldsymbol{x}$, with $\boldsymbol{v}(\boldsymbol{x}, t)$ being the velocity field in Eulerian coordinates. Two Lagrangian expressions for P_{d} can be shown:

$$P_{\mathrm{d}} = \int_{V_0} \bar{\boldsymbol{T}} : \dot{\boldsymbol{F}} \mathrm{d}V_0 = \int_{V_0} \boldsymbol{S} : \dot{\boldsymbol{E}} \mathrm{d}V_0 \tag{5}$$

where $\bar{\boldsymbol{T}}$ is known as the Piola stress:

$$\bar{\boldsymbol{T}} = J\boldsymbol{T} \cdot \left(\boldsymbol{F}^{-1}\right)^{\mathrm{T}} \tag{6}$$

and ***S*** the Kirchhoff stress:

$$\boldsymbol{S} = J\boldsymbol{F}^{-1} \cdot \boldsymbol{T} \cdot \left(\boldsymbol{F}^{-1}\right)^{\mathrm{T}} \tag{7}$$

with J denoting the Jacobian determinant, $J = \det \boldsymbol{F}$. In the sense of Equation (5), ***S*** is termed by Hill [16] the conjugate stress of the Lagrange strain ***E***, or ***S*** and ***E*** are work conjugates.

2.2. The Principle of Virtual Work in Lagrange Formulism

The Lagrangian equations of motion is given by:

$$\bar{T}_{iJ,J} + \rho_0 b_i = \rho_0 a_i \tag{8}$$

where ***a*** and ***b*** denote respectively acceleration and the prescribed body force per unit mass.

Consider two configurations $\boldsymbol{\chi}$ and $\boldsymbol{\chi} + \delta \boldsymbol{u}$, where $\delta \boldsymbol{u}$ is a kinematically admissible virtual displacement field, *i.e.*, $\delta \boldsymbol{u} = \boldsymbol{0}$ on S_u. (S is the boundary of V with ***n*** being the unit normal vector, and

$S = S_u + S_t$ such that $\boldsymbol{u} = \boldsymbol{u}^p$ on S_u and $\boldsymbol{n}\cdot\boldsymbol{T} = \boldsymbol{t}^{\mathrm{p}}$ on S_t, where $\boldsymbol{u}^p$ and $\boldsymbol{t}^{\mathrm{p}}$ are the prescribed displacement and surface traction per unit area.) The virtual strain field $\delta\boldsymbol{E}$ can be shown due to the definition of $\boldsymbol{E}$:

$$\delta E_{IJ} = \frac{1}{2}\left(F_{kI}\delta u_{k,J} + F_{kJ}\delta u_{k,I}\right) \tag{9}$$

Multiplying the left-hand side of Equation (8) by δu_i, integrating over the reference volume V_0 and applying the divergence theorem, we obtain one Lagrangian form of the principle of virtual work:

$$\int_{V_0} \boldsymbol{S} : \delta\boldsymbol{E}\,\mathrm{d}V_0 = \int_{V_0} \rho_0\left(\boldsymbol{b} - \boldsymbol{a}\right)\cdot\delta\boldsymbol{u}\,\mathrm{d}V_0 + \int_{S_{0t}} \overline{\boldsymbol{t}}^{\,\mathrm{p}}\cdot\delta\boldsymbol{u}\,\mathrm{d}S_0 \tag{10}$$

where $\overline{\boldsymbol{t}}^{\,\mathrm{p}}$ is the prescribed surface traction per unit reference area with $\overline{t}_i = n_{0J}\overline{T}_{iJ}$.

The principle of virtual work Equation (10), which almost epitomizes the integrated boundary value problem for large deformation bodies except for the constitutive relations, *i.e.*, in our case the relation of $\boldsymbol{S}$ and $\boldsymbol{E}$ discussed in the following section, serves as the starting point to find the generalized Hamilton's principle for inelastic bodies.

3. Thermodynamics with Internal Variables

The internal-variable theory of inelastic continua described by Rice [15] belongs to the domain of thermodynamics of irreversible processes, or non-equilibrium thermodynamics. A collection of discrete scalar internal variables are introduced to represent the extent of micro-structural rearrangement within the material sample with macroscopically homogeneous stress or strain and temperature. Thus, entropy and temperature may be defined at a non-equilibrium state associated with a fictitious "constrained equilibrium state" at which the internal variables are fixed someway on their actual values at the non-equilibrium state. It follows that the existence of a free-energy density can be assumed from which the entropy density and the stress can be derived. Some detailed and overall results on internal variable and constrained equilibrium state have been obtained by Kestin and Rice [17], Germain, Nguyen and Suquet [18], Maugin and Muschik [19], Chaboche [20,21] and Yang *et al.* [22,23] and others.

Consider a material sample with mass density ρ_0 and volume V_0 measured in a reference state and at a reference temperature ϑ_0. Its thermodynamic state can be fully described by the state variables $\left(\boldsymbol{E},\vartheta,\boldsymbol{\xi}\right)$, or alternatively $\left(\boldsymbol{S},\vartheta,\boldsymbol{\xi}\right)$. Our problem is discussed in the state space denoted $\sum\left(\boldsymbol{E},\vartheta,\boldsymbol{\xi}\right)$, where $\boldsymbol{E}$ is the Lagrange strain, ϑ temperature and $\boldsymbol{\xi}$ a set of scalar internal variables, $\boldsymbol{\xi} = \{\xi_1,\cdots,\xi_n\}$, describing the specific local rearrangements within the material sample.

3.1. Green-Inelastic Material

Inelastic materials whose free energy densities are point functions of internal variables were termed Green-inelastic materials by Yang *et al.* [1], which constitute Green-inelastic bodies. For a green-inelastic material sample, the specific free energy density ϕ in $\sum\left(\boldsymbol{E},\vartheta,\boldsymbol{\xi}\right)$ is introduced:

$$\phi = \phi\left(\boldsymbol{E},\vartheta,\boldsymbol{\xi}\right) \tag{11}$$

Neighboring constrained equilibrium states corresponding to different sets of internal variables are related by the following Gibbs equation:

$$\boldsymbol{S}:\delta\boldsymbol{E}-\rho_0\eta\delta\vartheta-\delta\omega=\rho_0\delta\phi \tag{12}$$

where $\boldsymbol{S}$ denotes the Kirchhoff stress:

$$\boldsymbol{S}=\rho_0\frac{\partial\phi}{\partial\boldsymbol{E}} \tag{13}$$

and η the specific entropy density:

$$\eta=-\frac{\partial\phi}{\partial\vartheta} \tag{14}$$

and $\delta\omega$ the variational dissipation work per unit reference volume:

$$\delta\omega=\boldsymbol{f}\cdot\delta\xi \tag{15}$$

where $\boldsymbol{f}$ denotes the set of thermodynamic forces conjugate to internal variables $\boldsymbol{\xi}$, $\boldsymbol{f}=\{f_1,\cdots,f_n\}$, which are termed internal forces as compared with external forces:

$$\boldsymbol{f}=-\rho_0\frac{\partial\phi}{\partial\xi} \tag{16}$$

Non-equilibrium thermodynamics extends the thermostatic relation (12) to dynamic processes with δ replaced by $\mathrm{d}/\mathrm{d}t$ where *t* denotes time:

$$\boldsymbol{S}:\dot{\boldsymbol{E}}-\rho_0\eta\dot{\vartheta}-\boldsymbol{f}\cdot\dot{\boldsymbol{\xi}}=\rho_0\dot{\phi} \tag{17}$$

It follows Equation (17) that the local Clausius-Duhem inequality becomes:

$$\mathrm{Div}\left(\bar{\boldsymbol{h}}\vartheta^{-1}\right)+\vartheta^{-1}\boldsymbol{f}\cdot\dot{\boldsymbol{\xi}}\geq 0 \tag{18}$$

where Div denotes the divergence operator with respect to Lagrangian coordinates, and $\bar{\boldsymbol{h}}=J\boldsymbol{F}^{-1}\cdot\boldsymbol{h}$ is the Lagrangian heat-flux vector and $\boldsymbol{h}$ the Eulerian heat-flux vector. This inequality must be obeyed in any process and at any state, particularly when the temperature gradient vanishes. Thus, the dissipation inequality, or Kelvin inequality should be imposed on the evolution of internal variables:

$$\boldsymbol{f}\cdot\dot{\boldsymbol{\xi}}\geq 0 \tag{19}$$

3.2. Non-Green-Inelastic Material

If the presumption of Green-inelastic material is violated in that ϕ is not a point function of $\boldsymbol{\xi}$ but instead depends on their path history, Equation (11) does not stand. Rice [15] took the transformed version of Equation (11), $\phi=\phi\left(\boldsymbol{E},\vartheta,PIR\right)$, where *PIR* symbolically denotes the pattern of internal rearrangement. However, the pattern of Equation (12), packed with Equations (13)–(15), remains *unchanged* under a more general interpretation of $\delta\xi_\alpha$, that at any given pattern of internal rearrangement of the material sample, a set of discrete scalar infinitesimals $\delta\boldsymbol{\xi}=\{\delta\xi_1,\cdots,\delta\xi_n\}$ characterizes all possible infinitesimal variations in internal arrangement. For the change in the pattern of internal rearrangement symbolically denoted δPIR resulting from the imposition of $\delta\boldsymbol{\xi}$, the internal forces $\boldsymbol{f}$ exists but cannot be simply determined by Equation (16). More detailed descriptions on this general case are provided by Hill and Rice [24] and Rice [25].

As shown in the next section, the derivation of the generalized Hamilton's principle is based on the unchanged Equations (12)–(15), so the result holds true even without the presumption of Green-inelasticity. But the altered meanings and characteristics of $\boldsymbol{\xi}$ must be recognized: ξ_α is no longer a state variable and the infinitesimal $\delta\xi_\alpha$ does not necessarily represent an infinitesimal change in ξ_α, although the above interpretation may include this as a special case, as indicated by Rice [15].

4. Generalized Hamilton's Principle for Green-Inelastic Bodies

To discuss Hamilton's principle, it is essential to clarify the system of generalized coordinates first. With the groundwork laid in Sections 2 and 3, a displaced configuration of an inelastic body can be described by the displacement field $\boldsymbol{u}$ and the internal variables $\boldsymbol{\xi}$ on each material point, which builds up the system of generalized coordinates. $\boldsymbol{u}$ and $\boldsymbol{\xi}$ respectively determine the external and internal configuration of the inelastic body and thus are term external and internal generalized coordinates. The idea of viewing the internal variables as generalized coordinates is inspired by Yang [26] who proposed Hamilton's principle of entropy production for creep and relaxation process of the Green-inelastic materials under macroscopically homogeneous and constant stress or strain and temperature.

Submitting Equation (12) into Equation (10) to eliminate $\boldsymbol{S}:\delta\boldsymbol{E}$, and then integrating over an isothermal process ($\delta\vartheta = 0$) with constant time duration $[t_0, t_1]$ leads to:

$$\delta\int_{t_0}^{t_1} \Phi - K\mathrm{d}t + \int_{t_0}^{t_1} \delta A\mathrm{d}t + \int_{t_0}^{t_1} \delta\Omega\mathrm{d}t = 0 \tag{20}$$

where, Φ is the free energy:

$$\Phi = \int_{V_0} \rho_0\phi\mathrm{d}V_0 \tag{21}$$

K is the kinetic energy:

$$K = \int_{V_0} \frac{1}{2}\rho_0\dot{\boldsymbol{u}}\cdot\dot{\boldsymbol{u}}\mathrm{d}V_0 \tag{22}$$

and in Lagrangian description it can be shown that:

$$\delta\int_{t_0}^{t_1} K\mathrm{d}t = \delta\int_{t_0}^{t_1}\int_{V_0} \frac{1}{2}\rho_0\dot{\boldsymbol{u}}\cdot\dot{\boldsymbol{u}}\mathrm{d}V_0\mathrm{d}t = -\int_{t_0}^{t_1}\int_{V_0} \rho_0\boldsymbol{a}\cdot\delta\boldsymbol{u}\mathrm{d}V_0\mathrm{d}t \tag{23}$$

$-\delta A$ is the variational work of the non-conservative external forces:

$$-\delta A = \int_{V_0} \rho_0\boldsymbol{b}\cdot\delta\boldsymbol{u}\mathrm{d}V_0 + \int_{S_{0t}} \bar{\boldsymbol{t}}^{\mathrm{p}}\cdot\delta\boldsymbol{u}\mathrm{d}S_0 \tag{24}$$

$\delta\Omega$ is the variational work done by the internal forces, or the variational dissipation work:

$$\delta\Omega = \int_{V_0} \delta\omega\mathrm{d}V_0 = \int_{V_0} \boldsymbol{f}\cdot\delta\boldsymbol{\xi}\mathrm{d}V_0 \tag{25}$$

If, as is usually assumed, the external forces are conservative, Equation (20) can be rewritten as:

$$\delta\int_{t_0}^{t_1} L\mathrm{d}t + \int_{t_0}^{t_1} \delta\Omega\mathrm{d}t = 0 \tag{26}$$

where L is called the Lagrangian function:

$$L = \Phi - K + A \tag{27}$$

and A is the potential energy of the external forces.

Equations (20) and (26) are the generalized Hamilton's principle for green-inelastic bodies subject to respectively non-conservative and conservative external forces. The results are brief and neat as we supposed the Hamilton's principle to be. It is emphasized again that these results hold true for non-Green inelastic bodies described in Section 3.2, if only the different connotation of $\delta\Omega$ being regarded.

5. Three Significant Special Cases

Now let's evaluate some special cases of certain significance, also to have a better understanding of the generalized form. External forces are assumed conservative in this section just for convenience.

5.1. Green-Elastic Bodies

For green-elastic bodies, the internal variables vanish and every local state is an equilibrium state. The free energy density ϕ reduces to the Helmholtz free energy density $\psi(\boldsymbol{E},T)$ where T is the absolute temperature.

The Hamilton's principle for Green-elastic bodies subject to conservative external forces in isothermal processes is:

$$\delta\int_{t_0}^{t_1} L\mathrm{d}t = \delta\int_{t_0}^{t_1} \Psi - K + A\mathrm{d}t = 0 \tag{28}$$

where $\boldsymbol{\Psi}$ is the Helmholtz free energy:

$$\boldsymbol{\Psi} = \int_{V_0} \rho_0\psi\mathrm{d}V_0 \tag{29}$$

To obtain the classical Hamilton's principle, the small (or infinitesimal) deformation situation is discussed. For small deformation elastic bodies (of which V and V_0, ρ and ρ_0 are not distinguished) in an isothermal process with T at a fixed value, $\boldsymbol{\Psi}$ is usually identified with the strain-energy W:

$$W = \int_V w\mathrm{d}V \tag{30}$$

where w is the strain-energy volume density such that the stress-strain relation is $\boldsymbol{\sigma} = \partial w/\partial\boldsymbol{\varepsilon}$. Then the classical Hamilton's principle is obtained by replacing $\boldsymbol{\Psi}$ with W:

$$\delta\int_{t_0}^{t_1} L\mathrm{d}t = \delta\int_{t_0}^{t_1} W - K + A\mathrm{d}t = 0 \tag{31}$$

It is clarified here, in passing, that in quite a few materials the presumption of linear elasticity, *i.e.*, $\boldsymbol{\sigma} = \boldsymbol{C} : \boldsymbol{\varepsilon}$, is taken in the demonstration of the classical Hamilton's principle, which is, as indicated by Equation (29), not a necessity. The classical Hamilton's principle stands for any non-linear elasticity if and only if the strain-energy exists.

Comparing Equation (31) with Equation (26), we can draw the conclusion that the classical Hamilton's principle for Green-elastic bodies can be extended to Green-inelastic bodies by replacing the strain energy with the specific free energy while considering the dissipation work done by the internal forces.

5.2. Green-Inelastic Bodies with Additive Decomposition of Strain

The additive decomposition of strain into elastic and inelastic parts is frequently assumed in studies of inelastic constitutive relations. It actually comes of a more general theory cored by the multiplicative decomposition in the domain of large deformation kinematics. The multiplicative theory, contributed by Green and Naghdi [27] and Casey [28] and others, unfortunately, has too large an inclusion and too many assumptions to be applied here, so we follow the way of Lubliner [29] and only consider small deformation cases.

The infinitesimal strain $\boldsymbol{\varepsilon}$ is assumed to be decomposable into elastic part depending on stress and temperature and inelastic part depending on internal variables only:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e\left(\boldsymbol{\sigma}, T\right) + \boldsymbol{\varepsilon}^i\left(\boldsymbol{\xi}\right) \tag{32}$$

In large deformation situations, a similar additive decomposition of strain with certain approximations, $\boldsymbol{E} = \boldsymbol{E}^e + \boldsymbol{E}^i$, is proposed by Green and Naghdi [27] and conditioned by Casey [28], which is adopted in the works of Simo and Ortiz [30] and others, though, not in ours. The decomposition Equation (32), as shown by Lubliner [29], accords with the existence of a free-energy density $\psi = \psi\left(\boldsymbol{\varepsilon}, T, \boldsymbol{\xi}\right)$ if and only if ψ can be decomposed as:

$$\psi\left(\boldsymbol{\varepsilon}, T, \boldsymbol{\xi}\right) = \psi^e\left(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^i\left(\boldsymbol{\xi}\right), T\right) + \psi^i\left(\boldsymbol{\xi}, T\right) \tag{33}$$

Applying the Gibbs equation to the elastic part ψ^e leads to:

$$\boldsymbol{\sigma} : \left(\delta\boldsymbol{\varepsilon} - \delta\boldsymbol{\varepsilon}^i\right) + \rho\frac{\partial \psi^e}{\partial T}\delta T = \rho\delta\psi^e \tag{34}$$

where $\delta\boldsymbol{\varepsilon}^i$ can be further shown to be:

$$\delta\boldsymbol{\varepsilon}^i = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\sigma}} \cdot \delta\boldsymbol{\xi} \tag{35}$$

Substituting Equation (35), Equation (34) becomes:

$$\boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon} - \rho\frac{\partial \psi^e}{\partial T}\delta T - \delta\tilde{\omega} = \rho\delta\psi^e \tag{36}$$

where:

$$\delta\tilde{\omega} = \tilde{\boldsymbol{f}} \cdot \delta\xi \text{ and } \tilde{\boldsymbol{f}} = \boldsymbol{\sigma} : \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\sigma}} \tag{37}$$

Likewise, considering isothermal processes, comparing Equation (36) with its general form (12), we obtain the generalized Hamilton's principle for Green-inelastic bodies with additive decomposition of strain:

$$\delta \int_{t_0}^{t_1} \tilde{L} \mathrm{d}t + \int_{t_0}^{t_1} \delta \tilde{\Omega} \mathrm{d}t = 0 \tag{38}$$

where, the Lagrange function $\tilde{L}$ is:

$$\tilde{L} = \Psi^e - K + A \tag{39}$$

with Ψ^e being the elastic part of free energy:

$$\Psi^e = \int_{V_0} \rho_0 \phi^e \mathrm{d}V_0 \tag{40}$$

and the transformed variational dissipation work $\delta\tilde{\Omega}$ is:

$$\delta\tilde{\Omega} = \int_{V_0} \delta\tilde{\omega} \mathrm{d}V_0 = \int_{V_0} \tilde{\boldsymbol{f}} \cdot \delta\boldsymbol{\xi} \mathrm{d}V_0 \tag{41}$$

Compared with the general form (26), the advantage of the current form (38) is obvious that only the elastic part of the free energy is directly involved, which in most cases lies shallower than the inelastic part.

5.3. Green-Inelastic Bodies Undergoing Isothermal Relaxation Processes

In isothermal relaxation processes, with the external state variables, *i.e.*, the strain and temperature fixed at prescribed values, the state space $\sum(\boldsymbol{E}, \vartheta, \boldsymbol{\xi})$ reduces to the relaxation subspace $\sum(\boldsymbol{E}^{\mathrm{p}}, \vartheta^{\mathrm{p}}, \boldsymbol{\xi})$ and the external configuration of the body changes only through rigid motions.

The free energy density becomes $\phi(\boldsymbol{E}^{\mathrm{p}}, \vartheta^{\mathrm{p}}, \boldsymbol{\xi})$ and it is quite visible that a potential energy density, denoted p, exists in the relaxation subspace:

$$p = p(\boldsymbol{E}^{\mathrm{p}}, \vartheta^{\mathrm{p}}, \boldsymbol{\xi}) = \rho_0 \int_0^{\xi} \boldsymbol{f} \cdot \mathrm{d}\boldsymbol{\xi} = -\int_0^{\xi} \frac{\partial \phi}{\partial \boldsymbol{\xi}} \cdot \mathrm{d}\boldsymbol{\xi} \tag{42}$$

and it is easily shown that:

$$\delta p = \rho_0 \boldsymbol{f} \cdot \delta\boldsymbol{\xi} \text{ and } \boldsymbol{f} = \frac{1}{\rho_0} \frac{\partial p}{\partial \boldsymbol{\xi}} \tag{43}$$

Obviously, p serves as the potential energy of the internal forces of the material sample and is thus termed the internal potential energy density. Following the definition (42), the relation between p and ϕ is $\partial p / \partial \boldsymbol{\xi} = -\partial \phi / \partial \boldsymbol{\xi}$ and in general $p \neq -\phi$.

Since the internal forces are potential forces, the dissipation work done by them through the constant time duration $[t_0, t_1]$ can be rewritten as:

$$\int_{t_0}^{t_1} \delta \Omega \mathrm{d}t = \delta \int_{t_0}^{t_1} P \mathrm{d}t \tag{44}$$

where P is the internal potential energy:

$$P = \int_{V_0} \rho_0 p \mathrm{d}V_0 \tag{45}$$

Submitting Equation (44) back to Equation (26), we obtain the generalized Hamilton's principle for Green-inelastic bodies undergoing isothermal relaxation processes:

$$\delta\int_{t_0}^{t_1} L\mathrm{d}t = \delta\int_{t_0}^{t_1} \Phi - K + A + P\mathrm{d}t = 0 \tag{46}$$

Compared with Equation (31), it takes a similar form to the classical Hamilton's principle for Green-elastic bodies, just by replacing the strain energy with the summation of the corresponding free energy and internal potential energy, as indicated by Yang *et al.* [1].

6. Conclusions

Within the thermodynamic framework with internal variables, the generalized Hamilton's principle for Green-inelastic bodies is established, brief and neat as expected. Compared with the classical Hamilton's principle for Green-elastic bodies, the strain energy is replaced by the specific free energy and the dissipation work done by the internal forces must be involved. The general result holds true even without the Green-inelasticity presumption under a more general interpretation of the infinitesimal internal rearrangement. For Green-inelastic bodies with additive decomposition of strain, only the elastic part of the free energy is directly involved in the generalized Hamilton's principle. For Green-inelastic bodies undergoing isothermal relaxation processes, internal forces prove to be potential forces and the Hamilton's principle takes a classical form.

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