*Entropy* **2012**, *14*, 1522-1538; doi:10.3390/e14081522

OPEN ACCESS

*entropy*

**ISSN 1099-4300**

www.mdpi.com/journal/entropy

*Article*

# The Entropy of a Discrete Real Variable

**Scott Funkhouser**

SPAWAR Systems Center Atlantic, Joint Base Charleston, North Charleston, SC 29406, USA;
E-Mail: scott.funkhouser@gmail.com

*Received: 12 June 2012; in revised form: 3 August 2012 / Accepted: 6 August 2012 / Published: 17 August 2012*

---

**Abstract:** The discrete Shannon entropy $H$ was formulated only to measure indeterminacy effected through a set of probabilities, but the indeterminacy in a real-valued discrete variable depends on both the allowed outcomes $\mathbf{x}$ and the corresponding probabilities $\mathbf{p}$. A fundamental measure that is sensitive to both $\mathbf{x}$ and $\mathbf{p}$ is derived here from the total differential entropy of a continuous real variable and its conjugate in the discrete limit, where the conjugate is universally eliminated. The asymptotic differential entropy recovers $H$ plus the new measure, named $\Xi$, which provides a novel probe of intrinsic organization in sequences of real numbers.

**Keywords:** Shannon entropy; information entropy; information theory

---

## 1. Introduction

Let $Y$ be a discrete variable with $n > 1$ generic outcomes $\mathbf{y} = \{y_1, \ldots, y_n\}$ and corresponding probabilities $\mathbf{p} = \{p_1, \ldots, p_n\}$ such that

$$\sum_{j=1}^{n} p_j = 1 \tag{1}$$

Expressed in terms of the natural logarithm, the Shannon entropy attributed to $Y$ in connection with $\mathbf{p}$ is [1]

$$H(\mathbf{p}) = -\sum_{j=1}^{n} p_j \log(p_j) \tag{2}$$

For convenience let

$$H_2(\mathbf{p}) \equiv \frac{H(\mathbf{p})}{\log(2)} \tag{3}$$

represent the original form defined in terms of the base-2 logarithm. $H = H(\mathbf{p})$ is bound according to

$$0 \leq H \leq \log(n) \tag{4}$$

The maximum is generated only when all outcomes are equally probable. $H(\mathbf{p})$ vanishes in the limit as some $p_j \in \mathbf{p}$ approaches unity, which represents a completely predetermined outcome.

Because it emerges naturally from the basic principles of encoded compression, the discrete Shannon entropy is widely recognized as an informatic measure. Specifically, consider a message $\mathbf{m}$ consisting of $N \gg 1$ symbols from a generalized alphabet $\mathbf{s} = \{s_1, \ldots, s_a\}$. Let $N_j$ represent the total number of occurrences of a particular symbol $s_j \in \mathbf{s}$ within $\mathbf{m}$. In the limit as $N$ becomes infinitely large suppose that each $N_j/N$ approaches a certain constant $w_j$ that is characteristic to the "language" in which $\mathbf{m}$ is composed. For sufficiently large $N$ we therefore have $N_j \simeq w_j N$. The total number $I$ of likely messages of length $N$ is consequently fixed by $\mathbf{w} = \{w_1, \ldots, w_a\}$ according to

$$I \simeq \prod_{j=1}^{n} w_j^{-N_j} \tag{5}$$

The right side of Equation (5) is simply the inverse of the probability for finding any one particular $\mathbf{m}$ among all messages of length $N$ subject to $\mathbf{w}$. The inventory of likely messages could be encoded as $I$ different integers. The number $\beta$ of bits required to register the encoded inventory is given by $\beta = \lceil \log_2(I) \rceil$, and we therefore have the average compression rate

$$\frac{\beta}{N} \simeq H_2(\mathbf{w}) \tag{6}$$

Notwithstanding its significance, Equation (6) represents only a particular example of the expansive utility of the discrete Shannon entropy. Interpreted most broadly, and in accordance with the purpose for which it was derived, $H(\mathbf{p})$ measures the indeterminacy in a random selection from among $n$ different options with corresponding probabilities $\mathbf{p}$ [1]. It is therefore appropriate to attribute a quantity of entropy $H(\mathbf{p})$ to a discrete variable with probabilities $\mathbf{p}$.

Consider a variable $Y$ whose $n \gg 1$ allowed outcomes $\mathbf{y}$ are governed by some $\mathbf{p}$ such that $H(\mathbf{p})$ is near $\log(n)$. The nearly maximal entropy implies that $Y$ exhibits no significant statistical preference toward any particular outcome or group of outcomes among $\mathbf{y}$. Consequently an ideal observer who studied the behavior of $Y$ could develop a successful model only to predict the most trivial details about future outcomes. Stated alternatively, the degree to which $Y$ behaves deterministically is minimal. In contrast suppose that $H(\mathbf{p})$ is much smaller than $\log(n)$. The nearly minimal entropy implies a strong statistical preference for some comparatively small number of $y_j \in \mathbf{y}$. As such $Y$ would exhibit an appreciable degree of deterministic behavior. A successful, non-trivial predictive model could be therefore developed from observations of $Y$.

It is important to emphasize that $H$ is a rigorous measure of indeterminacy in a variable only insofar as the indeterminacy depends on $\mathbf{p}$ alone. Let $X$ be a discrete variable whose $n$ allowed outcomes $\mathbf{x} = \{x_1, \ldots, x_n\}$ are $D$-dimensional real vectors, *i.e.* $x_j \in \Re^D$ for $j = 1, \ldots, n$. The elements of every one-dimensional $\mathbf{x}$ are implicitly listed in ascending order throughout the following. Analogously to the real nature of $\mathbf{p}$, the real nature of $\mathbf{x}$ endows the outcomes of $X$ with definite spatial attributes in which

non-trivial deterministic behaviors could manifest. Consequently, as illustrated in the following two examples, $H$ is not generally sufficient as a measure of indeterminacy in real-valued discrete variables.

Consider two variables $X_{ee}$ and $X_{er}$ whose outcomes are confined to the real axis. Let the allowed outcomes of $X_{ee}$ be *e*qually probable and *e*qually spaced, hence the subscript "$ee$", and let the allowed outcomes of $X_{er}$ be *e*qually probable but *r*andomly spaced, hence the subscript "$er$". More formally, the allowed outcomes of $X_{ee}$ are of the form $\mathbf{x}_e$, which is defined for $D = 1$ to be a set of $n$ equally spaced points on the real axis. Let $\gamma$ represent the arbitrary spacing in a given $\mathbf{x}_e$. For convenience we also define $\mathbf{p}_e \equiv \{1/n, \ldots, 1/n\}$ for a given $n$ to represent a set of equal probabilities. The allowed outcomes of $X_{er}$ are of the form $\mathbf{x}_r$, which is defined for $D = 1$ to be indistinguishable from a set of $n$ points distributed randomly over some simple finite segment $\mathfrak{X}_1$ of the real axis. The intrinsic spatial structure of $\mathbf{x}_e$ effects a commensurate degree of deterministic behavior in $X_{ee}$. For instance, the difference between successive outcomes of $X_{ee}$ is always an integer multiple of $\gamma$, and is therefore significantly more predictable than the difference between successive outcomes of $X_{er}$. Because $X_{ee}$ and $X_{er}$ differ only in the spatial arrangement of their allowed outcomes, $X_{ee}$ would behave more deterministically. By definition a smaller quantity of entropy should be therefore attributed to $X_{ee}$. The discrete Shannon entropy attributed to both variables, however, would be identically maximal.

Consider next a complementary scenario in which the allowed outcomes are identically valued but the respective probabilities differ. Let $X_{re}$ and $X_{ae}$ be two variables with $n \gg 1$ one-dimensional allowed outcomes of the form $\mathbf{x}_e$. The set of probabilities governing $X_{re}$ is of the form $\mathbf{p}_r$, which is defined generally to be indistinguishable from $n$ randomly selected positive real numbers, subsequently normalized to unity and listed in *r*andom order. The spatial assignment of $\mathbf{p}_r$ through $\mathbf{x}_e$ would be globally isotropic but locally irregular, despite the intrinsic structure of $\mathbf{x}_e$. The outcomes of $X_{ae}$ are matched to a set of probabilities of the form $\mathbf{p}_a$, which is defined generally to be indistinguishable from $n$ randomly selected positive real numbers, subsequently normalized to unity and listed in *a*scending order. The ordered arrangement of $\mathbf{p}_a$ effects a non-trivial deterministic bias favoring progressively larger outcomes in $X_{ae}$. Note that the specific form of $\mathbf{p}_a$ in this example could be constructed simply by sorting the elements of $\mathbf{p}_r$, in which case $X_{ae}$ would behave more deterministically but $H(\mathbf{p}_r)$ and $H(\mathbf{p}_a)$ would be identical.

The problems raised in the previous two paragraphs do not imply any flaw in $H$. The discrete Shannon entropy was formulated only to measure indeterminacy effected by a given $\mathbf{p}$ in a generic discrete variable. The net indeterminacy in a real-valued discrete variable, however, depends on the respective intrinsic characteristics of $\mathbf{x}$ and $\mathbf{p}$, and on the manner in which $\mathbf{x}$ and $\mathbf{p}$ are matched. A special measure is therefore required for discrete real variables. Note that Equation (6) is insensitive to the nature of $\mathbf{s}$, and the present considerations are therefore irrelevant in the context of the coding theorems. To be precise, the role of $H_2$ in Equation (6) is not truly as a measure of indeterminacy.

Although it included no special provisions for discrete real variables, Shannon's seminal paper introduced a separate formulation for the entropy of a continuous real variable. Let $\mathcal{X}$ be a probabilistically determined variable whose allowed outcomes span a continuum of real vectors within some region $\mathfrak{X} \in \Re^D$. Let the probability density $\rho(x)$ be defined such that $\rho(x)\mathrm{d}x$ gives the probability for an outcome of $\mathcal{X}$ to lie within an infinitesimal element $\mathrm{d}x$ centered on $x$, where $x \in \Re^D$ is an independent variable and $\mathrm{d}x = \mathrm{d}x^{(1)} \ldots \mathrm{d}x^{(D)}$ represents implicitly the product of the differentials of each

vector component of $x = (x^{(1)}, \ldots, x^{(D)})$. We therefore require $\rho = \rho(x)$ to vanish for all $x \notin \mathfrak{X}$. The normalization of $\rho$ is accordingly

$$\int_{\Re^D} \rho \mathrm{d}x = \int_{\mathfrak{X}} \rho \mathrm{d}x = 1 \tag{7}$$

Henceforth any integration is understood to span $\Re^D$ in the absence of explicit limits. The entropy $\mathcal{H} = \mathcal{H}(\rho)$ attributed to $\mathcal{X}$ in connection with $\rho$ is [1]

$$\mathcal{H} = -\int \rho \log(\rho)\, \mathrm{d}x \tag{8}$$

where $\rho\log(\rho)$ is defined to vanish for $\rho = 0$. The measure in Equation (8) is known as the *differential (Shannon) entropy*.

In contrast to $H$ a single quantity of $\mathcal{H}$ is subject to no finite bounds. Consequently a simple comparison between $H$ and $\mathcal{H}$ is not meaningful. For instance, the differential entropy of a normalized Gaussian

$$\rho_G = \rho_G(\nu, x) \equiv \left(\frac{\nu}{\sqrt{\pi}}\right)^D e^{-\nu^2 x^2} \tag{9}$$

is $D\log(\sqrt{\pi e}/\nu)$. For $\nu$ equal to $\sqrt{\pi e}$ the entropy vanishes, which signifies a predetermined outcome in the context of discrete real variables. Furthermore in the limit as $\nu$ becomes infinitely large $\rho_G$ behaves as a $D$-dimensional delta-distribution $\delta(x)$, which describes a single predetermined real outcome. $\mathcal{H}(\rho_G)$, however, decreases without bound as $\nu$ approaches infinity.

The apparent inconsistencies between $H$ and $\mathcal{H}$ are reconciled in the following manner. As a generalization of $\rho_G$ let $\rho_0 = \rho_0(\nu, x)$ be some well-behaved, unity-normalized function that becomes $\delta(x)$ in the limit as the positive, real parameter $\nu$ becomes infinitely large. In association with an arbitrary $X$, with $n$ allowed outcomes $\mathbf{x}$ and corresponding probabilities $\mathbf{p}$, let the probability density $\rho_X = \rho_X(\nu, x)$ be defined such that

$$\rho_X(\nu, x) \sim B \sum_{j=1}^{n} p_j \rho_0(\nu, x - x_j) \tag{10}$$

and thus

$$\lim_{\nu \to \infty} \rho_X = \sum_{j=1}^{n} p_j \delta(x - x_j) \tag{11}$$

where $B = B(\nu)$ is the appropriate normalization coefficient for a given $\nu$, and $x_j$ and $p_j$ span $\mathbf{x}$ and $\mathbf{p}$ respectively. The symbol "$\sim$" is used in Equation (10) and throughout, where convenient, to signify asymptotic equality as $\nu$ approaches infinity. The right side of Equation (11) describes the probabilistic behavior of $X$. Consider an originally continuous test variable $\mathcal{X}_X$ whose outcomes are governed by $\rho_X$. The entropy attributed to $\mathcal{X}_X$ is $\mathcal{H}(\rho_X)$, which behaves asymptotically as

$$\mathcal{H}(\rho_X) \sim -B \sum_{j=1}^{n} p_j \int \rho_0 \log\big(B p_j \rho_0\big) \mathrm{d}x \tag{12}$$

When $\nu$ is finite but very large the behavior of $\mathcal{X}_X$ is a hybrid of the continuous traits of $\rho_0$ and the discrete traits of $\mathbf{x}$ and $\mathbf{p}$. In the limit as $\nu \to \infty$ the two sets of traits become completely disassociated from one another, and $\mathcal{X}_X = X$ is characterized only by $\mathbf{x}$ and $\mathbf{p}$. The entropy associated with the probabilistic

behavior of $X$ should be therefore given by the asymptote of $\mathcal{H}(\rho_X) - \mathcal{H}(\rho_0)$. From Equation (12) we readily obtain

$$\lim_{\nu\to\infty} \left(\mathcal{H}(\rho_X) - \mathcal{H}(\rho_0)\right) = -\sum_{j=1}^{n} p_j \log(p_j) \tag{13}$$

thereby recovering $H$ from $\mathcal{H}$.

Although the analysis leading to Equation (13) may be instructive, $\mathcal{H}(\rho_X)$ is asymptotically independent of $\mathbf{x}$ and could not produce the desired new measure for discrete real variables. It is reasonable to suspect that some additional quantity of the form $\mathcal{H}$ could be considered in conjunction with $\mathcal{H}(\rho_X)$ to provide the required asymptotic sensitivity to both $\mathbf{x}$ and $\mathbf{p}$. In order to be meaningful, however, the additional measure should be intrinsically related to $\rho_X$.

Within the paradigm of wave mechanics two distinct quantities of differential entropy are naturally associated with a given $\rho$. More precisely, two separate probability density functions are naturally associated with a given probability amplitude $\psi = \psi(x)$, namely $\rho = |\psi|^2$ and the spectral density

$$\hat{\rho}(k) = \left|\hat{\psi}(k)\right|^2 \tag{14}$$

where

$$\hat{\psi} = \hat{\psi}(k) = \left(\frac{1}{\sqrt{2\pi}}\right)^D \int \psi e^{-ik\cdot x} \mathrm{d}x \tag{15}$$

is the Fourier-conjugate of $\psi$ and $k \in \Re^D$ is an independent variable. We may interpret $\hat{\rho} = \hat{\rho}(k)$ as the probability density governing the outcomes of a variable $\mathcal{K}$ that is the conjugate of the variable $\mathcal{X}$ governed by $\rho$. The entropy attributed to $\mathcal{K}$ in connection with $\hat{\rho}$ is $\mathcal{H}(\hat{\rho})$. The total differential entropy attributed to the conjugated pair is therefore

$$S(\psi) \equiv \mathcal{H}(|\psi|^2) + \mathcal{H}(|\hat{\psi}|^2) \tag{16}$$

A more formal derivation of $S = S(\psi)$ follows from the total phase-space distribution [9]

$$f = f(x,k) \equiv \rho\hat{\rho} \tag{17}$$

which is subject to the normalization

$$\int f \mathrm{d}r = 1, \tag{18}$$

where $r \equiv \left(x^{(1)}, \ldots, x^{(D)}, k^{(1)}, \ldots, k^{(D)}\right)$ for any given $x$ and $k$. Because $f$ is simply a probability density in a $2D$-dimensional vector space, it is appropriate to attribute a quantity of differential entropy

$$-\int f \log(f)\, \mathrm{d}r = S \tag{19}$$

to the conjugated pair.

The total differential entropy $S$ has been the subject of extensive research. (See, for instance, [2–8]). It is worthwhile to mention here two general characteristics that distinguish $S$ as a fundamentally important measure. Because $f$ is dimensionless $S$, like the relative entropy, is free from a potentially serious flaw to which the differential Shannon entropy is otherwise vulnerable [10]. Furthermore, whereas $\mathcal{H}(\rho)$ and $\mathcal{H}(\hat{\rho})$ may become infinitely negative, $S$ is subject to the lower bound [2]

$$S \geq D\log(\pi e) \tag{20}$$

Perhaps uniquely, a Gaussian amplitude generates the minimal $S$ allowed by Equation (20) [2].

The purpose of this article is to demonstrate that a measure of indeterminacy for discrete real variables that is sensitive to both $\mathbf{x}$ and $\mathbf{p}$ emerges from the total differential entropy of $\mathcal{X}_x$ and its conjugate in the limit as $\nu$ diverges. Specifically, the conjugated analogue of Equation (13) includes an additional measure $\Xi = \Xi(\mathbf{p}, \mathbf{x})$, expressed most generally in terms of an integral, that behaves naturally as a measure of indeterminacy in the spatial configuration of $\mathbf{x}$ and $\mathbf{p}$. Furthermore the conjugate is universally eliminated in the limit as $\mathcal{X}_x$ becomes discrete and the entire residual entropy $H+\Xi$, named $\eta$, is therefore attributed to $X$. Section 2 contains a derivation of $\eta$. The general characteristics of $\eta$ are examined in Section 3. Section 4 presents a quantitative study demonstrating the basic behaviors of $\eta$ as a measure of indeterminacy for discrete real variables. The primary conclusions are summarized and interpreted in Section 5.

## 2. Derivation of $\eta$

The terms defined in Section 1 retain their definitions throughout the following. Let $\mathcal{K}_x$ be the conjugate of a given $\mathcal{X}_x$. The probability density $\hat{\rho}_x = \hat{\rho}_x(\nu, k)$ governing the outcomes of $\mathcal{K}_x$ is formulated presently. Let $\psi_x = \psi_x(\nu, x)$ be defined such that $\rho_x = |\psi_x|^2$. The total differential entropy of the conjugated pair is thus $S(\psi_x) = \mathcal{H}(\rho_x) + \mathcal{H}(\hat{\rho}_x)$. This section contains a derivation of the asymptotic form of $S(\psi_x)$ expressed analogously to Equation (13).

The first step is to formulate $\psi_x$. Following the standard wave-mechanical prescription, and in accordance with natural law, we attribute to each allowed outcome a separate amplitude [11]. The total amplitude is simply the sum of the individual amplitude waveforms multiplied by an appropriate normalization coefficient. Let the probability amplitude $\phi_0 = \phi_0(\nu, x)$ be therefore defined such that $|\phi_0|^2 = \rho_0$, *i.e.*,

$$\lim_{\nu\to\infty} |\phi_0|^2 = \delta(x) \tag{21}$$

The waveform assigned to each $x_j \in \mathbf{x}$ is

$$\phi_j = \phi_j(\nu, x) \equiv \sqrt{p_j}\, \phi_0(\nu, x - x_j) \tag{22}$$

and the total amplitude is accordingly

$$\psi_x = A \sum_{j=1}^{n} \phi_j \tag{23}$$

where $A = A(\nu)$ is defined such that $\rho_x$ is unity-normalized for all $\nu$. Because the real part of $\phi_0$ becomes ever more sharply peaked with increasing $\nu$, the cross terms $|A|^2 \phi_j \phi_l^*$ in $|\psi_x|^2$ are negligible for sufficiently large $\nu$ and vanish asymptotically. We therefore have

$$\rho_x \sim |A|^2 \sum_{j=1}^{n} |\phi_j|^2 \tag{24}$$

which is equivalent to Equation (10). Note that

$$\lim_{\nu\to\infty} |A|^2 = \lim_{\nu\to\infty} B = 1 \tag{25}$$

The next step is to determine the respective forms of the conjugate amplitude $\hat{\psi}_X = \hat{\psi}_X(\nu, k)$ and density $\hat{\rho}_X = |\hat{\psi}_X|^2$. Let $\hat{\phi}_0 = \hat{\phi}_0(\nu, k)$ represent the conjugate of $\phi_0$, and let the associated density $|\hat{\phi}_0|^2$ be defined as $\hat{\rho}_0 = \hat{\rho}_0(\nu, k)$. It follows from Equation (15), with $x - x_j$ substituted for $x$, that

$$\hat{\phi}_j = \sqrt{p_j}\, e^{-\imath k \cdot x_j}\, \hat{\phi}_0 \tag{26}$$

where $\hat{\phi}_j = \hat{\phi}_j(\nu, k)$ is the conjugate of a given $\phi_j$. We therefore have

$$\hat{\psi}_X = A\hat{\phi}_0 \sum_{j=1}^{n} \sqrt{p_j}\, e^{-\imath k \cdot x_j} \tag{27}$$

and thus

$$\hat{\rho}_X = |A|^2\, \tau\, \hat{\rho}_0 \tag{28}$$

where

$$\tau \equiv \left| \sum_{j=1}^{n} \sqrt{p_j} e^{-\imath k \cdot x_j} \right|^2 \tag{29}$$

An equivalent expression for $\tau = \tau(k)$ following from the Euler identity is

$$\tau = 1 + 2 \sum_{j=1}^{n-1} \left\{ \sum_{l=j+1}^{n} \sqrt{p_j p_l} \cos\left( (x_l - x_j) \cdot k \right) \right\} \tag{30}$$

For convenience we define $\zeta = \zeta(k)$ such that $\tau = 1 + 2\zeta$.

The asymptotic behavior of $\hat{\rho}_X$ is ascertained readily from the explicit form of $\hat{\phi}_0$ in Equation (15) with $\phi_0$ expressed as $\rho_0/\phi_0^*$ for all non-vanishing $\phi_0^*$. It follows from the behavior ascribed to $\rho_0$ that

$$\hat{\phi}_0 \sim \left( \frac{1}{\sqrt{2\pi}} \right)^D \frac{1}{\phi_0^*(0)} \tag{31}$$

which is independent of $k$. Because $\int \rho_0 \mathrm{d}k$ must remain unity-normalized $\hat{\phi}_0$ must vanish everywhere asymptotically. As $\rho_0$ increasingly resembles $\delta(x)$ the real part of $\phi_0^*(0)$ must either increase or decrease without bound, which ensures that the right side of Equation (31) vanishes. Consequently $\hat{\psi}_X$ and $\hat{\rho}_X$ vanish asymptotically for all $k$.

It is also worthwhile to note that, because $\hat{\rho}_X$ and $\hat{\rho}_0$ are both unity-normalized for all $\nu$, Equations (28) and (30) imply

$$\lim_{\nu \to \infty} \langle \zeta \rangle_0 = 0 \tag{32}$$

where

$$\langle \varepsilon \rangle_0 \equiv \int \hat{\rho}_0 \varepsilon \mathrm{d}k \tag{33}$$

is the expected value of any $\varepsilon = \varepsilon(k)$ averaged against $\hat{\rho}_0$. Equation (32) may be understood as a consequence of the asymptotic uniformity of $\hat{\rho}_0$ in conjunction with the usual trigonometric property

$$\lim_{b \to \infty} \int_{-b}^{b} \cos(k) \mathrm{d}k = 0 \tag{34}$$

In the limit as $\nu$ diverges $\langle\zeta\rangle_0$ becomes simply the average of $\zeta$ over all space, which we denote by $\overline{\zeta}$, and thus vanishes term-by-term in accordance with Equation (34). Expressed more generally,

$$\lim_{\nu\to\infty}\langle\varepsilon\rangle_0 = \overline{\varepsilon}. \tag{35}$$

Returning to the derivation proper, the next step is to evaluate $\mathcal{H}(\hat{\rho}_X)$ in the limit as $\nu$ approaches infinity. It follows from Equations (25) and (28) that

$$\mathcal{H}(\hat{\rho}_X) \sim -\langle\tau\log(\hat{\rho}_0)\rangle_0 - \langle\tau\log(\tau)\rangle_0 \tag{36}$$

Substituting $1+2\zeta$ for $\tau$ in the left-most term on the right side of Equation (36) produces

$$\mathcal{H}(\hat{\rho}_X) \sim \mathcal{H}(\hat{\rho}_0) - 2\langle\zeta\log(\hat{\rho}_0)\rangle_0 - \langle\tau\log(\tau)\rangle_0 \tag{37}$$

Although $\log(\hat{\rho}_0)$ decreases without bound it does so with asymptotic uniformity, hence

$$\langle\zeta\log(\hat{\rho}_0)\rangle_0 \sim \langle\zeta\rangle_0 \log\big(\hat{\rho}_0(k_0)\big) \tag{38}$$

for any $k_0 \in \Re^D$. The middle term on the right side of Equation (37) is therefore asymptotically proportional to $\overline{\zeta}$ and thus vanishes. Equation (37) accordingly becomes

$$\mathcal{H}(\hat{\rho}_X) \sim \mathcal{H}(\hat{\rho}_0) - \langle\tau\log(\tau)\rangle_0 \tag{39}$$

The asymptotic form of the right-most term in Equation (39) is

$$\Xi(\mathbf{p},\mathbf{x}) \equiv -\overline{\tau\log(\tau)} \tag{40}$$

Because $\tau$ is periodic $\Xi = \Xi(\mathbf{p},\mathbf{x})$ is identical to the average value of $-\tau\log(\tau)$ taken over some finite region $\mathfrak{K}_0 \in \Re^D$ spanning one period in each dimension. We therefore have

$$\Xi = -\frac{1}{\kappa_0}\int_{\mathfrak{K}_0} \tau\log(\tau)\mathrm{d}k \tag{41}$$

where

$$\kappa_0 \equiv \int_{\mathfrak{K}_0} \mathrm{d}k \tag{42}$$

An analytical expression for $\int\tau\log(\tau)\mathrm{d}k$ is possible only for a few rudimentary configurations of $\mathbf{x}$ and $\mathbf{p}$, and Equation (41) is thus critical for quantifying $\Xi$.

Finally, Equation (39) may be expressed in terms of a proper limit as

$$\lim_{\nu\to\infty}\big(\mathcal{H}(\hat{\rho}_X) - \mathcal{H}(\hat{\rho}_0)\big) = \Xi \tag{43}$$

analogously to Equation (13). Combining Equations (13) and (43) produces the central result

$$\lim_{\nu\to\infty}\big(S(\psi_X) - S(\phi_0)\big) = \eta(\mathbf{p},\mathbf{x}) \tag{44}$$

where

$$\eta(\mathbf{p},\mathbf{x}) \equiv H(\mathbf{p}) + \Xi(\mathbf{p},\mathbf{x}) \tag{45}$$

The nature and behavior of $\eta = \eta(\mathbf{p},\mathbf{x})$ are examined in the following Sections.

## 3. General Characteristics of $\eta$

By definition $S(\psi_X)$ measures the net indeterminacy in the conjugated pair $\mathcal{X}_X$ and $\mathcal{K}_X$. In the limit as $\nu$ becomes infinitely large, however, the pair is characterized completely by $\mathbf{x}$ and $\mathbf{p}$, independently of $\phi_0$. The entropy attributed asymptotically to the pair is therefore $\eta$. Furthermore as $\mathcal{X}_X$ becomes discrete $\hat{\rho}_X$ vanishes everywhere and $\mathcal{K}_X$ becomes therefore a "ghost" variable, doomed to exist without ever generating an outcome. The indeterminacy measured by $\eta$ must be therefore evident in the probabilistic behaviors of $X$. Because $X$ and its ghost conjugate together are indistinguishable from $X$ alone we conclude that $\eta$ is a generally valid measure of indeterminacy in discrete real variables.

Given that $\Xi$ is the asymptotic form of $\mathcal{H}(\hat{\rho}_X) - \mathcal{H}(\hat{\rho}_0)$ its attribution to $X$ is perhaps counterintuitive, despite the elimination of the conjugate. In particular $\mathcal{H}(\rho)$ and $\mathcal{H}(\hat{\rho})$ are typically anti-correlated among different $\rho$ of a given generalized form. Recall, however, that $\mathcal{H}(\rho_X)$ is asymptotically insensitive to $\mathbf{x}$ and therefore is not generally correlated to $\mathcal{H}(\hat{\rho}_X)$ in any appreciable manner among different $\rho_X$. Not only is $\mathcal{H}(\hat{\rho}_X)$ sensitive to both $\mathbf{x}$ and $\mathbf{p}$, it is also naturally correlated to the intrinsic indeterminacy effected by the definite spatial attributes of $\mathbf{x}$. Stated most generally, greater spatial regularity in the configuration of $\mathbf{x}$ and $\mathbf{p}$ effects greater harmonic regularity in the spatial arrangement of the peaks in $\rho_X$, for $\nu \gg 1$. Greater harmonic regularity in $\rho_X$ implies greater localization in the associated Fourier spectrum, which accordingly generates a smaller $\mathcal{H}(\hat{\rho}_X)$, for a given $\phi_0$. In order to isolate the effects associated with $\mathbf{x}$ and $\mathbf{p}$ we exclude the contribution from $\mathcal{H}(\hat{\rho}_0)$. In that manner $\Xi$ measures the deterministic effects of real allowed outcomes.

Lower bounds on $\eta$ and $\Xi$ follow directly from the universal lower bound on $S$. Specifically, both $S(\psi_X)$ and $S(\phi_0)$ are subject to Equation (20) for all $\nu$. We therefore have

$$\eta \geq 0 \tag{46}$$

and thus

$$\Xi \geq -H \tag{47}$$

Regardless of the degree of spatial structure in the configuration of $\mathbf{x}$ and $\mathbf{p}$, no variable could behave more deterministically than a fixed real number. Equation (46) is therefore consistent with the attribution of zero entropy to a predetermined outcome.

Upper bounds on $\eta$ and $\Xi$ may be obtained by writing Equation (41) as

$$\Xi = -\int_{\mathfrak{K}_0} \frac{\tau}{\kappa_0} \log\left(\frac{\tau}{\kappa_0}\right) \mathrm{d}k - \log(\kappa_0) \tag{48}$$

where the right-most term has been reduced using

$$\overline{\tau} = \frac{1}{\kappa_0} \int_{\mathfrak{K}_0} \tau \mathrm{d}k = 1, \tag{49}$$

which is a consequence of Equation (34). The integral on the right side of Equation (48) is simply the differential Shannon entropy associated with a normalized density $\tau/\kappa_0$ defined over $\mathfrak{K}_0$. Furthermore $\log(\kappa_0)$ is the maximal differential Shannon entropy that could be associated with $\mathfrak{K}_0$. Equation (48) therefore implies

$$\Xi \leq 0 \tag{50}$$

hence

$$\eta \leq H \tag{51}$$

The upper bound on $\eta$ is consistent with the basic expectations about the predictive utility of real outcomes. Consider some $X$ that has been "disguised" by mapping the $n$ allowed outcomes in $\mathbf{x}$ to a set of $n$ generic outcomes, while keeping the corresponding probabilities $\mathbf{p}$ fixed. An ideal observer who studied only the generic outcomes could form a predictive model only to the extent allowed by $\mathbf{p}$. Suppose that such an observer were subsequently allowed to study the real outcomes. The numeric details could never detract from the observer's capability to develop a predictive model because, at very least, the outcomes from any $\mathbf{x}$ could be interpreted simply as non-numeric symbols. The entropy attributed to $X$, therefore, should not be greater than the entropy attributed to the generic analogue, which is precisely what Equation (51) ensures.

For any $\mathbf{p}$, $H$ and $\eta$ should be similar when the degree of deterministic structure within $\mathbf{x}$ is negligible. We therefore expect

$$\eta(\mathbf{p}, \mathbf{x}_r) \simeq H(\mathbf{p}) \tag{52}$$

which is validated in the following section. Among variables with spatially randomized outcomes, indeterminacy depends primarily on the intrinsic characteristics of $\mathbf{p}$, hence Equation (52). Conversely, among variables with equally probable outcomes, indeterminacy depends only on $\mathbf{x}$. In that manner $\eta(\mathbf{p}_e, \mathbf{x})$ measures the intrinsic indeterminacy in the spatial configuration of $\mathbf{x}$. It is therefore fitting, as explicitly evident in Equation (30), that $\tau$ is sensitive to each of the $n(n-1)/2$ uniquely defined positive intervals $x_l - x_j$ within a given $\mathbf{x}$.

Because there is no preferred region within $\Re^D$, any meaningful measure of indeterminacy for discrete real variables must be invariant under spatial transformations of the form $\mathbf{x}+\Delta x \equiv \{x_1+\Delta x, \ldots, x_n+\Delta x\}$ for any finite $\Delta x \in \Re^D$. By inspecting either Equation (29) or Equation (30) we readily find that $\tau$ is translationally invariant, hence

$$\eta(\mathbf{p}, \mathbf{x}+\Delta x) = \eta(\mathbf{p}, \mathbf{x}) \tag{53}$$

Similarly transformations of the form $\mathbf{R}\mathbf{x} \equiv \{\mathbf{R}x_1, \ldots, \mathbf{R}x_n\}$, where $\mathbf{R}$ is a $D$-dimensional unitary rotation matrix, could not change the indeterminacy in any $X$. Consequently the entropy attributed to $X$ must be rotationally invariant as well. The effects of such a rotation would be equivalent to replacing the original $\tau(k)$ with $\tau'(k) = \tau(k\mathbf{R})$, which we may write as $\tau'(k) = \tau(k')$ after the variable substitution $k' = k\mathbf{R}$. The rotation therefore amounts to nothing more than a cosmetic change of variables, which does not affect the average value of $\tau\log(\tau)$—whether taken over all space or taken over a finite region spanning a period in each dimension of $k'$. We therefore have

$$\eta(\mathbf{p}, \mathbf{R}\mathbf{x}) = \eta(\mathbf{p}, \mathbf{x}) \tag{54}$$

For the same reasons we also have

$$\eta(\mathbf{p}, \alpha\mathbf{x}) = \eta(\mathbf{p}, \mathbf{x}) \tag{55}$$

for any non-vanishing $\alpha \in \Re$, where $\alpha\mathbf{x} \equiv \{\alpha x_1, \ldots, \alpha x_n\}$. Contrary to Equation (55), suppose that the entropy of a real discrete variable were to increase under transformations of the form $\alpha\mathbf{x}$ for $|\alpha| > 1$. Consider applying such transformations successively to the allowed outcomes of some non-trivial $X$. Given that the entropy must never exceed $H$ the effects of the transformations must

become negligible at some point, which would imply the existence of some preferred spatial scale for a given $\mathbf{x}$ and $\mathbf{p}$. The indeterminacy, however, could not be sensitive to such a scale. The scale-invariance expressed in Equation (55) is therefore necessary for any meaningful measure of indeterminacy in discrete real variables.

## 4. Quantitative Study

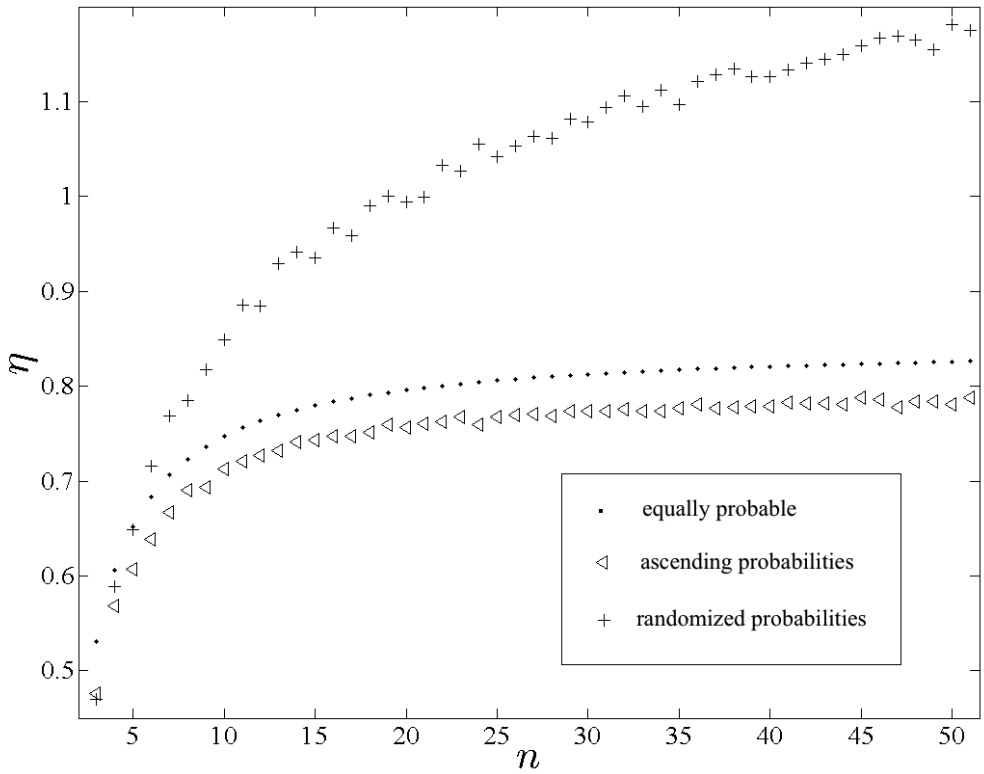
This section presents the results of numerical calculations of $\eta$ chosen to validate the identification of $\eta$ as the entropy of a discrete real variable and to demonstrate the utility of $\Xi$ as a probe of deterministic structure in sequences of real numbers or non-numeric symbols. For convenience this introductory study is restricted to scenarios where $D = 1$; the conclusions are readily generalized to higher-dimensional spaces. In each calculation of $\eta$ reported here the period-averaged value of $-\tau\log(\tau)$ was obtained from $2^5\lfloor\kappa_0/\kappa_*\rfloor$ samples, where $\kappa_* = 2\pi/(x_n - x_1)$ represents the finest periodicity in $\tau$. Note that $x_n - x_1$ is always the largest positive interval within a one-dimensional $\mathbf{x}$. All computations were precise to $15$ significant figures. The accuracy, in comparison to the "true" value of $\eta$ produced by an infinite number of samples, is always better than $10^{-6}\eta$.

The first sets of calculations examined below involve the basic types of $\mathbf{x}$ and $\mathbf{p}$ introduced in Section 1, matched in various instructive combinations. Each $\mathbf{p}_r$ was constructed from $n$ different real numbers with $15$ significant figures selected randomly from $(0, 1)$, and subsequently normalized. Each $\mathbf{x}_r$ was constructed by randomly selecting $n$ different integers from $[1, 100n]$. This produces approximately the same effect as using the full range available with $15$ significant figures, but dramatically reduces processing requirements. For each calculation involving $\mathbf{p}_r$ a corresponding calculation involving $\mathbf{p}_a$ is performed, and each $\mathbf{p}_a$ is constructed by sorting the elements of the corresponding $\mathbf{p}_r$. Finally, each reported quantity of $\eta$ involving a randomly generated parameter is the average of $100$ separate calculations with different randomizations.

Let us begin by examining how $\eta$ measures indeterminacy effected through different types of $\mathbf{p}$. Figure 1 displays calculations of $\eta(\mathbf{p}_e, \mathbf{x}_e)$, $\eta(\mathbf{p}_a, \mathbf{x}_e)$ and $\eta(\mathbf{p}_r, \mathbf{x}_e)$ plotted as functions of $n$. Recall that $\eta(\mathbf{p}_r, \mathbf{x}_e)$ and $\eta(\mathbf{p}_a, \mathbf{x}_e)$ should behave respectively as quantities of entropy attributed to the variables $X_{re}$ and $X_{ae}$ examined in Section 1. We find that $\eta(\mathbf{p}_r, \mathbf{x}_e)$ is appreciably greater than $\eta(\mathbf{p}_a, \mathbf{x}_e)$ for all but trivially small $n$, which is consistent with the expected result. Because the only difference between each $\mathbf{p}_a$ and $\mathbf{p}_r$ is the order of their respective terms, the differences between the associated quantities of entropy are entirely due to the contribution from $\Xi$.

Perhaps more importantly, and in opposition to the behavior of $H$ alone, we find that $\eta(\mathbf{p}_r, \mathbf{x}_e)$ is also appreciably greater than $\eta(\mathbf{p}_e, \mathbf{x}_e)$ for all non-trivial $n$. Among generic discrete variables any non-uniformity in $\mathbf{p}$ is a benefit to predictability, hence $H(\mathbf{p}_r) \leq H(\mathbf{p}_e)$. Among real-valued discrete variables, however, the predictive benefit of non-uniformities in $\mathbf{p}$ depends on the manner in which the probabilities are spatially assigned. In the case of $X_{re}$ the indeterminacy effected by the irregular spatial assignment of $\mathbf{p}_r$ evidently outweighs the intrinsic predictive benefit of $\mathbf{p}_r$, which implies that $X_{ee}$ behaves more deterministically than $X_{re}$. Though not necessarily expected *a priori*, that implication is not surprising.

**Figure 1.** Plots of $\eta(\mathbf{p}_e, \mathbf{x}_e)$, $\eta(\mathbf{p}_a, \mathbf{x}_e)$ and $\eta(\mathbf{p}_r, \mathbf{x}_e)$ using the respective symbols "•" , "◁" and "+".




As a complement to the previous scenario it is instructive to compare the effects of the three types of probabilities when matched to spatially randomized outcomes. Figure 2 displays calculations of $\eta(\mathbf{p}_e, \mathbf{x}_r)$, $\eta(\mathbf{p}_a, \mathbf{x}_r)$ and $\eta(\mathbf{p}_r, \mathbf{x}_r)$ plotted as functions of $n$. Note that $\eta(\mathbf{p}_a, \mathbf{x}_r)$ and $\eta(\mathbf{p}_r, \mathbf{x}_r)$ nearly coincide for each $n$, and the symbols "◇" and "$*$" are always superimposed in Figure 2. The entropy-difference $\eta(\mathbf{p}_r, \mathbf{x}_r) - \eta(\mathbf{p}_a, \mathbf{x}_r)$ is shown in Figure 3. In contrast to the previous scenario $\mathbf{p}_e$ consistently effects the largest entropy among the three, which is expected from Equation (52) given that $H(\mathbf{p}_e)$ is maximal. Note that $\eta(\mathbf{p}_e, \mathbf{x}_r)$ would be even closer to $H(\mathbf{p}_e)$ had the randomness of $\mathbf{x}_r$ not been restricted for the sake of computational feasibility. The similarity between $\eta(\mathbf{p}_r, \mathbf{x}_r)$ and $\eta(\mathbf{p}_a, \mathbf{x}_r)$ is also expected from Equation (52) given that $H(\mathbf{p}_r) = H(\mathbf{p}_a)$. Furthermore insofar as $\eta(\mathbf{p}_r, \mathbf{x}_r)$ and $\eta(\mathbf{p}_a, \mathbf{x}_r)$ differ the former should be typically greater for sufficiently large $n$, which is confirmed in Figure 3.

Let us next examine calculations demonstrating how $\eta$ measures indeterminacy effected through $\mathbf{x}$, which is perhaps the most significant attribute of $\eta$. The most basic comparison corresponds to the scenario involving $X_{ee}$ and $X_{er}$ in Section 1. It is immediately evident from Figures 1 and 2 that $\eta(\mathbf{p}_e, \mathbf{x}_r)$ is considerably larger than $\eta(\mathbf{p}_e, \mathbf{x}_e)$ for all $n$, which is expected.

**Figure 2.** Plots of $\eta(\mathbf{p}_e, \mathbf{x}_r)$, $\eta(\mathbf{p}_a, \mathbf{x}_r)$ and $\eta(\mathbf{p}_r, \mathbf{x}_r)$, using the respective symbols "$\times$", "$\diamond$" and "$*$". The solid line follows a plot of $\log(n)$ for reference.

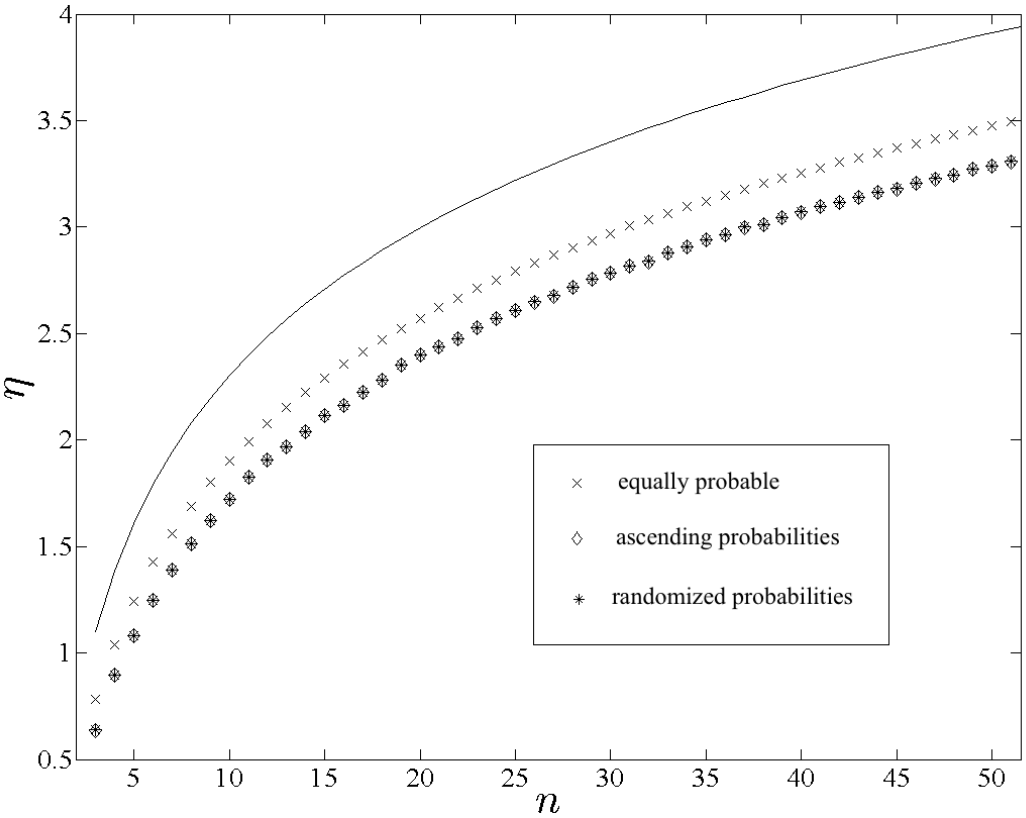
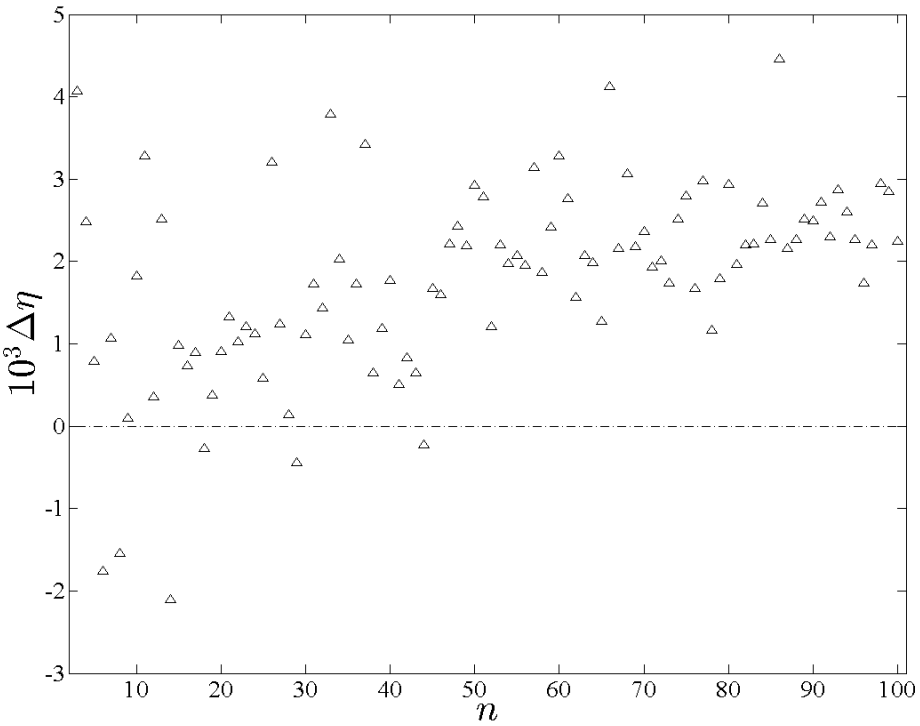



**Figure 3.** The entropy difference $\Delta\eta$ plotted here is $\eta(\mathbf{p}_r, \mathbf{x}_r) - \eta(\mathbf{p}_a, \mathbf{x}_r)$.



In order to test the sensitivity of $\eta$ to more subtle differences in spatial structure the following additional sets of type $\mathbf{x}$ are defined. For a given $n$ let $\mathbf{x}_p$ consist of the first $n$ prime numbers. It follows from the Prime Number Theorem (PNT) that the local average gap in the vicinity of a given $x_j \in \mathbf{x}_p$ varies as $\log(j)$ for sufficiently large $j$. The term *gap* is used throughout in reference to a positive interval between consecutive terms in any given one-dimensional $\mathbf{x}$ and is defined such that the $j$-th gap is

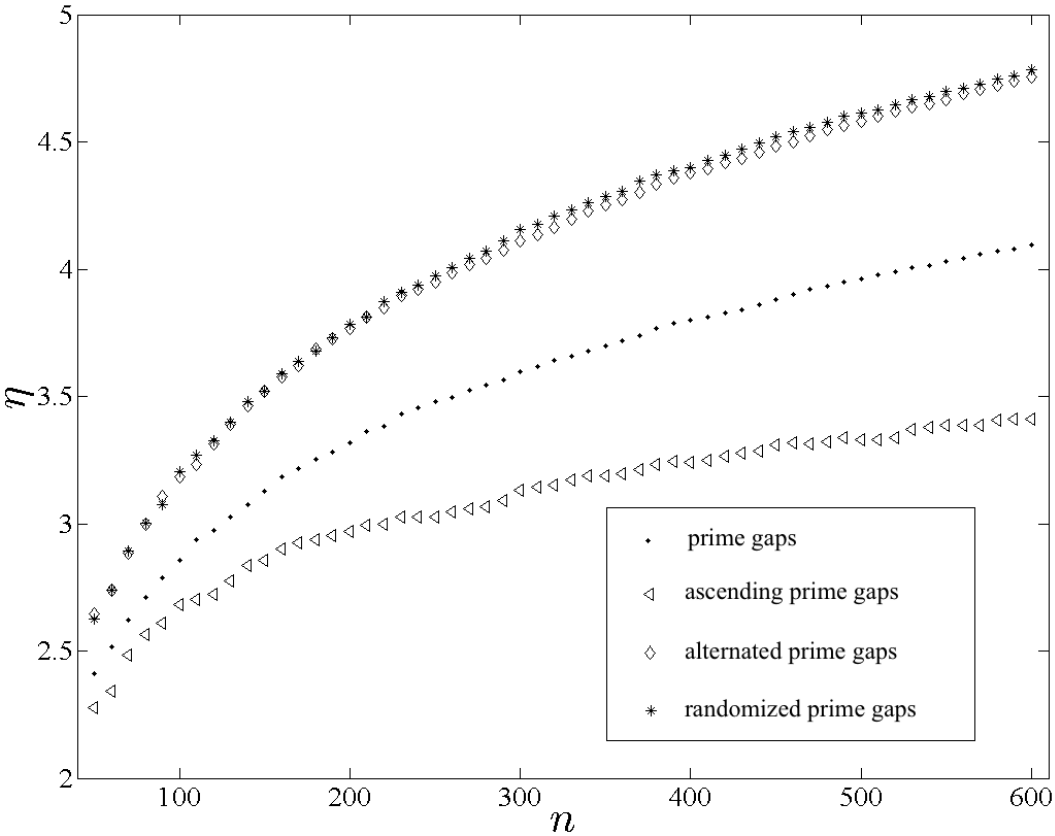
$$g_j \equiv x_{j+1} - x_j \tag{56}$$

The global anisotropy associated with the PNT constitutes an intrinsic deterministic spatial characteristic. For a more pronounced anisotropy of the same kind, let $\acute{\mathbf{x}}_p = \{2, 3, \dots\}$ be defined for a given $n$ such that its sequence of gaps consists of the first $n-1$ prime gaps arranged in ascending order. As an example, for $n = 6$ we have $\acute{\mathbf{x}}_p = \{2, 3, 5, 7, 9, 13\}$. Let $\widetilde{\mathbf{x}}_p = \{2, 3, \dots\}$ be defined for a given $n$ such that its sequence of gaps consists of the first $n-1$ prime gaps ordered randomly, with the exception that the first gap in $\widetilde{\mathbf{x}}_p$ is always 1. Finally let $\breve{\mathbf{x}}_p = \{2, 3, \dots\}$ be defined such that its sequence of gaps is $\{g'_1, g'_3, g'_2, g'_5, g'_4, \dots\}$, where $g'_j$ is the $j$-th prime gap. In other words, the sequence of gaps in $\breve{\mathbf{x}}_p$ is constructed simply by alternating the order of $g'_i$ and $g'_{i+1}$ for all even $i$ no greater than $n-2$. Consequently every even-numbered element of $\breve{\mathbf{x}}_p$ is guaranteed to be identical to the corresponding prime. As an example we have $\breve{\mathbf{x}}_p = \{2, 3, 5, 7, 9, 13, 15, 19\}$ for $n = 8$. Note that the average gap in the vicinity of some $x_j \in \breve{\mathbf{x}}_p$ differs only negligibly from the corresponding average gap in the primes, and $\breve{\mathbf{x}}_p$ therefore exhibits the same global anisotropy inherent to $\mathbf{x}_p$.

Consider four variables with $n \gg 1$ allowed outcomes $\mathbf{x}_p$, $\acute{\mathbf{x}}_p$, $\widetilde{\mathbf{x}}_p$ and $\breve{\mathbf{x}}_p$, respectively, all of which are equally probable. As $\widetilde{\mathbf{x}}_p$ is the least deterministically configured among the four, $\eta(\mathbf{p}_e, \widetilde{\mathbf{x}}_p)$ should be largest. Furthermore, given that the sequence of gaps in $\acute{\mathbf{x}}_p$ ascends uniformly, the deterministic nature of the associated anisotropy is significantly greater than in $\mathbf{x}_p$ and $\breve{\mathbf{x}}_p$. We therefore expect $\eta(\mathbf{p}_e, \acute{\mathbf{x}}_p)$ to be the smallest entropy among the four. Figure 4 shows $\eta(\mathbf{p}_e, \mathbf{x}_p)$, $\eta(\mathbf{p}_e, \acute{\mathbf{x}}_p)$, $\eta(\mathbf{p}_e, \widetilde{\mathbf{x}}_p)$ and $\eta(\mathbf{p}_e, \breve{\mathbf{x}}_p)$ plotted as functions of $n$. Although $\eta(\mathbf{p}_e, \widetilde{\mathbf{x}}_p)$ and $\eta(\mathbf{p}_e, \breve{\mathbf{x}}_p)$ differ only negligibly for $n$ smaller than approximately 200, for all larger $n$ the entropy due to $\widetilde{\mathbf{x}}_p$ is the greatest among the four. The small but non-negligible difference between $\eta(\mathbf{p}_e, \widetilde{\mathbf{x}}_p)$ and $\eta(\mathbf{p}_e, \breve{\mathbf{x}}_p)$ for large $n$ is attributed to the global anisotropy in $\breve{\mathbf{x}}_p$. The entropy due to $\acute{\mathbf{x}}_p$ is the smallest among the four. The behavior of $\eta$ in these calculations is therefore consistent with the stated expectations.

The dramatic difference between $\eta(\mathbf{p}_e, \mathbf{x}_p)$ and $\eta(\mathbf{p}_e, \breve{\mathbf{x}}_p)$ is noteworthy and unexpected. The smaller entropy due to $\mathbf{x}_p$ implies a greater degree of intrinsic determinacy in the arrangement of $\mathbf{x}_p$ along the real axis. Note that the implied deterministic structure could not be associated with the PNT. A detailed analysis of this finding is beyond the present scope and has become the subject of a separate investigation [12]. For the present purposes it is sufficient to mention, as the reader may verify readily, that the number of positive intervals of a given size $d$ among the first $n \gg 1$ primes is strongly correlated to the largest primordial factor of $d$ [12]. That regularity could represent the deterministic property intimated by $\eta(\mathbf{p}_e, \mathbf{x}_p)$.

**Figure 4.** Plots of $\eta(\mathbf{p}_e, \mathbf{x}_p)$, $\eta(\mathbf{p}_e, \acute{\mathbf{x}}_p)$, $\eta(\mathbf{p}_e, \breve{\mathbf{x}}_p)$ and $\eta(\mathbf{p}_e, \widetilde{\mathbf{x}}_p)$, shown with "•", "◁", "◇" and "∗", respectively, measuring intrinsic structure in the primes $\mathbf{x}_p$ and in three variants constructed by changing the order of the first $n-2$ even prime gaps.




## 5. Summary and Conclusions

For finite $\nu$ the entropy attributed to the variable $\mathcal{X}_x$ governed by $\rho_x(\nu, x)$ is simply $\mathcal{H}(\rho_x)$. In the limit as $\nu \to \infty$, however, $\mathcal{X}_x$ becomes a discrete variable $X$ whose behavior is insensitive to the function $\rho_0$ from which $\rho_x$ is constructed. The entropy attributed asymptotically to $X$ is therefore $\mathcal{H}(\rho_x) - \mathcal{H}(\rho_0)$, which recovers the discrete Shannon entropy. Similarly when $\nu$ is finite a quantity of entropy $S(\psi_x)$ is attributed to the conjugated pair $\mathcal{X}_x$ and $\mathcal{K}_x$. The nature of the pair, however, is asymptotically insensitive to $\phi_0$. The entropy attributed to the pair in the limit as $\nu$ diverges is therefore $S(\psi_x) - S(\phi_0)$, whose asymptotic form $H + \Xi$ is defined as $\eta$. Furthermore because the conjugate is asymptotically eliminated $\eta$ must be attributed to $X$ alone.

As $H$ is a function only of $\mathbf{p}$, the existence of some $\mathbf{x}$-dependent contribution to the entropy of a discrete real variable is expected independently of the introduction of $\Xi$. Whereas the form of $H$ uniquely exhibits the properties required for a self-consistent measure of indeterminacy effected by $\mathbf{p}$ [1], the complete set of requirements for a measure of indeterminacy effected by $\mathbf{x}$ and $\mathbf{p}$ is not readily ascertained *a priori*. Among those required properties are certainly translational invariance, scale-invariance, rotational invariance, non-negativity and being bound from above by $H$, all of which were proven for $\eta$ in Section 3. Furthermore the calculations presented in Section 4 demonstrate that $\eta$ behaves in the expected manner over a broad range of different configurations of $\mathbf{x}$ and $\mathbf{p}$.

The somewhat counterintuitive role of $\Xi$ is a natural consequence of the relationship between a given wave-mechanical probability density and the corresponding spectral density. More broadly interpreted, the wave-mechanical origin of $\eta$ may be understood as a natural consequence of a fundamental unity between information theory and quantum theory. Such an interpretation is well-motivated given the profound connections already known. For instance, the bounds on $S$ lead to an uncertainty relation that is stronger than the most general form of the Heisenberg Uncertainty Principle [2]. Even more striking, the basic precepts of quantum theory have been derived from informatic principles [13]. It is important to specify, however, that the validity of $\eta$ is contingent upon no quantum-mechanical premise.

The distinctive feature of $\eta$ is its dependence on $\mathbf{x}$, which is necessary for a complete measure of indeterminacy in discrete real variables. Because of the generality of its form, $\eta$ can be readily manipulated to probe intrinsic organization in sequences of real numbers. Section 4 presented calculations demonstrating the novel capability of $\eta$ in such applications. Future investigations will explore additional applications of $\eta$.

## Acknowledgments

This work was funded by the Naval Innovative Science and Engineering (NISE) Program under USA PUBLIC LAW 110-417, SECTION 219. The author is also grateful to the anonymous reviewers for their helpful suggestions.

## References

1. Shannon, C.E. A mathematical theory of communication. *Bell. Syst. Tech. J.* **1948**, *27*, 379.
2. Bialynicki-Birula, I.; Mycielski, J. Uncertainty relations for information entropy in wave mechanics. *Commun. Math. Phys.* **1975**, *44*, 129–132.
3. Gadre, S.R.; Bendale, R.D. Rigorous relationships among quantum-mechanical kinetic energy and atomic information entropies: Upper and lower bounds. *Phys. Rev. A* **1987**, *36*, 1932–1935.
4. Lalazissis, G.A.; Massen, S.E.; Panos, C.P.; Dimitrova, S.S. Information entropy as a measure of the quality of a nuclear density distribution. *Int. J. Mod. Phys. E* **1998**, *7*, 485–494.
5. Guevara, N.L.; Sagar, R.P.; Esquivel, R.O. Shannon-information entropy sum as a correlation measure in atomic systems. *Phys. Rev. A* **2003**, *67*, 012507.
6. Massen, S.E. Application of information entropy to nuclei. *Phys. Rev. C* **2003**, *67*, 014314.
7. Shi, Q.; Kais, S. Discontinuity of Shannon information entropy for two-electron atoms. *J. Chem. Phys.* **2004**, *309*, 127–131.
8. Bialynicki-Birula, I.; Rudnicki, L. Entropic uncertainty relations in quantum physics. In *Statistical Complexity*; Sen, K.D., Ed.; Springer: Berlin, Heidelberg, Germany, 2007; pp. 1–34.
9. Ghosh, S.K.; Berkowitz, M.; Parr, R.G. Transcription of ground-state density-functional theory into a local thermodynamics. *Proc. Natl. Acad. Sci. USA* **1984**, *81*, 8028.
10. Kullback, S.; Leibler, R.A. On information and sufficiency. *Annals Math. Stat.* **1951**, *22*, 79–86.
11. Feynman, R.P. *Theory of Fundamental Processes*; W. A. Benjamin, Inc.: New York, NY, USA, 1962; pp. 1–6.

12. Cartwright, C.; Funkhouser, S.; Sengupta, D.; Williams, B. Periodicity in the intervals between primes. *Am. J. Comput. Math.* **2012**, submitted for publication.
13. Chiribella, G.; D'Ariano, G.M.; Perinotti, P. Informational derivation of quantum theory. *Phys. Rev. A* **2011**, *84*, 012311.