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Article

Cosmology of $F(T)$ Gravity and k-Essence

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Abstract: This a brief review on $F(T)$ gravity and its relation with k-essence. Modified teleparallel gravity theory with the torsion scalar has recently gained a lot of attention as a possible explanation of dark energy. We perform a thorough reconstruction analysis on the so-called $F(T)$ models, where $F(T)$ is some general function of the torsion term, and deduce the required conditions for the equivalence between of $F(T)$ models with pure kinetic k-essence models. We present a new class of models of $F(T)$-gravity and k-essence.

Keywords: $F(T)$ gravity; k-essence; cosmology; dark energy

1. Introduction

Recent astrophysical data imply that the current expansion of the universe is accelerating [1]. There exist different candidates for this acceleration phase. The simplest one is the introduction of the Cosmological Constant Λ in the framework of General Relativity (ΛCDM model), namely an exotic form of energy (the dark energy) whose Equation of State (EoS) parameter w is equal to minus one and dynamically remains near this value, but in principle quintessence/phantom-fluid description is not excluded. Despite the fact that the ΛCDM is a good candidate to describe our universe, the finite but very small value of the Λ causes some well-known problems, such as the difference between the order of Λ predicted by quantum field theory (a.k.a., fine-tuning), as well as the time where such acceleration happen (a.k.a., the coincidence problem). Further, the origin of dark energy is an unsolved question. Also, the existence of an early accelerated epoch, namely the inflation, introduces a new problem to the standard cosmology, and various proposals have been made to construct acceptable inflationary model, including the scalar, spinor $SU(2)$, (non-)abelian vector theory $(SU(2))$ $U(1)$, *etc.*

Another alternative approach to the dark energy puzzle is the modified gravity theories. A typical modified gravity is a generalization of Einstein's gravity, where some combination of curvature invariants is added into the classical Hilbert–Einstein action of General Relativity. This modification may lead to an accelerated era without invoking the dark energy. The simplest theory of modified gravity is the $F(R)$ one, where the modification is given by a function of the Ricci scalar only. Another popular modification is given by the string-inspired Gauss–Bonnet modified theories, where a modification via the topological invariant four dimensional Gauss–Bonnet G appears (see the recent reviews [2–13]). Also it can be represented by the $f(R, T)$ models where T is the trace of the energy-momentum tensor [14–16]. The field equations of these theories are much more complicated with respect to the case of General Relativity, since they are fourth order differential equations and it is so difficult to obtain the exact solutions.

Recently a new type of gravity model, the $F(T)$-gravity, has been proposed. Its field equations are second order [17,18]. These models are based on the "teleparallel" equivalent of General Relativity (TEGR) [19–25], which, instead of using the curvature defined via the Levi–Civita connection, uses the Weitzenb$\ddot{o}$ck connection that has no curvature but only torsion (see [24,25] for applications to inflation). The fact that the field equations of $F(T)$ gravity are second order makes these theories simpler than the ones where modification is via curvature invariants, and a deeper investigation on this kind of models is of extreme interest (see [26–40] for recent developments).

In this paper we give a brief review on $F(T)$ gravity and its relation with k-essence. We study some $F(T)$-models and models of k-essence. In Sections 2 and 3, we present some basic facts on $F(T)$ gravity. In the Section 4, we study some models of $F(T)$ gravity for the FRW spacetime. Noether symmetry in $F(T)$ gravity was considered in the Section 5. In Section 6, we consider the torsion-scalar model. We investigate k-essence and its models in Sections 7 and 8. Section 9 is devoted to the study of the relation between $F(T)$ gravity and k-essence and in Section 10 we present some generalizations of $F(T)$ gravity. In the last section we give conclusions and general remarks.

2. General Aspects of $F(T)$ Gravity

The action of $F(T)$-gravity reads [17,18,26]

$$S = \int eLd^4x \tag{1}$$

where

$$L = \frac{1}{2\kappa^2}F(T) + L_m \tag{2}$$

Here T is the torsion scalar, $e = \det\left(e^i_\mu\right) = \sqrt{-g}$ and $\mathcal{L}_m$ is the matter Lagrangian. Here e^i_μ are the components of the vierbein vector field $\mathbf{e}_A$ in the coordinate basis $\mathbf{e}_A \equiv e^\mu_A \partial_\mu$. Note that in the teleparallel gravity, the dynamical variable is the vierbein field $\mathbf{e}_A(x^\mu)$. To derive the equations of motion we consider the metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \eta_{ab}\theta^a\theta^b \tag{3}$$

where

$$\theta^a = e^a{}_\mu dx^\mu \ , \ dx^\mu = e_a{}^\mu \theta^a \tag{4}$$

$g_{\mu\nu}$ being the metric of space-time, η_{ab} the Minkowski's metric, θ^a the tetrads and $e^a{}_\mu$ and their inverses $e_a{}^\mu$ the tetrads basis. We note that the tetrad basis satisfy the relations

$$e^a{}_\mu e_a{}^\nu = \delta^\nu_\mu, \quad e^a{}_\mu e_b{}^\mu = \delta^a_b \tag{5}$$

The root of the metric determinant is given by

$$e = \sqrt{-g} = det[e^a{}_\mu] \tag{6}$$

The standard Weitzenböck's connection reads

$$\Gamma^\alpha_{\mu\nu} = e_i{}^\alpha \partial_\nu e^i{}_\mu = -e^i{}_\mu \partial_\nu e_i{}^\alpha \tag{7}$$

Then the components of the torsion and the contorsion are given by

$$T^\alpha{}_{\mu\nu} = \Gamma^\alpha_{\nu\mu} - \Gamma^\alpha_{\mu\nu} = e_i{}^\alpha \left(\partial_\mu e^i{}_\nu - \partial_\nu e^i{}_\mu\right) \tag{8}$$

$$K^{\mu\nu}{}_\alpha = -\frac{1}{2}\left(T^{\mu\nu}{}_\alpha - T^{\nu\mu}{}_\alpha - T_\alpha{}^{\mu\nu}\right) \tag{9}$$

Now we define another tensor from the components of torsion and the contorsion as

$$S_\alpha{}^{\mu\nu} = \frac{1}{2}\left(K^{\mu\nu}{}_\alpha + \delta^\mu_\alpha T^{\beta\nu}{}_\beta - \delta^\nu_\alpha T^{\beta\mu}{}_\beta\right) \tag{10}$$

Finally, we define the torsion scalar as usual

$$T = T^\alpha{}_{\mu\nu} S_\alpha{}^{\mu\nu} \tag{11}$$

Let us derive the equations of motion from the Euler–Lagrange equations. In order to use these equations we first write the quantities

$$\frac{\partial L}{\partial e^a{}_\mu} = F(T) e e_a{}^\mu + e F_T(T) 4 e_a{}^\alpha T^\sigma{}_{\nu\alpha} S_\sigma{}^{\mu\nu} + \frac{\partial L_m}{\partial e^a{}_\mu} \tag{12}$$

and

$$\partial_\alpha\left[\frac{\partial L}{\partial(\partial_\alpha e^a{}_\mu)}\right] = -4F_T(T)\partial_\alpha\left(e e_a{}^\sigma S_\sigma{}^{\mu\nu}\right) - 4 e e_a{}^\sigma S_\sigma{}^{\mu\alpha}\partial_\alpha T\, F_{TT}(T) + \partial_\alpha\left[\frac{\partial L_m}{\partial(\partial_\alpha e^a{}_\mu)}\right] \tag{13}$$

where $F_T(T) = dF(T)/dT$ and $F_{TT}(T) = d^2F(T)/dT^2$. Now we use the Euler–Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial e^a{}_\mu} - \partial_\alpha\left[\frac{\partial \mathcal{L}}{\partial(\partial_\alpha e^a{}_\mu)}\right] = 0 \tag{14}$$

Substituting the expressions (12) and (13) into the later equation, we get the equations of motion of the $F(T)$ gravity (after multiplying by $e^{-1}e^a{}_\beta/4$)

$$S_\beta{}^{\mu\alpha}\partial_\alpha T\, f_{TT}(T) + \left[e^{-1}e^a{}_\beta\partial_\alpha\left(e e_a{}^\sigma S_\sigma{}^{\mu\alpha}\right) + T^\sigma{}_{\nu\beta}S_\sigma{}^{\mu\nu}\right] f_T(T) + \frac{1}{4}\delta^\mu_\beta f(T) = 4\pi\mathcal{T}^\mu_\beta \tag{15}$$

where

$$\mathcal{T}^\mu_\beta = -\frac{e^{-1}e^a{}_\beta}{16\pi}\left\{\frac{\partial \mathcal{L}_{Matter}}{\partial e^a{}_\mu} - \partial_\alpha\left[\frac{\partial \mathcal{L}_{Matter}}{\partial(\partial_\alpha e^a{}_\mu)}\right]\right\} \tag{16}$$

is the gravitational energy momentum tensor.

3. The FRW Space-Time

We will assume a flat homogeneous and isotropic FRW universe with the metric

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1}^{3} (dx^i)^2 \tag{17}$$

where t is cosmic time and $a(t)$ is the scale factor. Then the modified Friedmann equations and the continuity equation read (see, e.g., [17,18,26])

$$\begin{aligned} -2TF_T + F &= 2k^2\rho_m & (18)\\ -8\dot{H}TF_{TT} + (2T - 4\dot{H})F_T - F &= 2k^2 p_m & (19)\\ \dot{\rho}_m + 3H(\rho_m + p_m) &= 0 & (20) \end{aligned}$$

This set can be rewritten as

$$\begin{aligned} -T - 2Tf_T + f &= 2k^2\rho_m & (21)\\ -8\dot{H}Tf_{TT} + (2T - 4\dot{H})(1 + f_T) - T - f &= 2k^2 p_m & (22)\\ \dot{\rho}_m + 3H(\rho_m + p_m) &= 0 & (23) \end{aligned}$$

if we consider the following equivalent form of the action

$$S = \int d^4xe[\frac{1}{2\kappa^2}(T + f(T)) + \mathcal{L}_m] \tag{24}$$

where $f = F - T$. Some properties of $F(T)$-gravity were studied in [18–36]. The field equations (21)–(23) are equivalent to

$$\begin{aligned} \hat{M}_1 F &= 2k^2\rho_m & (25)\\ \hat{M}_2 F &= -\hat{M}_3\hat{M}_1 F = 2k^2 p_m & (26)\\ \hat{M}_3 \rho_m &= -p_m & (27) \end{aligned}$$

where

$$\begin{aligned} \hat{M}_1 &= -2T\partial_T + 1 & (28)\\ \hat{M}_2 &= -8\dot{H}T\partial^2_{TT} + (2T - 4\dot{H})\partial_T - 1 = (4\dot{H}\partial_T - 1)\hat{M}_1 = -(\frac{1}{3H}\partial_t + 1)\hat{M}_1 = -\hat{M}_3\hat{M}_1 & (29)\\ \hat{M}_3 &= \frac{1}{3H}\partial_t + 1 & (30) \end{aligned}$$

By using these equations we may construct high hierarchy of $F(T)$ gravity. For the case $\rho_m = p_m = 0$ such hierarchy is written as

$$\hat{M}_1^n F_n = 0 \tag{31}$$

where $F_1 = F$ and (for $n = 1, 2, 3$)

$$\begin{aligned} -2TF_{1T} + F_1 &= 0 & (32)\\ 4T^2F_{2TT} + F_2 &= 0 & (33)\\ -8T^3F_{3TTT} - 12T^2F_{3TT} - 2TF_{3T} + F_3 &= 0 & (34) \end{aligned}$$

and so on. From the system (25)–(27) one has that any solution of the Equation (25) automatically solves the Equations (26) and (27). It means that by solving the Equation (25), we have also a solution for the Equations (26) and (27). Finally we introduce the effective EoS parameter

$$w_{eff} = -1 - 3^{-1}H^{-1}[\ln(\hat{M}_1 F)]_t = -1 - 3^{-1}[\ln(\hat{M}_1 F)]_N \tag{35}$$

4. Specific Models of $F(T)$ Gravity in FRW Universe

Some explicit models of $F(T)$ gravity have recently appeared in the literature (see, e.g., [17,18,26,27,30,31,34,37]). Here, we would like to present some new models of modified teleparallel gravity.

4.1. Example 1: The M_{13}-Model

Let us consider the M_{13}-model. Its Lagrangian is

$$F(T) = \sum_{j=-m}^{n} \nu_j(t)T^j = \nu_{-m}(t)T^{-m} + ... + \nu_{-1}(t)T^{-1} + \nu_0(t) + \nu_1(t)T + ... + \nu_n(t)T^n \tag{36}$$

We consider the particular case where $m = n = 1$ and $\nu_j =$ **consts**. Thus,

$$F = \nu_{-1}T^{-1} + \nu_0 + \nu_1 T \quad F_T = -\nu_{-1}T^{-2} + \nu_1, \quad F_{TT} = 2\nu_{-1}T^{-3} \tag{37}$$

By substituting these expressions into (18) and (19) we obtain

$$3k^{-2}H^2 = \rho_{eff} + \rho_m \tag{38}$$

$$-k^{-2}(2\dot{H} + 3H^2) = p_{eff} + p_m \tag{39}$$

where

$$\rho_{eff} = k^{-2}[3H^2 - 1.5\nu_{-1}T^{-1} + 0.5\nu_1 T - 0.5\nu_0] \tag{40}$$

$$p_{eff} = k^{-2}[6\nu_{-1}\dot{H}T^{-2} + 1.5\nu_{-1}T^{-1} - 0.5\nu_1 T + 0.5\nu_0 + 2(\nu_1 - 1)\dot{H} - 3H^2] \tag{41}$$

The effective EoS parameter is given by

$$w_{eff} = \frac{p_{eff}}{\rho_{eff}} = \frac{6\nu_{-1}\dot{H}T^{-2} + 1.5\nu_{-1}T^{-1} - 0.5\nu_1 T + 0.5\nu_0 + 2(\nu_1 - 1)\dot{H} - 3H^2}{3H^2 - 1.5\nu_{-1}T^{-1} + 0.5\nu_1 T - 0.5\nu_0} \tag{42}$$

Let us set $\nu_1 = 1$. Thus,

$$\rho_{eff} = k^{-2}[-1.5\nu_{-1}T^{-1} - 0.5\nu_0] \quad p_{eff} = k^{-2}[6\nu_{-1}\dot{H}T^{-2} + 1.5\nu_{-1}T^{-1} + 0.5\nu_0] \tag{43}$$

and

$$w_{eff} = \frac{p_{eff}}{\rho_{eff}} = \frac{6\nu_{-1}\dot{H}T^{-2} + 1.5\nu_{-1}T^{-1} + 0.5\nu_0}{-1.5\nu_{-1}T^{-1} - 0.5\nu_0} = -1 - \frac{6\nu_{-1}\dot{H}T^{-2}}{1.5\nu_{-1}T^{-1} + 0.5\nu_0} \tag{44}$$

4.2. Example 2: The M_{21}-Model

Our next example is the M_{21}-model

$$F = T + \alpha T^{\delta} \ln T \tag{45}$$

Now

$$F_T = 1 + \alpha\delta T^{\delta-1} \ln T + \alpha T^{\delta-1}, \quad F_{TT} = \alpha\delta(\delta-1)T^{\delta-2}\ln T + \alpha(2\delta-1)T^{\delta-2} \tag{46}$$

As a consequence, Equations (18) and (19) take the form

$$-T - 2\alpha T^{\delta} - \alpha(2\delta-1)T^{\delta}\ln T = 2k^2\rho_m \tag{47}$$

$$\alpha(2\delta-1)(T-4\delta\dot{H})T^{\delta-1}\ln T + T - 4\dot{H} + 2\alpha T^{\delta} - 4\alpha\dot{H}(4\delta-1)T^{\delta-1} = 2k^2 p_m \tag{48}$$

One has

$$\rho_{eff} = 0.5k^{-2}[2\alpha T^{\delta} + \alpha(2\delta-1)T^{\delta}\ln T] \tag{49}$$

$$p_{eff} = -0.5k^{-2}\alpha T^{\delta-1}[(2\delta-1)(T-4\delta\dot{H})\ln T + 2T - 4(4\delta-1)\dot{H}] \tag{50}$$

The special case $\delta = 1/2$ deserves a separate consideration. In this case the above equations take a simpler form

$$-T - 2\alpha T^{0.5} = 2k^2\rho_m \quad T - 4\dot{H} + 2\alpha T^{0.5} - 4\alpha\dot{H}T^{-0.5} = 2k^2 p_m \tag{51}$$

For the effective energy density and pressure we get

$$\rho_{eff} = k^{-2}\alpha T^{0.5}, \quad p_{eff} = -k^{-2}\alpha T^{-0.5}(T - 2\dot{H}) \tag{52}$$

4.3. Example 3: The M_{22}-Model

Now we consider the M_{22}-model

$$F = T + f(y), \quad y = \tanh[T] \tag{53}$$

Thus

$$F_T = 1 + f_y(1-y^2) \quad F_{TT} = f_{yy}(1-y^2)^2 - 2y(1-y^2)f_y \tag{54}$$

so that Equations (18) and (19) take the form

$$-T - 2(1-y^2)Tf_y + f = 2k^2\rho_m \tag{55}$$

$$T - 4\dot{H} - 8(1-y^2)^2 T\dot{H}f_{yy} + (16y\dot{H}T + 2T - 4\dot{H})(1-y^2)f_y - f = 2k^2 p_m \tag{56}$$

We have

$$\rho_{eff} = 0.5k^{-2}[2(1-y^2)Tf_y - f] \tag{57}$$

$$p_{eff} = 0.5k^{-2}[8(1-y^2)^2T\dot{H}f_{yy} - (16y\dot{H}T + 2T - 4\dot{H})(1-y^2)f_y + f] \tag{58}$$

The EoS parameter reads

$$w_{eff} = \frac{8(1-y^2)^2T\dot{H}f_{yy} - (16y\dot{H}T + 2T - 4\dot{H})(1-y^2)f_y + f}{2(1-y^2)Tf_y - f} =$$

$$= -1 + \frac{8(1-y^2)^2T\dot{H}f_{yy} - (16y\dot{H}T - 4\dot{H})(1-y^2)f_y + f}{2(1-y^2)Tf_y - f} \tag{59}$$

4.4. Example 4: The M_{25}-Model

In this subsection we will consider the M_{25}-model

$$F = \sum_{-m}^{n} \nu_j(t)\xi^j \tag{60}$$

where $\xi = \ln T$. We take the case $m = n = 1$ and $\nu_j =$ **consts**, namely

$$F = \nu_{-1}\xi^{-1} + \nu_0 + \nu_1\xi \tag{61}$$

Thus

$$F_\xi = -\nu_{-1}\xi^{-2} + \nu_1 \quad F_{\xi\xi} = 2\nu_{-1}\xi^{-3} \tag{62}$$

and

$$F_T = (-\nu_{-1}\xi^{-2} + \nu_1)e^{-\xi} \quad F_{TT} = (2\nu_{-1}\xi^{-3} + \nu_{-1}\xi^{-2} - \nu_1)e^{-2\xi} \tag{63}$$

In this case, Equations (18) and (19) lead to

$$2\nu_{-1}\xi^{-2} + \nu_{-1}\xi^{-1} + \nu_0 - 2\nu_1 + \nu_1\xi = 2k^2\rho_m \tag{64}$$

$$-4\dot{H}(4\nu_{-1}\xi^{-3} + \nu_{-1}\xi^{-2} - \nu_1)e^{-\xi} - 2\nu_{-1}\xi^{-2} - \nu_{-1}\xi^{-1} + 2\nu_1 - \nu_0 - \nu_1\xi = 2k^2p_m \tag{65}$$

5. Noether Symmetry in $F(T)$ Gravity

In this section we want to present a brief review on Noether symmetry in $F(T)$ gravity following to the paper [41]. Generally speaking, Noether symmetry is a power method to select models motivated at a fundamental level. It also allows to construct the exact solution of the model. We start from the point-like Lagrangian of $F(T)$ gravity:

$$\mathcal{L}(a, \dot{a}, T, \dot{T}) = a^3\left(f - f_T T\right) - 6f_T a\dot{a}^2 - \rho_{m0} \tag{66}$$

We now use the Euler–Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \tag{67}$$

where q_i are the generalized coordinates of the phase space and $q_i = a$ and T. Then we have

$$a^3 f_{TT}\left(T + 6\frac{\dot{a}^2}{a^2}\right) = 0 \tag{68}$$

$$f - f_T T + 2f_T H^2 + 4\left(f_T\frac{\ddot{a}}{a} + Hf_{TT}\dot{T}\right) = 0 \tag{69}$$

Hence as $f_{TT} \neq 0$ we obtain

$$T = -6\frac{\dot{a}^2}{a^2} = -6H^2 \tag{70}$$

that is the Euler constraint of the dynamics. Next using $\ddot{a}/a = H^2 + \dot{H}$, we obtain

$$48H^2 f_{TT}\dot{H} - 4f_T\left(3H^2 + \dot{H}\right) - f = 0 \tag{71}$$

i.e., the modified second Friedmann equation. Now let us consider the Hamiltonian corresponding to Lagrangian $\mathcal{L}$ [41]:

$$\mathcal{H} = \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\dot{q}_i - \mathcal{L} \tag{72}$$

so that

$$\mathcal{H}(a, \dot{a}, T, \dot{T}) = a^3\left(-6f_T\frac{\dot{a}^2}{a^2} - f + f_T T + \frac{\rho_{m0}}{a^3}\right) \tag{73}$$

Assuming that the total energy $\mathcal{H} = 0$ (Hamiltonian constraint) and from Equation (70), we get

$$12H^2 f_T + f = \frac{\rho_{m0}}{a^3} \tag{74}$$

that is nothing but the first Friedmann equation.

Now we want to present the Noether symmetry for $F(T)$ gravity in the FRW metric case. To do it, we introduce the generator of Noether symmetry as [41]

$$\mathbf{X} = \alpha\frac{\partial}{\partial a} + \beta\frac{\partial}{\partial T} + \dot{\alpha}\frac{\partial}{\partial \dot{a}} + \dot{\beta}\frac{\partial}{\partial \dot{T}} \tag{75}$$

where $\alpha = \alpha(a, T)$ and $\beta = \beta(a, T)$. As is well known, Noether symmetry exists if the equation

$$L_{\mathbf{X}}\mathcal{L} = \mathbf{X}\mathcal{L} = \alpha\frac{\partial \mathcal{L}}{\partial a} + \beta\frac{\partial \mathcal{L}}{\partial T} + \dot{\alpha}\frac{\partial \mathcal{L}}{\partial \dot{a}} + \dot{\beta}\frac{\partial \mathcal{L}}{\partial \dot{T}} = 0 \tag{76}$$

has solution. Here $L_{\mathbf{X}}\mathcal{L}$ is the Lie derivative of the Lagrangian $\mathcal{L}$ with respect to the vector $\mathbf{X}$. The corresponding Noether charge reads as

$$Q_0 = \sum_i \alpha_i\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \alpha\frac{\partial \mathcal{L}}{\partial \dot{a}} + \beta\frac{\partial \mathcal{L}}{\partial \dot{T}} = const \tag{77}$$

From Equation (76) and using the relations $\dot{\alpha} = (\partial\alpha/\partial a)\,\dot{a} + (\partial\alpha/\partial T)\,\dot{T}$, $\dot{\beta} = (\partial\beta/\partial a)\,\dot{a} + (\partial\beta/\partial T)\,\dot{T}$, we come to the equation

$$3\alpha a^2\left(f - f_T T\right) - \beta a^3 f_{TT}T - 6\dot{a}^2\left(\alpha f_T + \beta a f_{TT} + 2af_T\frac{\partial\alpha}{\partial a}\right) - 12a\dot{a}\dot{T}\frac{\partial\alpha}{\partial T} = 0 \tag{78}$$

Now we impose that the coefficients of $\dot{a}^2$, $\dot{T}^2$ and $\dot{a}\dot{T}$ in Equation (78) to be zero. Then we get

$$a\frac{\partial\alpha}{\partial T} = 0 \tag{79}$$

$$\alpha f_T + \beta a f_{TT} + 2af_T\frac{\partial\alpha}{\partial a} = 0 \tag{80}$$

$$3\alpha a^2\left(f - f_T T\right) - \beta a^3 f_{TT}T = 0 \tag{81}$$

As is known, the constraint (81) is sometimes called Noether condition. The corresponding Noether charge looks like

$$Q_0 = -12\alpha f_T a\dot{a} = const \tag{82}$$

From Equation (79) it follows that $\alpha = \alpha(a)$. On the other hand, Equation (81) gives us

$$\beta a f_{TT}T = 3\alpha\left(f - f_T T\right) \tag{83}$$

Hence we have

$$f_T T\left(2a\frac{d\alpha}{da}-2\alpha\right)+3\alpha f=0 \tag{84}$$

which we recast as

$$1-\frac{a}{\alpha}\frac{d\alpha}{da}=\frac{3f}{2f_T T} \tag{85}$$

We split the last equations into two equations as

$$nf-f_T T = 0 \tag{86}$$

$$1-\frac{a}{\alpha}\frac{d\alpha}{da}-\frac{3}{2n} = 0 \tag{87}$$

These equations have the solutions [41]

$$f(T) = \mu T^n \tag{88}$$

$$\alpha(a) = \alpha_0\, a^{1-3/(2n)} \tag{89}$$

where μ and α_0 are real constants. So from Equation (83), we get

$$\beta(a,T)=-\frac{3\alpha_0}{n}\,a^{-3/(2n)}\,T \tag{90}$$

Finally we can conclude that the existence of explicit non-zero solutions of $f(T)$, α and β, implies the existence of Noether symmetry. Note that Noether symmetry allows us to construct the exact solution of $a(t)$ for the given $f(T)$ model. For example, from Equation (82) it follows [41]

$$a^{c_1}\,\dot{a}=c_2 \tag{91}$$

where

$$c_1=\frac{3}{2n}-1\,,\qquad c_2=\left[\frac{Q_0}{-12\alpha_0\mu n(-6)^{n-1}}\right]^{1/(2n-1)} \tag{92}$$

Its solution reads as

$$a(t)=-(1+c_1)(c_3-c_2t)^{1/(1+c_1)}=(-1)^{1+2n/3}\cdot\frac{3}{2n}\,(c_2t-c_3)^{2n/3} \tag{93}$$

where $c_3=$ **conts**. This solution describes the accelerated expansion of the universe as

$$a(t)\sim t^{2n/3} \tag{94}$$

where its prefactor $(-1)^{1+2n/3}\cdot\frac{3}{2n}\,c_2^{2n/3}$ is not important. As is well-known, in order to get the expanding universe, the constraint $n>0$ is required.

6. The Torsion–Scalar Model

In this section we would like to study the $F(T)$ gravity in the presence of matter whose Lagrangian is

$$L_m=\frac{1}{2}\epsilon\dot{\phi}^2-V(\phi) \tag{95}$$

where ϕ is a scalar field and $V(\phi)$ is the potential depending on ϕ. The equations of motion assume the form

$$-2TF_T + F = 2k^2[\frac{1}{2}\epsilon\dot{\phi}^2 + V] \quad (96)$$

$$-8\dot{H}TF_{TT} + (2T - 4\dot{H})F_T - F = 2k^2[\frac{1}{2}\epsilon\dot{\phi}^2 - V] \quad (97)$$

$$\ddot{\phi} + 3H\dot{\phi} + \epsilon\frac{\partial V}{\partial \phi} = 0 \quad (98)$$

where $\epsilon = 1$ for the usual case and $\epsilon = -1$ for the phantom case. From this system we get

$$\epsilon\dot{\phi}^2 = -8\dot{H}TF_{TT} - 4\dot{H}F_T, \quad V = 4\dot{H}TF_{TT} - 2(T - \dot{H})F_T + F \quad (99)$$

where dot denotes the derivative with respect to the time. If we compare these equations with (23)–(25) we have

$$\rho = \frac{1}{2}\epsilon\dot{\phi}^2 + V, \quad p = \frac{1}{2}\epsilon\dot{\phi}^2 - V \quad (100)$$

For simplicity we restrict ourself to the case $F = \alpha T + \beta T^{0.5}$. Thus,

$$\epsilon\dot{\phi}^2 = -4\alpha\dot{H}, \quad V = -\alpha T + 2\alpha\dot{H} \quad (101)$$

and

$$w = -1 + 4\frac{\dot{H}}{T} \quad (102)$$

Let us consider some examples.

6.1. Example 1: $a = \delta\sinh^m[\mu t]$

In our first example we consider the following form for the scale factor

$$a = \delta\sinh^m[\mu t] \quad (103)$$

As a consequence

$$H = \mu m\coth[\mu t], \quad \dot{H} = -\frac{\mu^2 m}{\sinh^2[\mu t]}, \quad \dot{\phi}^2 = \frac{4\alpha\mu^2 m}{\epsilon\sinh^2[\mu t]} \quad (104)$$

So we obtain

$$\phi = \phi_0 \pm 2\sqrt{\frac{\alpha\mu^2 m}{\epsilon}}\log[\tanh[\frac{\mu t}{2}]], \quad V = 6\alpha m^2\mu^2\coth^2[\mu t] - \frac{2\alpha\mu^2 m}{\sinh^2[\mu t]} \quad (105)$$

and the potential takes the form ($\tanh[\frac{\mu t}{2}] = e^{\pm\frac{\phi-\phi_0}{2\sqrt{\alpha\mu^2 m\epsilon^{-1}}}}$)

$$V = 3\alpha m^2\mu^2[\frac{1 + e^{\pm\frac{\phi-\phi_0}{\sqrt{\alpha\mu^2 m\epsilon^{-1}}}}}{e^{\pm\frac{\phi-\phi_0}{2\sqrt{\alpha\mu^2 m\epsilon^{-1}}}}}] - \frac{\alpha\mu^2 m(1 - e^{\pm\frac{\phi-\phi_0}{\sqrt{\alpha\mu^2 m\epsilon^{-1}}}})^2}{2e^{\pm\frac{\phi-\phi_0}{\sqrt{\alpha\mu^2 m\epsilon^{-1}}}}} \quad (106)$$

6.2. Example 2: $a = a_0 e^{\beta t^m}$

Let us consider the case $a = a_0 e^{\frac{\delta}{m+1} t^{m+1}}$. Thus, $H = \delta t^m$ and we have

$$t = [\frac{(\phi - \phi_0)(m+1)}{\pm 4\sqrt{-\alpha m \delta \epsilon^{-1}}}]^{\frac{2}{m+1}}, \quad \epsilon \dot{\phi}^2 = -4\alpha m \delta t^{m-1} \tag{107}$$

such that

$$\phi = \phi_0 \pm \frac{4\sqrt{-\alpha m \delta \epsilon^{-1}}}{m+1} t^{\frac{m+1}{2}}, \quad V = 6\alpha\delta^2 t^{2m} + 2\alpha m \delta t^{m-1} \tag{108}$$

We finally get

$$V = 6\alpha\delta^2 [\frac{(\phi - \phi_0)(m+1)}{\pm 4\sqrt{-\alpha m \delta \epsilon^{-1}}}]^{\frac{4m}{m+1}} + 2\alpha m \delta [\frac{(\phi - \phi_0)(m+1)}{\pm 4\sqrt{-\alpha m \delta}}]^{\frac{2(m-1)}{m+1}} \tag{109}$$

6.3. Example 3: $a = a_0 t^n$

The next example is given by

$$a = a_0 t^n \tag{110}$$

for which

$$H = \frac{n}{t}, \quad \dot{H} = -\frac{n}{t^2}, \quad \epsilon\dot{\phi}^2 = \frac{4\alpha n}{t^2}, \quad \phi - \phi_0 = \pm 2\sqrt{\alpha n \epsilon^{-1}} \ln[t], \quad t = e^{\pm \frac{\phi - \phi_0}{2\sqrt{\alpha n \epsilon^{-1}}}} \tag{111}$$

and

$$V = 2\alpha n(3n-1)t^{-2} \tag{112}$$

The potential assumes the final form

$$V = 2\alpha n(3n-1)e^{\mp \frac{\phi - \phi_0}{\sqrt{\alpha n \epsilon^{-1}}}} \tag{113}$$

7. The k-Essence

The action of k-essence reads [42–45]

$$S = \int d^4x \sqrt{-g}[\frac{1}{2\kappa^2}R + K(X,\phi) + L_m] \tag{114}$$

The corresponding (closed) set of equations for FRW metric (17) is

$$3k^{-2}H^2 = 2XK_X - K + \rho_m \tag{115}$$

$$-k^{-2}(2\dot{H} + 3H^2) = K + p_m \tag{116}$$

$$(K_X + 2XK_{XX})\dot{X} + 6HXK_X - K_\phi = 0 \tag{117}$$

$$\dot{\rho_m} + 3H(\rho_m + p_m) = 0 \tag{118}$$

where $X = -(1/2)\dot{\phi}^2$. The equation for the scalar field ϕ is given by

$$-(a^3\dot{\phi}K_X)_t = a^3K_\phi \tag{119}$$

which corresponds to the Equation (117). In the pure kinetic k-essence case we have $K_\phi = 0$ and from the last equation one has (see, e.g., [46])

$$a^3\dot{\phi}K_X = a^3\sqrt{-2X}K_X = \sqrt{\kappa} = const \tag{120}$$

8. Models of k-Essence for FRW Universe

In what follows we will present some new models of k-essence. All of them may give rise to cosmic acceleration.

8.1. Example 1: The M_{12}-Model

Let us consider the M_{12}-model with the following Lagrangian

$$K = \nu_{-m}(N)N^{-m} + ... + \nu_{-1}(N)N^{-1} + \nu_0(N) + \nu_1(N)N + ... + \nu_n(N)N^n \tag{121}$$

where in general $\nu_j = \nu_j(\phi) = \nu_j(N)$ and $N = \ln{(aa_0^{-1})}$. We study the case $m = 0, n = 2, \nu_j = const$. The M_{12}-model becomes

$$K = \nu_0 + \nu_1 N + \nu_2 N^2 \tag{122}$$

To find ν_j and X we look for H in the form

$$H = \mu_0 + \mu_1 N \tag{123}$$

where $\mu_j =$ **consts** [in general $\mu_j = \mu_j(t)$]. This solution corresponds to the scale factor

$$a = a_0 e^N \tag{124}$$

Finally, we obtain the following parametric form of the M_{12}-model (parametric pure kinetic k-essence)

$$K = -(2\mu_0\mu_1 + 3\mu_0^2) - 2\mu_1(\mu_1 + 3\mu_0)N - 3\mu_1^2 N^2 \tag{125}$$

$$X = k^{-1}a_0^6\mu_1^2(\mu_0 + \mu_1 N)^2 e^{6N} \tag{126}$$

8.2. Example 2: The M_1-Model

Our next example is the M_1-model, whose Lagrangian assumes the form

$$K = \nu_{-m}(t)t^{-m} + ... + \nu_{-1}(t)t^{-1} + \nu_0(t) + \nu_1(t)t + ... + \nu_n(t)t^n \tag{127}$$

where in general $\nu_j = \nu_j(\phi) = \nu_j(t)$. Let us explore this model for the case: $m = 0, n = 2$ and $\nu_j = consts$. In this case the M_1-model takes the form

$$K = \nu_0 + \nu_1 t + \nu_2 t^2 \tag{128}$$

To find ν_j and X we look for the following form of H,

$$H = \mu_0 + \mu_1 t \tag{129}$$

so that

$$a = a_0 e^{\mu_0 t + 0.5\mu_1 t^2} \tag{130}$$

where $\mu_j = consts$ [in general $\mu_j = \mu_j(t)$]. As a consequence, we obtain the following explicit form of the k-essence Lagrangian

$$K = -(2\mu_1 + 3\mu_0^2) - 6\mu_0\mu_1 t - 3\mu_1^2 t^2 \tag{131}$$

We also have

$$2XK_X = 3H^2 + K = -2\dot{H} = -2\mu_1 \tag{132}$$

For X we get the following expression

$$X = \gamma_2^{-1} e^{6\mu_0 t + 3\mu_1 t^2}, \quad \gamma_2^{-1} = \kappa^{-1} a_0^6 \mu_1^2 \tag{133}$$

from which

$$t = \frac{1}{3\mu_1}[-3\mu_0 \pm \sqrt{9\mu_0^2 + 3\mu_1 \ln(\gamma_2 X)}] \tag{134}$$

Finally, we reconstruct the M_{23}-model

$$K = -2\mu_1 - 3\mu_0^2 - \mu_1 \ln[\gamma_2 X] = \nu_0 + \nu_1 \ln X \tag{135}$$

We recall that in general the M_{23}-model is read as

$$K = \nu_{-m}(t)\zeta^{-m} + ... + \nu_{-1}(t)\zeta^{-1} + \nu_0(t) + \nu_1(t)\zeta + ... + \nu_n(t)\zeta^n \tag{136}$$

where $\zeta = \ln X$.]

8.3. Example 3: The M_{24}-Model

Here we present the M_{24}-model

$$K = \frac{2m\lambda\sigma^2(-2\beta v + \lambda v^2 + \lambda)(1 - v^2)}{(\beta - \lambda v)^2} - 3[n - \frac{m\lambda\sigma(1 - v^2)}{\beta - \lambda v}]^2 \tag{137}$$

$$X = \gamma_3(2\beta v - \lambda v^2 - \lambda)^2(1 - v^2)^2(\beta - \lambda v)^{6m-4} \tag{138}$$

where $\gamma_3 = \kappa^{-1}\alpha^6 m^2\lambda^2\sigma^6$, $v = \tanh[\sigma t]$ and $\lambda, \sigma, \alpha, \beta, n, m$ are some constants. Solving the Equation (116) we obtain

$$H = n - \frac{m\lambda\sigma(1-v^2)}{\beta - \lambda v} \tag{139}$$

from which we derive the scale factor as

$$a = \alpha[\beta - \lambda v]^m e^{nt} \tag{140}$$

Note that

$$\dot{H} = \frac{m\lambda\sigma^2(2\beta v - \lambda v^2 - \lambda)(1-v^2)}{(\beta - \lambda v)^2} \tag{141}$$

9. The Relation between $F(T)$-Gravity and k-Essence in the FRW Universe

In this section, we want to analyze the relation between modified teleparallel gravity and pure kinetic k-essence. Note that we can also consider this relation in the context of general modified gravity theories.

9.1. General Case

9.1.1. Version-I

Let us consider the following transformation

$$K = 8\dot{H}Tf_{TT} - 2(T - 2\dot{H})f_T + f \tag{142}$$

$$X = \kappa^{-1}k^{-4}a^6[\dot{H} + 0.5k^2(\rho_m + p_m)]^2 \tag{143}$$

where $T = -6H^2$. Thus Equations (21)–(23) take the form

$$0 = -3k^{-2}H^2 + 2XK_X - K + \rho_m \tag{144}$$

$$0 = k^{-2}(2\dot{H} + 3H^2) + K + p_m \tag{145}$$

$$(K_X + 2XK_{XX})\dot{X} + 6HXK_X = 0 \tag{146}$$

$$\dot{\rho_m} + 3H(\rho_m + p_m) = 0 \tag{147}$$

These are the equations of motion of pure kinetic k-essence. This result shows that the field equations of modified teleparallel gravity and pure kinetic k-essence are equivalent to each other. This equivalence permits to construct a new class of pure kinetic k-essence models starting from some models of modified teleparallel gravity. Let us see it for the following modified teleparallel gravity model: $f(T) = \alpha T^n$ [17,18]. In this case, we have

$$f_T = \alpha n T^{n-1}, \quad f_{TT} = \alpha n(n-1)T^{n-2} \tag{148}$$

Substituting these expressions into the Equations (142) and (143) we get

$$K = 8\alpha n(n-1)\dot{H}T^{n-1} - 2\alpha n(T - 2\dot{H})T^{n-1} + \alpha T^n \tag{149}$$

$$X = \kappa^{-1}k^{-4}a^6[\dot{H} + 0.5k^2(\rho_m + p_m)]^2 \tag{150}$$

Let us consider some specific cases.

(i) If the scale factor behaves as $a = a_0 e^{g(t)}$ so that $H = \dot{g}, \dot{H} = \ddot{g}$, K and X take the form

$$K = 8\alpha n(n-1)\ddot{g}(-6)^{n-1}\dot{g}^{2(n-1)} - 2\alpha n(-6\dot{g}^2 - 2\ddot{g})(-6)^{n-1}\dot{g}^{2(n-1)} + \alpha(-6)^n\dot{g}^{2n} \tag{151}$$

$$X = \kappa^{-1}k^{-4}a^6\ddot{g}^2 \tag{152}$$

If we now consider the simplest case $g = t$ (it means, $\dot{g} = 1, \ddot{g} = 0$), we get

$$K = -2\alpha n(-6)^n + \alpha(-6)^n = (1-2n)\alpha(-6)^n \tag{153}$$

$$X = 0 \tag{154}$$

(ii) A non-trivial model may be obtained from $a = a_0 t^m$. In this case $H = mt^{-1}, \dot{H} = -mt^{-2}, T = \frac{-6m^2}{t^2}$ and K and X take the form

$$K = 8\alpha n(n-1)\dot{H}(\frac{-6m^2}{t^2})^{n-1} - 2\alpha n(\frac{-6m^2}{t^2} - 2\dot{H})(\frac{-6m^2}{t^2})^{n-1} + \alpha(\frac{-6m^2}{t^2})^n \tag{155}$$

$$X = \kappa^{-1}k^{-4}a_0^6 m^2 t^{6m-4} \tag{156}$$

or

$$K = 2\alpha m(-6m^2)^{n-1}[-4n(n-1) + 2n(1-3m) + 3m]t^{-2n} \tag{157}$$

$$X = \kappa^{-1}k^{-4}a_0^6 m^2 t^{6m-4} = \gamma_5^{-1}t^{6m-4} \tag{158}$$

Since $t = (\gamma_5 X)^{\frac{1}{6m-4}}$ we finally get the following pure kinetic k-essence model

$$K = 2\alpha m(-6m^2)^{n-1}[-4n(n-1) + 2n(1-3m) + 3m](\gamma_5 X)^{\frac{n}{2-3m}} \tag{159}$$

9.1.2. Version-II

Let us rewrite Equations (21)–(23) as

$$3k^{-2}H^2 = \rho_{eff} + \rho_m \tag{160}$$

$$-k^{-2}(2\dot{H} + 3H^2 = p_{eff} + p_m \tag{161}$$

$$\dot{\rho}_m + 3H(\rho_m + p_m) = 0 \tag{162}$$

where

$$\rho_{eff} = 2Tf_T - f, \quad p_{eff} = 8\dot{H}Tf_{TT} - 2(T - 2\dot{H})f_T + f \tag{163}$$

We introduce the following two functions K and X,

$$K = 8\dot{H}Tf_{TT} - 2(T - 2\dot{H})f_T + f, \quad X = \frac{4\dot{H}^2(2Tf_{TT} + f_T)^2}{\kappa a^{-6}} \tag{164}$$

These functions belong to the system of the Equations (144)–(147).

9.2. Specific Case: $\phi = \phi_0 + \ln a^{\pm\sqrt{12}}$

One specific interesting case is given by

$$\phi = \phi_0 + \ln a^{\pm\sqrt{12}} \tag{165}$$

It deserves separate investigation. In fact for this case $\dot{\phi} = \pm\sqrt{12}H$ so that $X = -0.5\dot{\phi}^2 = -6H^2 = T$. The corresponding continuity equation is

$$\ddot{\phi}(f_T - \dot{\phi}^2 f_{TT}) + 3H\dot{\phi}f_T = 0 \tag{166}$$

or, in terms of T,

$$(f_T + 2Tf_{TT})\dot{T} + 6HTf_T = 0 \tag{167}$$

where $\rho' = 2Tf_T - f, \quad p' = f$ and $\dot{\rho}' + 3H(\rho' + p') = 0$. Let us split the Equation (22) into two separate equations,

$$4\dot{H}Tf_{TT} - (T - 2\dot{H})f_T = 0 \tag{168}$$

and

$$-4\dot{H} + T - f = 2k^2 p_m \tag{169}$$

Equation (168) is automatically satisfied since it is just an another form for the continuity Equation (166). So we finally obtain the equation system for $F(T)$-gravity, which takes the form

$$-T - 2Tf_T + f = 2k^2\rho_m \tag{170}$$

$$-4\dot{H} + T - f = 2k^2 p_m \tag{171}$$

$$(f_T + 2T f_{TT})\dot{T} + 6HT f_T = 0 \tag{172}$$

$$\dot{\rho}_m + 3H(\rho_m + p_m) = 0 \tag{173}$$

After the identification $T = X = -6H^2$ and $f = 2k^2 K$, we recover Equations (144)–(147). So we can conclude that for the special case (165) both $F(T)$-gravity and pure kinetic k-essence are equivalent to each other at least at the level of the dynamical equations. Some remarks can be observed from the continuity Equation (166)[=(167)=(168)]. Two integrals of motion ($I_{jT} = 0$) appear:

$$I_1 = a_0^{-3} a^3 T^{0.5} f_T, \quad I_2 = f - a^3 T^{0.5} f_T \partial_T^{-1}(a^{-3} T^{-0.5}) \tag{174}$$

Their general solution is given by

$$f = C_2 + iC_1 a_0^2 \partial_T^{-1}(a^{-3} T^{-0.5}), \quad C_j = const \tag{175}$$

Finally we would like to present an exact solution for both $F(T)$-gravity and pure kinetic k-essence. Let us consider the ΛCDM model for which $a^{-3} = -\frac{1}{2\rho_0}(T + 2\Lambda) = -\frac{1}{2\rho_0}(X + 2\Lambda)$ so that

$$f = f(X) = f(T) = C_2 - \frac{iC_1 a_0^3}{3\rho_0}(T^{1.5} + 6\Lambda T^{0.5}) = C_2 - \frac{iC_1 a_0^3}{3\rho_0}(X^{1.5} + 6\Lambda X^{0.5}) \tag{176}$$

which is the M_{32}-model. This is the exact solution of the equations of motion of pure kinetic k-essence and $F(T)$-gravity simultaneously.

10. $F(R, T)$ Gravity

We have just considered one generalization of $F(T)$ in the presence of scalar field. In this section we would like to present another possible generalization of $F(T)$ gravity, namely the so-called $F(R, T)$ gravity.

10.1. The M_{37}-Model

The action of M_{37}-gravity is given by [36]

$$S_{37} = \int d^4x \sqrt{-g}[F(R, T) + \mathcal{L}_m] \tag{177}$$

where $\mathcal{L}_m$ is the matter Lagrangian, $\epsilon_i = \pm 1$ (signature) and

$$\begin{aligned} R &= u + \epsilon_1 g^{\mu\nu} R_{\mu\nu} && (178) \\ T &= v + \epsilon_2 S_\rho{}^{\mu\nu} T^\rho{}_{\mu\nu} && (179) \end{aligned}$$

Here $u = u(x_i; g_{ij}, \dot{g_{ij}}, \ddot{g_{ij}}, ...; f_j)$ and $v = v(x_i; g_{ij}, \dot{g_{ij}}, \ddot{g_{ij}}, ...; g_j)$ are some functions to be defined. Now we work in the FRW universe with the metric (18). In this case the curvature and torsion scalars can be written as

$$\begin{aligned} R &= u + 6\epsilon_1(\dot{H} + 2H^2) && (180) \\ T &= v + 6\epsilon_2 H^2 && (181) \end{aligned}$$

where, $u = u(t, a, \dot{a}, \ddot{a}, \dddot{a}, ...; f_i)$ and $v = v(t, a, \dot{a}, \ddot{a}, \dddot{a}, ...; g_i)$ are some real functions, $H = (\ln a)_t$, while f_i and g_i are some unknown functions related with the geometry of the spacetime. By introducing the Lagrangian multipliers we can now rewrite the action (221) as

$$S_{37} = \int dt\, a^3 \left\{ F(R,T) - \lambda \left[T - v - 6\epsilon_2 \frac{\dot{a}^2}{a^2} \right] - \gamma \left[R - u - 6\epsilon_1 \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) \right] + L_m \right\} \tag{182}$$

where λ and γ are Lagrange multipliers. If we take the variations with respect to T and R of this action we get

$$\lambda = F_T, \quad \gamma = F_R \tag{183}$$

Therefore, the action (182) can be rewritten as

$$S_{37} = \int dt\, a^3 \left\{ F(R,T) - F_T \left[T - v - 6\epsilon_2 \frac{\dot{a}^2}{a^2} \right] - F_R \left[R - u - 6\epsilon_1 \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) \right] + L_m \right\} \tag{184}$$

Then the corresponding point-like Lagrangian reads as

$$L_{37} = a^3[F - (T-v)F_T - (R-u)F_R + L_m] - 6(\epsilon_1 F_R - \epsilon_2 F_T)a\dot{a}^2 - 6\epsilon_1(F_{RR}\dot{R} + F_{RT}\dot{T})a^2\dot{a} \tag{185}$$

We finally obtain the following equations of the M$_{37}$-model [36]:

$$\begin{aligned} D_2 F_{RR} + D_1 F_R + J F_{RT} + E_1 F_T + KF &= -2a^3\rho \\ U + B_2 F_{TT} + B_1 F_T + C_2 F_{RRT} + C_1 F_{RTT} + C_0 F_{RT} + MF &= 6a^2 p \\ \dot{\rho} + 3H(\rho + p) &= 0 \end{aligned} \tag{186}$$

Here

$$D_2 = -6\epsilon_1 \dot{R} a^2 \dot{a} \tag{187}$$
$$D_1 = 6\epsilon_1 a^2 \ddot{a} + a^3 u_{\dot{a}} \dot{a} \tag{188}$$
$$J = -6\epsilon_1 a^2 \dot{a} \dot{T} \tag{189}$$
$$E_1 = 12\epsilon_2 a \dot{a}^2 + a^3 v_{\dot{a}} \dot{a} \tag{190}$$
$$K = -a^3 \tag{191}$$

and

$$U = A_3 F_{RRR} + A_2 F_{RR} + A_1 F_R \tag{192}$$
$$A_3 = -6\epsilon_1 \dot{R}^2 a^2 \tag{193}$$
$$A_2 = -12\epsilon_1 \dot{R} a \dot{a} - 6\epsilon_1 \ddot{R} a^2 + a^3 \dot{R} u_{\dot{a}} \tag{194}$$
$$A_1 = 12\epsilon_1 \dot{a}^2 + 6\epsilon_1 a \ddot{a} + 3a^2 \dot{a} u_{\dot{a}} + a^3 \dot{u}_{\dot{a}} - a^3 u_a \tag{195}$$
$$B_2 = 12\epsilon_2 \dot{T} a \dot{a} + a^3 \dot{T} v_{\dot{a}} \tag{196}$$
$$B_1 = 24\epsilon_2 \dot{a}^2 + 12\epsilon_2 a \ddot{a} + 3a^2 \dot{a} v_{\dot{a}} + a^3 \dot{v}_{\dot{a}} - a^3 v_a \tag{197}$$
$$C_2 = -12\epsilon_1 a^2 \dot{R} \dot{T} \tag{198}$$
$$C_1 = -6\epsilon_1 a^2 \dot{T}^2 \tag{199}$$
$$C_0 = -12\epsilon_1 \dot{T} a \dot{a} + 12\epsilon_2 \dot{R} a \dot{a} - 6\epsilon_1 a^2 \ddot{T} + a^3 \dot{R} v_{\dot{a}} + a^3 \dot{T} u_{\dot{a}} \tag{200}$$
$$M = -3a^2 \tag{201}$$

The M_{37}-model (221) admits some interesting particular and physically important cases. Let us see some example.

(i) $F(R)$-*gravity*. If the model is independent of the torsion, namely $F = F(R,T) = F(R)$, and we assume that $u = 0$, the action (221) takes the form

$$\mathcal{S}_R = \int d^4xe[F(R) + L_m] \tag{202}$$

where

$$R = \epsilon_1 g^{\mu\nu} R_{\mu\nu} \tag{203}$$

is the curvature scalar. We work with the FRW metric (222). In this case R assumes the form

$$R = 6(\dot{H} + 2H^2) \tag{204}$$

We rewrite the action as

$$\mathcal{S}_R = \int dtL_R \tag{205}$$

where the Lagrangian is given by

$$L_R = a^3(F - RF_R + L_m) - 6\epsilon_1 F_R a\dot{a}^2 - 6\epsilon_1 F_{RR}\dot{R}a^2\dot{a} \tag{206}$$

The corresponding field equations of $F(R)$ gravity read

$$\begin{aligned} 6\dot{R}HF_{RR} - (R - 6H^2)F_R + F &= \rho && (207)\\ -2\dot{R}^2F_{RRR} + [-4\dot{R}H - 2\ddot{R}]F_{RR} + [-2H^2 - 4a^{-1}\ddot{a} + R]F_R - F &= p && (208)\\ \dot{\rho} + 3H(\rho + p) &= 0 && (209) \end{aligned}$$

(ii) $F(T)$-*gravity*. Now we assume that the function $F = F(R,T)$ is independent of the curvature scalar R and $v = 0$. In this case we get the modified teleparallel gravity—$F(T)$ gravity. Its gravitational action is

$$\mathcal{S}_T = \int d^4xe[F(T) + L_m] \tag{210}$$

where $e = \det\,(e^i_\mu) = \sqrt{-g}$. The torsion scalar T is defined as

$$T = \epsilon_2 S_\rho{}^{\mu\nu} T^\rho{}_{\mu\nu} \tag{211}$$

where

$$\begin{aligned} T^\rho{}_{\mu\nu} &\equiv -e^\rho_i \left(\partial_\mu e^i_\nu - \partial_\nu e^i_\mu\right) && (212)\\ K^{\mu\nu}{}_\rho &\equiv -\frac{1}{2}\left(T^{\mu\nu}{}_\rho - T^{\nu\mu}{}_\rho - T_\rho{}^{\mu\nu}\right) && (213)\\ S_\rho{}^{\mu\nu} &\equiv \frac{1}{2}\left(K^{\mu\nu}{}_\rho + \delta^\mu_\rho T^{\theta\nu}{}_\theta - \delta^\nu_\rho T^{\theta\mu}{}_\theta\right) && (214) \end{aligned}$$

For a spatially flat FRW metric (222), we have that the torsion scalar assumes the form

$$T = T_s = -6H^2 \tag{215}$$

The action (210) can be written as

$$\mathcal{S}_T = \int dt L_T \tag{216}$$

where the point-like Lagrangian reads

$$L_T = a^3 \left(F - F_T T\right) - 6F_T a\dot{a}^2 - a^3 L_m \tag{217}$$

The equations of F(T) gravity look like

$$\begin{aligned} 12H^2 F_T + F &= \rho && (218)\\ 48H^2 F_{TT}\dot{H} - F_T\left(12H^2 + 4\dot{H}\right) - F &= p && (219)\\ \dot{\rho} + 3H(\rho + p) &= 0 && (220) \end{aligned}$$

10.2. The M$_{43}$-Model

In this subsection we consider the M$_{43}$-model which is one of the representatives of $F(R,T)$ gravity. The action of M$_{43}$-model reads as

$$\begin{aligned} S_{43} &= \int d^4x\sqrt{-g}[F(R,T) + L_m]\\ R &= R_s = \epsilon_1 g^{\mu\nu}R_{\mu\nu} && (221)\\ T &= T_s = \epsilon_2 S_\rho{}^{\mu\nu}T^\rho{}_{\mu\nu} \end{aligned}$$

where L_m is the matter Lagrangian, $\epsilon_i = \pm 1$ (signature), R is the curvature scalar, T is the torsion scalar. Let us consider the spacetime where the curvature and torsion are written by using the connection $G^\lambda{}_{\mu\nu}$ as a sum of the curvature and torsion, namely

$$G^\lambda{}_{\mu\nu} = e_i{}^\lambda\partial_\mu e^i{}_\nu + e_j{}^\lambda e^i{}_\nu\omega^j{}_{i\mu} = \Gamma^\lambda{}_{\mu\nu} + K^\lambda{}_{\mu\nu} \tag{222}$$

Here $\Gamma^j_{i\mu}$ is the Levi–Civita connection and $K^j_{i\mu}$ is the contorsion. The quantities $\Gamma^j_{i\mu}$ and $K^j_{i\mu}$ are defined as

$$\Gamma^l{}_{jk} = \frac{1}{2}g^{lr}\{\partial_k g_{rj} + \partial_j g_{rk} - \partial_r g_{jk}\} \tag{223}$$

and

$$K^\lambda_{\mu\nu} = -\frac{1}{2}\left(T^\lambda{}_{\mu\nu} + T_{\mu\nu}{}^\lambda + T_{\nu\mu}{}^\lambda\right) \tag{224}$$

respectively. Here the components of the torsion tensor are given by

$$\begin{aligned} T^\lambda{}_{\mu\nu} &= e_i{}^\lambda T^i{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu} && (225)\\ T^i{}_{\mu\nu} &= \partial_\mu e^i{}_\nu - \partial_\nu e^i{}_\mu + \Gamma^i{}_{j\mu}e^j{}_\nu - \Gamma^i{}_{j\nu}e^j{}_\mu && (226) \end{aligned}$$

The curvature $R^\rho{}_{\sigma\mu\nu}$ is defined as

$$\begin{aligned} R^\rho{}_{\sigma\mu\nu} &= e_i{}^\rho e^j{}_\sigma R^i{}_{j\mu\nu} = \partial_\mu G^\rho{}_{\sigma\nu} - \partial_\nu G^\rho{}_{\sigma\mu} + G^\rho{}_{\lambda\mu}G^\lambda{}_{\sigma\nu} - G^\rho{}_{\lambda\nu}G^\lambda{}_{\sigma\mu}\\ &= \bar{R}^\rho{}_{\sigma\mu\nu} + \partial_\mu K^\rho{}_{\sigma\nu} - \partial_\nu K^\rho{}_{\sigma\mu} + K^\rho{}_{\lambda\mu}K^\lambda{}_{\sigma\nu} - K^\rho{}_{\lambda\nu}K^\lambda{}_{\sigma\mu}\\ &\quad + \Gamma^\rho_{\lambda\mu}K^\lambda{}_{\sigma\nu} - \Gamma^\rho_{\lambda\nu}K^\lambda{}_{\sigma\mu} + \Gamma^\lambda_{\sigma\nu}K^\rho{}_{\lambda\mu} - \Gamma^\lambda_{\sigma\mu}K^\rho{}_{\lambda\nu} && (227) \end{aligned}$$

where the Riemann curvature is defined in the standard way

$$\bar{R}^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}_{\sigma\nu} - \partial_{\nu}\Gamma^{\rho}_{\sigma\mu} + \Gamma^{\rho}_{\lambda\mu}\Gamma^{\lambda}_{\sigma\nu} - \Gamma^{\rho}_{\lambda\nu}\Gamma^{\lambda}_{\sigma\mu} \tag{228}$$

Now we introduce the curvature and torsion scalars,

$$R = g^{ij}R_{ij} \tag{229}$$
$$T = S^{\mu\nu}_{\rho}T^{\rho}_{\mu\nu} \tag{230}$$

where

$$S^{\mu\nu}_{\rho} = \frac{1}{2}\left(K^{\mu\nu}_{\rho} + \delta^{\mu}_{\rho}T^{\theta\nu}_{\theta} - \delta^{\nu}_{\rho}T^{\theta\mu}_{\theta}\right) \tag{231}$$

Now the M_{43}-model is written in the form of (221).

Now we want to present the M_{43}-model for the spatially flat FRW spacetime. In this case the metric assumes the form

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \tag{232}$$

where $a(t)$ is the scale factor. In this case, the non-vanishing components of the Levi–Civita connection are

$$\begin{aligned}
\Gamma^0_{00} &= \Gamma^0_{0i} = \Gamma^0_{i0} = \Gamma^i_{00} = \Gamma^i_{jk} = 0 \\
\Gamma^0_{ij} &= a^2H\delta_{ij} \\
\Gamma^i_{jo} &= \Gamma^i_{0j} = H\delta^i_j
\end{aligned} \tag{233}$$

where $H = (\ln a)_t$ and $i, j, k, ... = 1, 2, 3$. Let us calculate the components of torsion tensor. The non-vanishing components are given by:

$$\begin{aligned}
T_{110} &= T_{220} = T_{330} = a^2h \\
T_{123} &= T_{231} = T_{312} = 2a^3f,
\end{aligned} \tag{234}$$

where h and f are some real functions. Note that the indices of the torsion tensor are raised and lowered with the metric, namely

$$T_{ijk} = g_{kl}T_{ij}{}^{l}. \tag{235}$$

Now we can find the contortion components. We get

$$\begin{aligned}
K^1{}_{10} &= K^2{}_{20} = K^3{}_{30} = 0 \\
K^1{}_{01} &= K^2{}_{02} = K^3{}_{03} = h \\
K^0{}_{11} &= K^0{}_{22} = K^0{}_{22} = a^2h \\
K^1{}_{23} &= K^2{}_{31} = K^3{}_{12} = -af \\
K^1{}_{32} &= K^2{}_{13} = K^3{}_{21} = af.
\end{aligned} \tag{236}$$

The non-vanishing components of the curvature $R^{\rho}{}_{\sigma\mu\nu}$ are given by

$$\begin{aligned}
R^0{}_{101} &= R^0{}_{202} = R^0{}_{303} = a^2(\dot{H} + H^2 + Hh + \dot{h}) \\
R^0{}_{123} &= -R^0{}_{213} = R^0{}_{312} = 2a^3f(H + h) \\
R^1{}_{203} &= -R^1{}_{302} = R^2{}_{301} = -a(Hf + \dot{f}) \\
R^1{}_{212} &= R^1{}_{313} = R^2{}_{323} = a^2[(H + h)^2 - f^2].
\end{aligned} \tag{237}$$

Similarly, we write the non-vanishing components of the Ricci curvature tensor $R_{\mu\nu}$ as

$$\begin{aligned} R_{00} &= -3\dot{H} - 3\dot{h} - 3H^2 - 3Hh \\ R_{11} &= R_{22} = R_{33} = a^2(\dot{H} + \dot{h} + 3H^2 + 5Hh + 2h^2 - f^2) \end{aligned} \tag{238}$$

The non-vanishing components of the tensor $S_\rho^{\mu\nu}$ are

$$S_1^{10} = \frac{1}{2}\left(K_1^{10} + \delta_1^1 T_\theta^{\theta 0} - \delta_1^0 T_\theta^{\theta\nu}\right) = \frac{1}{2}(h + 2h) = h \tag{239}$$

$$S_1^{10} = S_2^{20} = S_3{}^3 0 = 2h \tag{240}$$

$$S_1^{23} = \frac{1}{2}\left(K_1^{23} + \delta_1^2 + \delta_1^3\right) = -\frac{f}{2a} \tag{241}$$

$$S_1^{23} = S_2^{31} = S_3^{21} = -\frac{f}{2a} \tag{242}$$

and

$$T = T_{10}^1 S_1^{10} + T_{20}^2 S_2^{20} + T_{30}^3 S_3^{30} + T_1^{23} S_{23}^1 + T_{31}^2 S_2^{31} + T_{12}^3 S_3^{12} \tag{243}$$

Now we can write the explicit forms of the curvature and torsion scalars. One has

$$R = 6(\dot{H} + 2H^2) + 6\dot{h} + 18Hh + 6h^2 - 3f^2 \tag{244}$$

$$T = 6(h^2 - f^2) \tag{245}$$

For FRW metric, the M_{43}-model takes the form

$$\begin{aligned} S_{43} &= \int d^4x\sqrt{-g}[F(R,T) + L_m] \\ R &= 6(\dot{H} + 2H^2) + 6\dot{h} + 18Hh + 6h^2 - 3f^2 \\ T &= 6(h^2 - f^2) \end{aligned} \tag{246}$$

In this way, we have derived the M_{43}-model as one of geometrical realizations of $F(R,T)$ gravity by starting from the pure geometrical point of view.

11. Conclusions

In this work we have presented a brief review on $F(T)$ gravity. We have investigated generalized $F(T)$ modified torsion models, that is, models in which the torsion gravity equations are extended to scalar fields. This study is a continuation of our investigation program of $F(T)$ gravity [31]. We note that the GR case corresponds not only to the model $F(T) = T$, but also to our specific model $F(T) = \alpha T + \beta T^{1/2}$, for which we obtain the same results.

We also considered the recently developed $F(T)$ gravity, which is a new modified gravity capable of accounting for the present cosmic accelerating expansion. In particular, we presented some new models of $F(T)$ gravity and k-essence. We analyzed the relation between $F(T)$ gravity and k-essence. We also studied some new parametric models of pure kinetic k-essence, presented a short review on Noether symmetry of $F(T)$ gravity, and considered some generalizations of $F(T)$ gravity. Finally we note that it is interesting to extend these results for the knot universe case [47,48].

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