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Article

Classes of *N*-Dimensional Nonlinear Fokker-Planck Equations Associated to Tsallis Entropy

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Abstract: Several previous results valid for one-dimensional nonlinear Fokker-Planck equations are generalized to *N*-dimensions. A general nonlinear *N*-dimensional Fokker-Planck equation is derived directly from a master equation, by considering nonlinearities in the transition rates. Using nonlinear Fokker-Planck equations, the H-theorem is proved; for that, an important relation involving these equations and general entropic forms is introduced. It is shown that due to this relation, classes of nonlinear *N*-dimensional Fokker-Planck equations are connected to a single entropic form. A particular emphasis is given to the class of equations associated to Tsallis entropy, in both cases of the standard, and generalized definitions for the internal energy.

Keywords: nonlinear Fokker-Planck equations; H-theorem; nonextensive thermostatistics

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1. Introduction

The Boltzmann-Gibbs (BG) theory of statistical mechanics represents one of the most successful theoretical frameworks of physics [1,2]. The proposal of different types of statistical ensembles, and their equivalence in the thermodynamic limit, makes it appropriate for a description of a wide variety of many-particle physical situations. These statistical ensembles can be defined in a very elegant manner, by the introduction of a probability density $P(\vec{x},t)$, associated with the occurrence of a given physical quantity $\vec{x}$ at a time t. Particularly, in the present paper, $\vec{x}$ will denote the position of a particle in a N-dimensional space; then, one can define the BG entropy,

$$S_{\rm BG}[P] = -k_{\rm B} \int d^N x \; P(\vec{x},t) \ln P(\vec{x},t) \tag{1}$$

where $k_{\rm B}$ is the Boltzmann constant, and the integral

$$\int d^N x \equiv \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \cdots \int_{-\infty}^{\infty} dx_N \cdots \tag{2}$$

will represent herein an integration over all possible positions of the particle [3]. By maximizing $S_{\rm BG}[P]$ with respect to $\{P(\vec{x},t)\}$ under certain constraints, one obtains the equilibrium distribution $P_{\rm eq}(\vec{x})$ associated to the different statistical ensembles (see, e.g., [1]). The BG theory is based on a fundamental assumption, namely, the ergodic hypothesis; only if ergodicity holds is that one can replace a given time average by the corresponding average over a statistical ensemble. Additionally, the fact that all thermodynamic quantities are either intensive or extensive is directly related to the short-range character of the interactions among the microscopic constituents of the system. Considering the constraint

$$\int d^N x \; P(\vec{x},t) = 1 \tag{3}$$

the maximization of $S_{\rm BG}[P]$ leads to the equilibrium microcanonical distribution. The same procedure carried by imposing Equation (3), as well as an additional constraint for the total energy,

$$U = \int d^N x \; E(\vec{x}) P(\vec{x},t) \tag{4}$$

yields the usual Boltzmann weight of the equilibrium canonical distribution.

However, the applicability of the BG theory becomes questionable for systems that violate the ergodic hypothesis, which may occur in systems characterized by long-range and/or competing interactions. The nonextensive statistical mechanics [4,5] appears nowadays as a possible alternative for describing physical situations where the BG theory fails. The former emerged from the introduction of the entropic form [6],

$$S_q[P] = k \, \frac{1 - \int d^N x \; [P(\vec{x},t)]^q}{q-1} \qquad (q \in \Re) \tag{5}$$

where k represents a constant with units of entropy, and the entropic index q is responsible for deviations from BG; notice that $S_q[P] \to S_{\rm BG}[P]$, when $q \to 1$.

The maximization of $S_q[P]$ under the constraints in Equations (3) and (4) yields the stationary distribution [4],

$$P_{\rm st}^{(1)}(\vec{x}) = B^{(1)} [1 - \beta^{(1)}(q-1) E(\vec{x})]_+^{1/(q-1)} \tag{6}$$

where $[y]_+ = y$, for $y > 0$ and zero otherwise. As usual, $B^{(1)}$ represents a normalization factor, whereas $\beta^{(1)}$ is the Lagrange multiplier associated with the energy constraint [Equation (4)]. However, if one uses Equation (3) and instead of Equation (4), one considers a generalized definition for the internal energy [4,7],

$$U = \int d^N x \; E(\vec{x}) [P(\vec{x},t)]^q \tag{7}$$

one obtains the Tsallis distribution,

$$P_{\rm st}^{(2)}(\vec{x}) = B^{(2)}[1 - \beta^{(2)}(1-q)E(\vec{x})]_{+}^{1/(1-q)} \tag{8}$$

One should notice that the distributions in Equations (6) and (8) present a "duality" property, $q \to (2-q)$, which appears frequently in nonextensive statistical mechanics. A third proposal was introduced in terms of the escort distribution [8],

$$U = \frac{\int d^N x \; E(\vec{x})[P(\vec{x},t)]^q}{\int d^N x \; [P(\vec{x},t)]^q} \tag{9}$$

yielding a stationary distribution similar to the one in Equation (8), with both q and the corresponding Lagrange multiplier $\beta^{(3)}$ redefined in terms of the quantities appearing in Equation (8) [4,7–9]. Therefore, the three proposals above for the extremization of $S_q[P]$ are equivalent to one another, by a proper redefinition of its parameters.

The linear differential equations in physics are, in many cases, valid for media characterized by specific conditions, like homogeneity, isotropy, and translational invariance, with particles interacting through short-range forces and with a dynamical behavior characterized by short-time memories. One of the most important equations of non-equilibrium statistical mechanics is the linear Fokker-Planck equation (FPE); its time-dependent solution, for an external harmonic potential, is given by a Gaussian distribution [2]. However, many real systems—specially the ones within the realm of complex systems—do not fulfill these requirements, e.g., those characterized by spatial disorder and/or long-range interactions. In some of these cases, the associated equations have to be modified, and very frequently, nonlinear terms are considered in order to take into account such effects. Nowadays the study of nonlinear differential equations has gained a lot of interest and a considerable advance in this area has been attained essentially due to latest advances in computer technology.

In the recent years the linear FPE has been modified in a way to introduce nonlinear terms, so that the study of nonlinear Fokker-Planck equations (NLFPEs) [10] became an area of a wide interest, particularly due to their applications in many physical phenomena, like those related to anomalous diffusion [11]. These type of phenomena may be found in the motion of particles in porous media [12–15], the dynamics of surface growth [15], the diffusion of polymer-like breakable micelles [16], the dynamics of interacting vortices in disordered superconductors [17,18], and the motion of overdamped particles through narrow channels [19], among others. An interesting aspect about the Tsallis distribution is that it appears also as a a stable solution of a NLFPE; in its simplest, one-dimensional form, this equation is given by [20–22]

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial [A(x)P(x,t)]}{\partial x} + D\mu \frac{\partial}{\partial x}\left\{[P(x,t)]^{\mu-1}\frac{\partial P(x,t)}{\partial x}\right\} \tag{10}$$

where the external force $A(x)$ is associated with a potential $\phi(x)$ $[A(x) = -d\phi(x)/dx]$ and μ is a real number [notice that Equation (10) recovers the linear FPE in the limit $\mu \to 1$]. The stationary solution of Equation (10), for an arbitrary confining potential $\phi(x)$, is given by Equation (6) for $\mu = q$, or by Equation (8) for $\mu = 2-q$ [considering in both cases, $E(\vec{x}) \equiv \phi(x)$] [20]. However, for an external force $A(x) = k_1 - k_2 x$ (k_1 and k_2 constants, $k_2 \geq 0$), and the initial condition $P(x,0) = \delta(x)$, the time-dependent solution of Equation (10) is given by a q-Gaussian distribution [20,21],

$$P(x,t) = B(t)\{1 - \beta(t)(1-q)[x - x_0(t)]^2\}_{+}^{1/(1-q)} \tag{11}$$

in the case $\mu = 2 - q$. In the equation above, $x_0(t)$ is related to the average value of $P(x,t)$, following the same behavior as for $q = 1$ [21], whereas for the preservation of the norm at all times, the time dependent quantities $B(t)$ and $\beta(t)$ should obey,

$$\frac{\beta(t)}{\beta(0)} = \left[\frac{B(t)}{B(0)}\right]^2 \tag{12}$$

For $x_0(t) = 0$, considering times much smaller than those necessary for an approach to the stationary state, one gets a time evolution completely dominated by the diffusion term, which gives $\langle x^2 \rangle \propto t^{2/(q+1)}$, leading to an anomalous diffusion for any $q \neq 1$ [23]. In the limit $t \to \infty$ one approaches a stable stationary state characterized by $P_{\rm st}(x)$, in which the time-dependent parameters of Equation (12) take their corresponding stationary values, $\beta(t) \to \beta^*$ and $B(t) \to B^*$.

A typical general NLFPE can be written in the following form [10,23–31],

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial \{A(x)\Psi[P(x,t)]\}}{\partial x} + \frac{\partial}{\partial x}\left\{\Omega[P(x,t)]\frac{\partial P(x,t)}{\partial x}\right\} \tag{13}$$

which recovers Equation (10) if $\Psi[P(x,t)] = P(x,t)$ and $\Omega[P(x,t)] = D\mu[P(x,t)]^{\mu-1}$. In Equation (13) the functionals $\Psi[P(x,t)]$ and $\Omega[P(x,t)]$ should satisfy certain mathematical requirements, like positiveness, integrability, and differentiability (at least once) with respect to the probability distribution $P(x,t)$ [29,30]. Moreover, in order to preserve the probability normalization for all times, one should impose the probability distribution, together with its first derivative, as well as the product $A(x)\Psi[P(x,t)]$, to be zero in the limit $x \to \pm\infty$,

$$P(x,t)|_{x\to\pm\infty} = 0\ ; \quad \left.\frac{\partial P(x,t)}{\partial x}\right|_{x\to\pm\infty} = 0\ ; \quad A(x)\Psi[P(x,t)]|_{x\to\pm\infty} = 0 \quad (\forall t) \tag{14}$$

The H-theorem represents one of the most important results of nonequilibrium statistical mechanics, providing a well-defined sign for the time-derivative of the free-energy (or the entropy), allowing for an approach to an equilibrium state. In BG statistical mechanics one may prove the H-theorem by making use of the linear FPE [2]; generalizations of this procedure for NLFPEs have called the attention of many workers in the recent years [10,23,25–30,32,33]. The H-theorem in the case of a system subject to an external potential $\phi(x)$ corresponds to a well-defined sign for the time derivative of the free-energy functional,

$$F = U - \gamma S\ ; \qquad U = \int_{-\infty}^{\infty} dx\ \phi(x)P(x,t) \tag{15}$$

where γ represents a positive Lagrange multiplier. The entropy may be considered in the general form,

$$S[P] = \int_{-\infty}^{\infty} dx\ g[P(x,t)]\ ; \quad g(0) = g(1) = 0\ ; \quad \frac{d^2 g}{dP^2} \leq 0 \tag{16}$$

with the condition that $g[P(x,t)]$ should be at least twice differentiable. One should remind that the entropy may be further formulated in more general forms than the definition above, e.g., by means of a functional of $S[P]$ [29], or even taking into account the possibility of a nonlocal diffusion coefficient, as done in [33]; however, since herein our main purpose is to discuss properties related to NLFPEs associated with Tsallis entropy, we will restrict ourselves to Equation (16).

Imposing a well-defined sign for the time derivative of the free energy (which was taken as $(dF/dt) \leq 0$ in [23,29,30]), and making use of the NLFPE of Equation (13), one gets the relation

$$-\gamma \frac{d^2 g[P]}{dP^2} = \frac{\Omega[P]}{\Psi[P]} \tag{17}$$

involving the functionals $\Omega[P]$ and $\Psi[P]$ of the FPE and the entropy defined in Equation (16). Examples of some entropic forms known in the literature, and their associated FPEs were worked out in [10,24,29–31].

As a simple illustration, one can see easily that the NLFPE of Equation (10) in the case $\mu = q$, for which $\Psi[P(x,t)] = P(x,t)$ and $\Omega[P(x,t)] = Dq[P(x,t)]^{q-1}$, is related to Tsallis entropy. Substituting these quantities in Equation (17), integrating and using the conditions of Equation (16), one obtains [23,29],

$$g_q[P] = k\, \frac{P(x,t) - [P(x,t)]^q}{q-1} \tag{18}$$

where k is defined through $D = k\gamma$ in the present formalism. Then, Tsallis entropy is recovered by $S_q[P] = \int_{-\infty}^{\infty} dx\; g_q[P(x,t)]$.

The one-dimensional NLFPE of Equation (10) has been generalized to N-dimensions [34,35],

$$\frac{\partial P(\vec{x},t)}{\partial t} = -\vec{\nabla} \cdot [\vec{A}(\vec{x}) P(\vec{x},t)] + D\mu \vec{\nabla} \cdot \{[P(\vec{x},t)]^{\mu-1}\, \vec{\nabla} P(\vec{x},t)\} \tag{19}$$

with the solution of Equation (11) turning into a N-dimensional q-Gaussian distribution,

$$P(\vec{x},t) = B(t)\{1 - \beta(t)(1-q)[\vec{x} - \vec{x}_0(t)]^2\}_+^{1/(1-q)} \tag{20}$$

for $\mu = 2 - q$ and the external force given by $\vec{A}(\vec{x}) = \vec{k}_1 - k_2\vec{x}$ ($k_2 \geq 0$). One should mention that for the N-dimensional q-Gaussian distribution above there are two typical values of q which depend on N, namely, the value of q below which the distribution is normalizable, $q_{\max} = (2+N)/N$ and the one at which the second moment diverges, $q_c = (4+N)/(2+N)$ [36].

In this paper we extend some well known results of one-dimensional NLFPEs to N dimensions, as described below. In Section II we derive N-dimensional NLFPEs from a master equation, following the same lines of [37,38]. In Section III we prove the H-theorem by making use of N-dimensional NLFPEs, deriving relations involving terms of the corresponding NLFPE and the entropic form, using both standard [*cf.* Equation (4)], and generalized definitions for the internal energy. In this later case, it is shown that the corresponding NLFPEs have to be modified accordingly. In Section IV we give an special emphasis the class of NLFPEs associated with Tsallis entropy and in particular, to those modified due to generalized definitions of the internal energy. Finally, in Section V we present our conclusions.

2. From Master Equation to Fokker-Planck Equation

The one-dimensional NLFPEs in Equations (10) and (13) may be derived directly from a master equation, by introducing nonlinear terms in the corresponding transition probabilities, as done in [29,32,37–39]. Herein we follow closely the procedure used in [29,37,38] in order to derive the N-dimensional FPE of Equation (19).

Let us then consider a system described in terms of discrete N-dimensional stochastic variables, for which $P(\vec{n},t) \equiv P(n_1,n_2,\cdots,n_N,t)$ represents the probability for finding it in a state characterized by the variable $\vec{n}$ at time t. The corresponding master equation is given by

$$\frac{\partial P(\vec{n},t)}{\partial t} = \sum_{\vec{m}}[P(\vec{m},t)w_{\vec{m},\vec{n}}(t) - P(\vec{n},t)w_{\vec{n},\vec{m}}(t)] \tag{21}$$

where

$$\sum_{\vec{m}} \ldots \equiv \sum_{m_1,m_2,\cdots,m_N=-\infty}^{\infty} \ldots$$

and $w_{\vec{k},\vec{l}}(t)$ represents the transition probability rate from state $\vec{k}$ to state $\vec{l}$ (*i.e.*, $w_{\vec{k},\vec{l}}(t)\delta t$ is the probability for a transition from state $\vec{k}$ to state $\vec{l}$ to occur during the time interval $t \to t+\delta t$). For a random walk in an isotropic space, *i.e.*, the same step size Δ for all directions, one can write the master equation above as

$$\frac{\partial P(\vec{n}\Delta,t)}{\partial t} = \sum_{\vec{m}}[P(\vec{m}\Delta,t)w_{\vec{m},\vec{n}}(\Delta,t) - P(\vec{n}\Delta,t)w_{\vec{n},\vec{m}}(\Delta,t)] \tag{22}$$

In [37] the following ansatz for the transition rate was introduced,

$$w_{k,l}(\Delta,t) = -\frac{1}{\Delta}\delta_{k,l+1}A(k\Delta) + \frac{1}{\Delta^2}(\delta_{k,l+1}+\delta_{k,l-1})[aP^{\mu-1}(k\Delta,t) + bP^{\nu-1}(l\Delta,t)] \tag{23}$$

where a and b are constants that may depend, in principle, on the system under consideration, and $A(k\Delta)$ represents an external force. The nonlinear contributions, $P^{\mu-1}(k\Delta,t)$ and $P^{\nu-1}(l\Delta,t)$, correspond to dependences on the probabilities associated to the outgoing and ingoing states, respectively; the motivations for these terms were already presented in [37]. One should remind that this transition rate recovers the one in the usual derivation of the linear FPE [2], either for $(a=D, b=0, \mu=1)$, or $(a+b=D, \mu=\nu=1)$, where D represents the diffusion constant.

In principle, one may modify the ansatz above to a very general form, as done in [29]; however, herein we are mostly interested in N-dimensional NLFPEs associated with Tsallis entropy [like the one of Equation (19)], and so, we shall introduce a slight modification in the force term of Equation (23),

$$w_{k,l}(\Delta,t) = -\frac{1}{\Delta}\delta_{k,l+1}A(k\Delta)\alpha[P(k\Delta,t)] + \frac{1}{\Delta^2}(\delta_{k,l+1}+\delta_{k,l-1})[aP^{\mu-1}(k\Delta,t) + bP^{\nu-1}(l\Delta,t)] \tag{24}$$

where $\alpha[P(k\Delta,t)]$ represents a functional of the probability associated to the outgoing state. The two-dimensional extension of Equation (24) is given by,

$$\begin{aligned} w_{k,l,k',l'}(\Delta,t) = & - \frac{1}{\Delta}\{\delta_{k,k'+1}\delta_{l,l'}A_x(k\Delta,l\Delta) + \delta_{k,k'}\delta_{l,l'+1}A_y(k\Delta,l\Delta)\}\alpha[P(k\Delta,l\Delta,t)] \\ & + \frac{1}{\Delta^2}(\delta_{k,k'+1}\delta_{l,l'} + \delta_{k,k'-1}\delta_{l,l'} + \delta_{k,k'}\delta_{l,l'+1} + \delta_{k,k'}\delta_{l,l'-1}) \\ & \times [aP^{\mu-1}(k\Delta,l\Delta,t) + bP^{\nu-1}(k'\Delta,l'\Delta,t)] \end{aligned} \tag{25}$$

whereas its N-dimensional form may be written as,

$$\begin{aligned} w_{\vec{k},\vec{l}}(\Delta,t) = & -\frac{1}{\Delta}\sum_{i=1}^{N}\prod_{j\neq i}^{N}\delta_{k_j,l_j}\delta_{k_i,l_i+1}A_i(\vec{k}\Delta)\alpha[P(\vec{k}\Delta,t)] + \frac{1}{\Delta^2}\sum_{i=1}^{N}\prod_{j\neq i}^{N}[\delta_{k_j,l_j}(\delta_{k_i,l_i+1}+\delta_{k_i,l_i-1})] \\ & \times \left[aP^{\mu-1}(\vec{k}\Delta,t) + bP^{\nu-1}(\vec{l}\Delta,t)\right] \end{aligned} \tag{26}$$

Substituting this transition rate in Equation (22) one gets, after carrying out the summations over the states $\vec{m}$,

$$
\begin{aligned}
&\frac{\partial P(\vec{n}\Delta,t)}{\partial t} = \\
&-\frac{1}{\Delta}\sum_{i=1}^{N}\{A_i[\{n_j\Delta\}_{j\neq i},(n_i+1)\Delta]P[\{n_j\Delta\}_{j\neq i},(n_i+1)\Delta,t]\alpha\{P[\{n_j\Delta\}_{j\neq i},(n_i+1)\Delta,t]\} \\
&-A_i(\vec{n}\Delta)P(\vec{n}\Delta,t)\alpha[P(\vec{n}\Delta,t)\alpha]\} \\
&+\frac{a}{\Delta^2}\sum_{i=1}^{N}\{P^{\mu}[\{n_j\Delta\}_{j\neq i},(n_i+1)\Delta,t]+P^{\mu}[\{n_j\Delta\}_{j\neq i},(n_i-1)\Delta,t]\}-\frac{2Na}{\Delta^2}P^{\mu}(\vec{n}\Delta,t) \\
&+\frac{b}{\Delta^2}P^{\nu-1}(\vec{n}\Delta,t)\sum_{i=1}^{N}\{P[\{n_j\Delta\}_{j\neq i},(n_i+1)\Delta,t]+P[\{n_j\Delta\}_{j\neq i},(n_i-1)\Delta,t]\} \\
&-\frac{b}{\Delta^2}P(\vec{n}\Delta,t)\sum_{i=1}^{N}\left\{P^{\nu-1}[\{n_j\Delta\}_{j\neq i},(n_i+1)\Delta,t]+P^{\nu-1}[\{n_j\Delta\}_{j\neq i},(n_i-1)\Delta,t]\right\}
\end{aligned}
\tag{27}
$$

where $\{n_j\Delta\}_{j\neq i} \equiv (n_1\Delta, n_2\Delta, \cdots, n_{i-1}\Delta, n_{i+1}\Delta, \cdots, n_N\Delta)$. Defining $\vec{x} = \vec{n}\Delta$, $x_i = n_i\Delta$, and considering the limit $\Delta \to 0$, one obtains,

$$
\begin{aligned}
\frac{\partial P(\vec{x},t)}{\partial t} &= -\sum_{i=1}^{N}\frac{\partial\{A_i(\vec{x},t)P(\vec{x},t)\alpha[P(\vec{x},t)]\}}{\partial x_i}+a\sum_{i=1}^{N}\frac{\partial^2 P^{\mu}(\vec{x},t)}{\partial x_i^2} \\
&+ bP^{\nu-1}(\vec{x},t)\sum_{i=1}^{N}\frac{\partial^2 P(\vec{x},t)}{\partial x_i^2}-bP(\vec{x},t)\sum_{i=1}^{N}\frac{\partial^2 P^{\nu-1}(\vec{x},t)}{\partial x_i^2}
\end{aligned}
\tag{28}
$$

or yet,

$$
\begin{aligned}
\frac{\partial P(\vec{x},t)}{\partial t} &= -\vec{\nabla}\cdot\{[\vec{A}(\vec{x})P(\vec{x},t)\alpha[P(\vec{x},t)]\}+a\nabla^2 P^{\mu}(\vec{x},t)+bP^{\nu-1}(\vec{x},t)\nabla^2 P(\vec{x},t) \\
&- bP(\vec{x},t)\nabla^2 P^{\nu-1}(\vec{x},t)
\end{aligned}
\tag{29}
$$

One should notice that the equation above corresponds to the N-dimensional generalization of Equation (2.4) in [37]. Now, following a procedure similar to the one of [38], this equation may be rewritten as,

$$
\frac{\partial P(\vec{x},t)}{\partial t} = -\vec{\nabla}\cdot\{\vec{A}(\vec{x})\Psi[P(\vec{x},t)]\}+\vec{\nabla}\cdot\{\Omega[P(\vec{x},t)]\vec{\nabla}P(\vec{x},t)\} \tag{30}
$$

$$
\Omega[P(\vec{x},t)] = a\mu P^{\mu-1}(\vec{x},t)+b(2-\nu)P^{\nu-1}(\vec{x},t) \tag{31}
$$

$$
\Psi[P(\vec{x},t)] = \alpha[P(\vec{x},t)]P(\vec{x},t) \tag{32}
$$

As in the one-dimensional case, the two functionals $\Psi[P(\vec{x},t)]$ and $\Omega[P(\vec{x},t)]$ should be both positive finite quantities, integrable, as well as differentiable (at least once) with respect to $P(\vec{x},t)$; additionally, $\Psi[P(\vec{x},t)]$ should also be monotonically increasing with respect to $P(\vec{x},t)$. Considering $\alpha[P(\vec{x},t)] = 1$, *i.e.*, $\Psi[P(\vec{x},t)] = P(\vec{x},t)$, the equations above correspond to the N-dimensional generalization of Equations (1.5) in [38]; also, they recover Equation (19) of the present paper in the particular cases, $(b = 0, a \to D)$, $(a = 0, \nu \to \mu, b = D\mu/(2-\mu))$, $(\mu = \nu, a\mu + b(2-\mu) = D\mu)$, for which the N-dimensional q-Gaussian distribution of Equation (20) is a solution.

3. H-Theorem

To our knowledge, the first proof of the H-theorem making use of a NLFPE appeared in the literature more than 20 years ago [40]. After that, such proof has been extended for more general NLFPEs by many authors (see, e.g., [10,23,25–30,32,33,41,42]), leading to important relations involving general entropic forms and terms of the NLFPEs. In this section we will prove the H-theorem using the N-dimensional NLFPE derived in the previous section. We do this by using both the standard [*cf.* Equation (4)], and generalized definitions for the internal energy. We follow closely the steps of [29,30].

3.1. Standard Definition of the Internal Energy

Let us consider a system subjected to an external N-dimensional potential $\phi(\vec{x})$ (for which $\vec{A}(\vec{x}) = -\vec{\nabla}\phi(\vec{x})$) with the free-energy functional of Equation (15), such that

$$U = \int d^N x \; \phi(\vec{x}) P(\vec{x},t) \tag{33}$$

whereas the entropy is defined similarly to Equation (16), *i.e.*,

$$S[P] = \int d^N x \; g[P(\vec{x},t)] \; ; \quad g(0) = g(1) = 0 \; ; \quad \frac{d^2 g}{dP^2} \leq 0 \tag{34}$$

The H-theorem for the present system is expressed through a well-defined sign for the time derivative of the free energy, which will be considered herein as $(dF/dt) \leq 0$. This derivative is given by

$$\frac{dF}{dt} = \frac{d}{dt} \int d^N x \; \left(\phi(\vec{x}) P(\vec{x},t) - \gamma g[P] \right) = \int d^N x \; \left(\phi(\vec{x}) - \gamma \frac{dg[P]}{dP} \right) \frac{\partial P(\vec{x},t)}{\partial t} \tag{35}$$

and making use of the NLFPE of Equation (30) for the time derivative of $P(\vec{x},t)$ one gets,

$$\begin{aligned} \frac{dF}{dt} &= \int d^N x \; \left(\phi(\vec{x}) - \gamma \frac{dg[P]}{dP} \right) \\ &\times \left(-\vec{\nabla} \cdot \{ \vec{A}(\vec{x}) \Psi[P(\vec{x},t)] \} + \vec{\nabla} \cdot \{ \Omega[P(\vec{x},t)] \vec{\nabla} P(\vec{x},t) \} \right) \\ &= \int d^N x \left(\phi(\vec{x}) - \gamma \frac{dg[P]}{dP} \right) \\ &\quad \times \sum_{i=1}^{N} \left(-\frac{\partial}{\partial x_i} \{ A_i(\vec{x}) \Psi[P(\vec{x},t)] \} + \frac{\partial}{\partial x_i} \left\{ \Omega[P(\vec{x},t)] \frac{\partial}{\partial x_i} P(\vec{x},t) \right\} \right) \end{aligned} \tag{36}$$

In the equation above one has a sum of integrals, $(dF/dt) = \sum_{i=1}^{N} \mathcal{I}_i$, where

$$\begin{aligned} \mathcal{I}_i &= \int d^{N-1} x \int_{-\infty}^{\infty} dx_i \left(\phi(\vec{x}) - \gamma \frac{dg[P]}{dP} \right) \\ &\times \left(-\frac{\partial}{\partial x_i} \{ A_i(\vec{x}) \Psi[P(\vec{x},t)]) \} + \frac{\partial}{\partial x_i} \left\{ \Omega[P(\vec{x},t)] \frac{\partial}{\partial x_i} P(\vec{x},t) \right\} \right) \end{aligned} \tag{37}$$

Carrying an integration by parts for the integral $\int_{-\infty}^{\infty} dx_i \cdots$, and using

$$P(\vec{x},t)|_{x_i \to \pm\infty} = 0 \; ; \quad \left. \frac{\partial P(\vec{x},t)}{\partial x_i} \right|_{x_i \to \pm\infty} = 0 \; ; \quad A_i(\vec{x}) \Psi[P(\vec{x},t)])|_{x_i \to \pm\infty} = 0 \quad (\forall t) \tag{38}$$

one gets,

$$\mathcal{I}_i = -\int d^{N-1}x \int_{-\infty}^{\infty} dx_i \left(\frac{\partial\phi(\vec{x})}{\partial x_i} - \gamma\frac{d^2g[P]}{dP^2}\frac{\partial P(\vec{x},t)}{\partial x_i}\right) \times \left(\frac{\partial\phi(\vec{x})}{\partial x_i}\Psi[P(\vec{x},t)] + \Omega[P(\vec{x},t)]\frac{\partial P(\vec{x},t)}{\partial x_i}\right) \quad (39)$$

The condition

$$-\gamma\frac{d^2g[P]}{dP^2} = \frac{\Omega[P]}{\Psi[P]} \quad (40)$$

leads to

$$\frac{dF}{dt} = \sum_{i=1}^{N}\mathcal{I}_i = -\sum_{i=1}^{N}\int d^{N-1}x \int_{-\infty}^{\infty} dx_i \Psi[P(\vec{x},t)]\left(\frac{\partial\phi(\vec{x})}{\partial x_i} + \frac{\Omega[P]}{\Psi[P]}\frac{\partial P}{\partial x_i}\right)^2 \leq 0 \quad (41)$$

proving the H-theorem associated with the NLFPE of Equation (30).

3.2. Generalized Definition for the Internal Energy

In Section I, we have introduced three different definitions for the internal energy [*cf.* Equations (4), (7) and (9)]. The distributions obtained from the extremization of $S_q[P]$ under the normalization constraint of Equation (3) and one of these definitions for the internal energy are essentially equivalent, being related to one another by redefining the index q and the associated Lagrange multiplier [4,8,9]. Next we shall prove the H-theorem considering a general definition for the internal energy,

$$U = \int d^N x\phi(\vec{x})\Gamma[P(\vec{x},t)] \quad (42)$$

where $\Gamma[P]$ is a positive, monotonically increasing functional and at least $\Gamma[P] \in C^2$. Particularly, in the definition for the internal energy like the one of Equation (7), one has that $\Gamma[P] = [P(\vec{x},t)]^q$. Notice that in this case the equation above may be rewritten in the form of Equation (33), *i.e.*,

$$U = \int d^N x\tilde{\phi}(\vec{x},t)P(\vec{x},t)\,; \quad \tilde{\phi}(\vec{x},t) \equiv \phi(\vec{x})[P(\vec{x},t)]^{q-1} \quad (43)$$

where $\tilde{\phi}(\vec{x},t)$ represents now a time-dependent "potential". Similarly to the one-dimensional case [29], the NLFPE of Equation (30) needs to be slightly modified; lets us then consider,

$$\frac{\partial P(\vec{x},t)}{\partial t} = \vec{\nabla}\cdot\left(\Psi[P(\vec{x},t)]\vec{\nabla}\left\{\phi(\vec{x})\chi[P(\vec{x},t)]\right\}\right) + \vec{\nabla}\cdot\left\{\Omega[P(\vec{x},t)]\vec{\nabla}P(\vec{x},t)\right\} \quad (44)$$

Proceeding as above,

$$\frac{dF}{dt} = \frac{d}{dt}\int d^N x\left(\phi(\vec{x})\Gamma[P(\vec{x},t)] - \gamma g[P]\right) = \int d^N x\left(\phi(\vec{x})\frac{d\Gamma[P]}{dP} - \gamma\frac{dg[P]}{dP}\right)\frac{\partial P(\vec{x},t)}{\partial t} \quad (45)$$

and using the NLFPE of Equation (44) one gets,

$$\frac{dF}{dt} = \int d^N x\left(\phi(\vec{x})\frac{\partial\Gamma[P]}{\partial P} - \gamma\frac{dg[P]}{dP}\right) \times \left[\vec{\nabla}\cdot\left(\Psi[P(\vec{x},t)]\vec{\nabla}\left\{\phi(\vec{x})\chi[P(\vec{x},t)]\right\}\right) + \vec{\nabla}\cdot\left\{\Omega[P(\vec{x},t)]\vec{\nabla}P(\vec{x},t)\right\}\right] \quad (46)$$

Writing the equation above as $(dF/dt) = \sum_{i=1}^{N} \mathcal{I}_i$, carrying an integration by parts for a given $\mathcal{I}_i$, and using the conditions of Equation (38), one obtains,

$$\begin{aligned} \mathcal{I}_i &= -\int d^{N-1}x \int_{-\infty}^{\infty} dx_i \left[\frac{\partial}{\partial x_i} \left(\phi(\vec{x}) \frac{d\Gamma[P]}{dP} \right) - \gamma \frac{d^2 g[P]}{dP^2} \frac{\partial P(\vec{x},t)}{\partial x_i} \right] \\ &\times \left\{ \Psi[P(\vec{x},t)] \frac{\partial}{\partial x_i} \left(\phi(\vec{x}) \chi[P(\vec{x},t)] \right) + \Omega[P(\vec{x},t)] \frac{\partial P(\vec{x},t)}{\partial x_i} \right\} \end{aligned} \tag{47}$$

If the condition of Equation (40) is still imposed, one readily notices that an additional restriction is necessary, namely,

$$\chi[P(\vec{x},t)] = \frac{d\Gamma[P]}{dP} \tag{48}$$

leading to

$$\mathcal{I}_i = -\int d^{N-1}x \int_{-\infty}^{\infty} dx_i \Psi[P(\vec{x},t)] \left\{ \frac{\partial}{\partial x_i} \left(\phi(\vec{x}) \chi[P(\vec{x},t)] \right) + \frac{\Omega[P(\vec{x},t)]}{\Psi[P(\vec{x},t)]} \frac{\partial P}{\partial x_i} \right\}^2 \leq 0 \tag{49}$$

Therefore a sum of N negative contributions,

$$\frac{dF}{dt} = \sum_{i}^{N} \mathcal{I}_i \leq 0 \tag{50}$$

guarantees the validity of the H-theorem associated with Equation (44).

It should be mentioned that the proof of the H-theorem for general definitions of the internal energy may become a difficult task, as already mentioned in previous works (see, e.g., the final discussion in [33]). The demonstration above represents, to our knowledge, the first one using an internal energy different from the standard definition of Equation (33). For that, the NLFPE of Equation (44) was modified accordingly, taking into account the definition of Equation (42). However, a proof of the H-theorem for the normalized energy definition in Equation (9) represents a hard task, remaining as an open problem up to the moment. Nevertheless, as already mentioned in Section I, since the three proposals [Equations (4), (7), and (9)] are all equivalent to one another, in what concerns the extremization of the entropy $S_q[P]$ [4,7–9], we believe that some type of equivalence may also occur in the proof of the H-theorem. From now on, a special emphasis will be given herein for the internal energy definition with $\Gamma[P] = [P(\vec{x},t)]^q$, which corresponds to one of the possible formulations of nonextensive statistical mechanics.

3.3. MaxEntropy Principle

Supposing a unique equilibrium state, a direct consequence of the H-theorem is that the system should reach equilibrium after a sufficiently long time (see next subsection); in what follows, we will refer to this equilibrium state. An important result concerns the fact that, at equilibrium, the relations of Equations (40) and (48) are fully compatible with the MaxEntropy principle. This can be seen through the definition of the functional,

$$\Im[P(\vec{x},t)] = S[P(\vec{x},t)] + \alpha \left(1 - \int d^N x P(\vec{x},t) \right) + \beta \left(U - \int d^N x \phi(\vec{x}) \Gamma[P(\vec{x},t)] \right) \tag{51}$$

where in the last term we have considered the energy constraint in terms of the general definition of Equation (42). The extremization of this functional yields,

$$\left(\frac{dg[P]}{dP} - \alpha - \beta\phi(\vec{x}) \frac{d\Gamma[P]}{dP} \right) \Bigg|_{P=P_{\rm eq}(\vec{x})} = 0 \tag{52}$$

where $P_{\rm eq}(\vec{x})$ represents the probability distribution at equilibrium.

Now, considering Equation (44) in equilibrium one gets,

$$\vec{\nabla} \left\{ \phi(\vec{x}) \chi[P_{\rm eq}(\vec{x})] \right\} = -\frac{\Omega[P_{\rm eq}(\vec{x})]}{\Psi[P_{\rm eq}(\vec{x})]} \vec{\nabla} P_{\rm eq}(\vec{x}) \tag{53}$$

which holds for each vector component,

$$\frac{\partial}{\partial x_i} \left\{ \phi(\vec{x}) \chi[P_{\rm eq}(\vec{x})] \right\} = -\frac{\Omega[P_{\rm eq}(\vec{x})]}{\Psi[P_{\rm eq}(\vec{x})]} \frac{\partial}{\partial x_i} \left\{ P_{\rm eq}(\vec{x}) \right\} \tag{54}$$

Carrying an integration over x_i,

$$\begin{aligned} \phi(\vec{x}) \chi[P_{\rm eq}(\vec{x})] &= -\int_{x_{i0}}^{x_i} dx_i \, \frac{\Omega[P_{\rm eq}(\vec{x})]}{\Psi[P_{\rm eq}(\vec{x})]} \frac{\partial}{\partial x_i} \left\{ P_{\rm eq}(\vec{x}) \right\} + C \\ &= -\int_{P_{\rm eq}(\vec{x}_0)}^{P_{\rm eq}(\vec{x})} \frac{\Omega[P_{\rm eq}(\vec{x}\,')]}{\Psi[P_{\rm eq}(\vec{x}\,')]} \, dP_{\rm eq}(\vec{x}\,') + C \end{aligned} \tag{55}$$

where C is a constant and x_{i0} represents a reference value for x_i.

Similarly, considering the condition of Equation (40) at equilibrium and integrating,

$$-\gamma \frac{dg[P]}{dP} \Bigg|_{P=P_{\rm eq}(\vec{x})} = \int_{P_{\rm eq}(\vec{x}_0)}^{P_{\rm eq}(\vec{x})} \frac{\Omega[P_{\rm eq}(\vec{x}\,')]}{\Psi[P_{\rm eq}(\vec{x}\,')]} \, dP_{\rm eq}(\vec{x}\,') + C_2 \tag{56}$$

Comparing Equations (55) and (56) one concludes that

$$\gamma \frac{dg[P]}{dP} \Bigg|_{P=P_{\rm eq}(\vec{x})} = \phi(\vec{x}) \chi[P_{\rm eq}(\vec{x})] + C' = \phi(\vec{x}) \frac{d\Gamma[P]}{dP} \Bigg|_{P=P_{\rm eq}(\vec{x})} + C' \tag{57}$$

where C' is another constant, and in the last equality we have used the relation in Equation (48). For the equation above [which comes as a consequence of a condition imposed for the H-theorem to hold, namely, Equation (40)] to be consistent with the MaxEnt principle, one should impose for the Lagrange multiplier β introduced above, $\beta = 1/\gamma$.

3.4. Lower Bound for the Free Energy

In the previous section we have proved the H-theorem, *i.e.*, $(dF/dt) \leq 0$, for both NLFPEs of Equations (30) and (44), which are associated to the energy definitions of Equations (33) and (42), respectively. We have also verified that some conditions are necessary for the H-theorem to be valid, namely Equation (40) in the first case, whereas in the later case one needs, in addition to Equation (40), the extra condition given in Equation (48). An important complementary property required for a functional satisfying the H-theorem is that it should be bounded from below,

$$F(P(\vec{x}, t)) \geq F(P_{\rm eq}(\vec{x})) \quad (\forall t) \tag{58}$$

where $P_{\rm eq}(\vec{x})$ represents the probability distribution at equilibrium. It is worth mentioning that, due to the nonlinearity of our equations, there is a possibility of a multiplicity of equilibrium solutions; herein we assume an important condition for our free-energy functional, *i.e.*, that it should present a unique equilibrium state. Up to the moment, all numerical analyses carried in the one-dimensional case [Equation (10)] are consistent with the supposition of a unique equilibrium solution (*cf.*, e.g., [17,18,23]). This assumption, together with its decrease in time imposed by H-Theorem, will drive the system towards this equilibrium state, in such a way that after a sufficiently long time the system will always reach equilibrium. Below, we shall prove the result of Equation (58) for both definitions of the internal energy given in Equations (33) and (42).

Let us then consider first the internal energy of Equation (33), associated with the NLFPE of Equation (30); the entropy will be defined in its general form of Equation (34). One has that,

$$F(P) - F(P_{\rm eq}) = \int d^N x \, [P(\vec{x},t) - P_{\rm eq}(\vec{x})]\phi(\vec{x}) - \gamma \int d^N x \, \left(g[P(\vec{x},t)] - g[P_{\rm eq}(\vec{x})]\right) \tag{59}$$

In order to study the behavior of $F(P)$ near equilibrium, we shall develop the functional $g[P(\vec{x},t)]$ as a power series of $[P(\vec{x},t) - P_{\rm eq}(\vec{x})]$, up to second order. Carrying out this expansion one gets,

$$\begin{aligned} F(P) - F(P_{\rm eq}) &= \int d^N x \, [P(\vec{x},t) - P_{\rm eq}(\vec{x})] \left(\phi(\vec{x}) - \gamma \left. \frac{dg[P]}{dP} \right|_{P=P_{\rm eq}} \right) \\ &- \frac{\gamma}{2} \int d^N x \, [P(\vec{x},t) - P_{\rm eq}(\vec{x})]^2 \left. \frac{d^2 g[P]}{dP^2} \right|_{P=P_{\rm eq}} + \cdots \end{aligned} \tag{60}$$

Considering Equation (52) in the case $\Gamma[P] = P$ and $\beta = 1/\gamma$, one sees that the first integral vanishes; additionally, using Equation (40) at equilibrium, one gets that

$$F(P) - F(P_{\rm eq}) = \frac{1}{2} \int d^N x \, [P(\vec{x},t) - P_{\rm eq}(\vec{x})]^2 \left(\frac{\Omega[P]}{\Psi[P]} \right)_{P=P_{\rm eq}} \geq 0 \tag{61}$$

which is positive by definition, since the functionals $\Omega[P]$ and $\Psi[P]$ are positive quantities.

Next we repeat the same procedure for the internal energy definition of Equation (42), related to the NLFPE of Equation (44). In this case one has,

$$\begin{aligned} F(P) - F(P_{\rm eq}) &= \int d^N x \, \left(\Gamma[P(\vec{x},t)] - \Gamma[P_{\rm eq}(\vec{x})]\right)\phi(\vec{x}) \\ &- \gamma \int d^N x \, \left(g[P(\vec{x},t)] - g[P_{\rm eq}(\vec{x})]\right) \end{aligned} \tag{62}$$

Now, considering the functionals $g[P(\vec{x},t)]$ and $\Gamma[P(\vec{x},t)]$ near equilibrium, one may expand them in power series of $[P(\vec{x},t) - P_{\rm eq}(\vec{x})]$ up to second order,

$$\begin{aligned} F(P) &- F(P_{\rm eq}) = \int d^N x \, [P(\vec{x},t) - P_{\rm eq}(\vec{x})] \left(\phi(\vec{x}) \left. \frac{d\Gamma[P]}{dP} \right|_{P=P_{\rm eq}} - \gamma \left. \frac{dg[P]}{dP} \right|_{P=P_{\rm eq}} \right) \\ &+ \frac{1}{2} \int d^N x \, [P(\vec{x},t) - P_{\rm eq}(\vec{x})]^2 \left(\phi(\vec{x}) \left. \frac{d^2\Gamma[P]}{dP^2} \right|_{P=P_{\rm eq}} - \gamma \left. \frac{d^2 g[P]}{dP^2} \right|_{P=P_{\rm eq}} \right) \end{aligned} \tag{63}$$

The first integral vanishes due to Equation (52) with $\beta = 1/\gamma$, whereas for the second one, using Equation (40) at equilibrium yields,

$$F(P) - F(P_{\rm eq}) = \frac{1}{2}\int d^N x\, [P(\vec{x},t) - P_{\rm eq}(\vec{x})]^2 \left(\phi(\vec{x})\, \frac{d^2\Gamma[P]}{dP^2}\bigg|_{P=P_{\rm eq}} + \frac{\Omega[P]}{\Psi[P]}\bigg|_{P=P_{\rm eq}} \right) \tag{64}$$

As before, the ratio $\Omega[P]/\Psi[P]$ is positive definite; hence, in order to have the free-energy bounded from below, ensuring the approach to the equilibrium state for sufficiently long times, one must have either $\phi(\vec{x})(d^2\Gamma[P]/dP^2)|_{P=P_{\rm eq}}$ positive, or if not positive, at least $(\Omega[P]/\Psi[P])|_{P=P_{\rm eq}} \geq -\phi(\vec{x})(d^2\Gamma[P]/dP^2)|_{P=P_{\rm eq}}$. Therefore, in the case of the internal energy defined according to Equation (42), the present property for the free energy depends on both external potential $\phi(\vec{x})$ and functional $\Gamma[P]$. In particular, in the example of a harmonic potential, $\phi(\vec{x}) = \frac{1}{2}\alpha x^2$ $(\alpha > 0)$, and $\Gamma[P] = P^q$, *i.e.*, $(d^2\Gamma[P]/dP^2) = q(q-1)P^{q-2}$, which appear very frequently in nonextensive statistical mechanics, one gets $F(P) - F(P_{\rm eq}) \geq 0$, for $q > 1$, whereas one should have $(\Omega[P]/\Psi[P])|_{P=P_{\rm eq}} \geq -\phi(\vec{x})q(q-1)P^{q-2}|_{P=P_{\rm eq}}$ for $q < 1$.

4. The Family of NLFPEs Associated to Tsallis Entropy

In this section we shall restrict ourselves to those NLFPEs associated with Tsallis entropy. As shown above, these connections are provided by the H-theorem, through relations involving quantities of the NLFPEs and entropic forms. In the case of Equation (30), the relevant relation is given in Equation (40), whereas for Equation (44), the relations are Equations (40) and (48). Hence, considering the functional $g_q[P]$ of Equation (18) in Equation (40), one gets,

$$\frac{\Omega[P]}{\Psi[P]} = k\gamma q[P(\vec{x},t)]^{q-2} \tag{65}$$

which, by choosing $\Psi[P(\vec{x},t)] = P(\vec{x},t)$ leads to the form of NLFPE in Equation (19), with $\mu = q$, $\Omega[P(\vec{x},t)] = q[P(\vec{x},t)]^{q-1}$ and a diffusion constant $D = k\gamma$. This corresponds to the simplest N-dimensional NLFPE associated with Tsallis entropy, whose time-dependent solution is given by the q-Gaussian distribution of Equation (20) [34]. However, it is possible to define a whole class of FPEs satisfying Equation (65), as discussed in [23,29], essentially by introducing functionals $a[P]$ and $b[P]$ such that $\Omega[P] = a[P]b[P]$ and $\Psi[P] = a[P]P(\vec{x},t)$. This class of FPEs satisfy the H-theorem for the standard definition of the internal energy in Equation (33).

A similar analysis follows for those NLFPEs associated with generalized energy definitions. In particular, considering the choice $\Gamma[P] = [P(\vec{x},t)]^q$, for which $\chi[P(\vec{x},t)] = (d\Gamma[P]/dP) = q[P(\vec{x},t)]^{q-1}$, the NLFPE of Equation (44) turns into

$$\frac{\partial P(\vec{x},t)}{\partial t} = \vec{\nabla}\cdot\{\Psi[P(\vec{x},t)]\vec{\nabla}[\phi(\vec{x})qP^{q-1}]\} + \vec{\nabla}\cdot\{\Omega[P(\vec{x},t)]\vec{\nabla}P(\vec{x},t)\} \tag{66}$$

which becomes similar to Equation (30) if one uses the time-dependent "potential" of Equation (43). By defining conveniently the functionals $\Omega[P(\vec{x},t)]$ and $\Psi[P(\vec{x},t)]$, one introduces the following family of NLFPEs,

$$\frac{\partial P(\vec{x},t)}{\partial t} = \vec{\nabla}\cdot\{a[P]P(\vec{x},t)\vec{\nabla}[\phi(\vec{x})qP^{q-1}]\} + \vec{\nabla}\cdot\{a[P]b[P]\vec{\nabla}P(\vec{x},t)\} \tag{67}$$

which, by imposing the functional $g_q[P]$ of Equation (18), becomes directly associated with Tsallis entropy,

$$\frac{\Omega[P]}{\Psi[P]} = \frac{b[P]}{P(\vec{x},t)} = k\gamma q[P(\vec{x},t)]^{q-2} \tag{68}$$

for an arbitrary functional $a[P(\vec{x},t)]$ restricted to the conditions already imposed for the functionals $\Omega[P(\vec{x},t)]$ and $\Psi[P(\vec{x},t)]$.

Although finding a time-dependent solution of Equation (67) may be a hard task, its equilibrium solution can be found easily by using $g_q[P]$ and $\Gamma[P] = [P(\vec{x},t)]^q$ in Equation (57),

$$\phi(\vec{x})qP_{\text{eq}}^{q-1} + C' = \gamma \left.\frac{dg[P]}{dP}\right|_{P=P_{\text{eq}}} = k\gamma\frac{1-qP_{\text{eq}}^{q-1}}{q-1} \tag{69}$$

Now, solving for P_{eq},

$$P_{\text{eq}}(\vec{x}) = \frac{1}{Z}\left[1-(1-q)\frac{\phi(\vec{x})}{k\gamma}\right]^{\frac{1}{1-q}} \tag{70}$$

where $Z = \int d^N x\ [1-(1-q)\phi(\vec{x})/(k\gamma)]^{\frac{1}{1-q}}$ is a normalization factor. Notice that the equilibrium solution above applies for any confining potential $\phi(\vec{x})$; particularly, for an external force $\vec{A}(\vec{x}) = \vec{k}_1 - k_2\vec{x}$ ($k_2 \geq 0$) one gets the same N-dimensional q-Gaussian solution of Equation (19), calculated in [34], which is valid for all times. Therefore, we have shown that Equation (67), which defines a class of NLFPEs directly related to Tsallis entropy and a generalized definition for the energy, may present complicated time-dependent solutions, although in the long-time limit these solutions should approach the above equilibrium solution, given in terms of a q-exponential for any confining potential.

Let us now proceed inversely, *i.e.*, given the solution of Equation (70) we shall obtain relation (68) at equilibrium. One has that,

$$\vec{\nabla}P_{\text{eq}}(\vec{x}) = -Z^{q-1}[P_{\text{eq}}(\vec{x})]^q\,\frac{\vec{\nabla}\phi(\vec{x})}{k\gamma}\ ;\ \Rightarrow\ \frac{\vec{\nabla}\phi(\vec{x})}{k\gamma} = -Z^{1-q}[P_{\text{eq}}(\vec{x})]^{-q}\,\vec{\nabla}P_{\text{eq}}(\vec{x}) \tag{71}$$

which shows that $\vec{\nabla}P_{\text{eq}}(\vec{x}) \neq 0$, for $\vec{A}(\vec{x}) = -\vec{\nabla}\phi(\vec{x}) \neq 0$. In addition to this,

$$\vec{\nabla}\left\{\frac{\phi(\vec{x})}{k\gamma}q[P_{\text{eq}}(\vec{x})]^{q-1}(\vec{x})\right\} = q[P_{\text{eq}}(\vec{x})]^{q-1}\frac{\vec{\nabla}\phi(\vec{x})}{k\gamma} + q(q-1)\frac{\phi(\vec{x})}{k\gamma}[P_{\text{eq}}(\vec{x})]^{q-2}\vec{\nabla}P_{\text{eq}}(\vec{x}) \tag{72}$$

Substituting Equation (71) into Equation (72), one gets,

$$\vec{\nabla}\left\{\frac{\phi(\vec{x})}{k\gamma}q[P_{\text{eq}}(\vec{x})]^{q-1}(\vec{x})\right\} = q\left\{-[ZP_{\text{eq}}(\vec{x})]^{1-q} + (q-1)\frac{\phi(\vec{x})}{k\gamma}\right\}[P_{\text{eq}}(\vec{x})]^{q-2}\,\vec{\nabla}P_{\text{eq}}(\vec{x}) \tag{73}$$

and using the equilibrium solution of Equation (70) one sees that the potential-dependent term inside braces on the r.h.s. disappears. Now substituting this result in Equation (53), one recovers Equation (68) at equilibrium,

$$\frac{\Omega[P_{\text{eq}}(\vec{x})]}{\Psi[P_{\text{eq}}(\vec{x})]} = k\gamma q[P_{\text{eq}}(\vec{x})]^{q-2} \tag{74}$$

showing the association of the equilibrium solution of Equation (67) with Tsallis entropy.

5. Conclusions

We have generalized several previous results of one-dimensional nonlinear Fokker-Planck equations to N-dimensions, giving a particular emphasis to those equations related to Tsallis entropy. A derivation of a nonlinear N-dimensional Fokker-Planck equation was carried, by considering the continuous limit of a discrete master equation with nonlinearities in its associated transition rates. Using general N-dimensional nonlinear Fokker-Planck equations, the H-theorem was proved for both standard and generalized definitions of the internal energy. In the former case, an important relation involving the dynamics, *i.e.*, the Fokker-Planck equation, and a general entropic form appeared naturally, whereas in the later, apart from this relation an additional one was also defined. It was shown that, due to these relations, classes of nonlinear N-dimensional Fokker-Planck equations are connected to a single entropic form. For the class of equations related to Tsallis entropy, a N-dimensional Fokker-Planck equation associated to a generalized internal energy definition was studied in detail for the first time. It was verified that its corresponding equilibrium solution is given by a q-exponential for an arbitrary confining potential. The theoretical background for most physical applications related to nonlinear Fokker-Planck equations, like the motion of particles in porous media [12–15], the dynamics of surface growth [15], and the dynamics of interacting vortices in disordered superconductors [17,18], which usually take place in physical spaces of dimensions 2 and 3, was presented.

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