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Article

Reshaping the Science of Reliability with the Entropy Function†

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Abstract: The present paper revolves around two argument points. As first, we have observed a certain parallel between the reliability of systems and the progressive disorder of thermodynamical systems; and we import the notion of reversibility/irreversibility into the reliability domain. As second, we note that the reliability theory is a very active area of research which although has not yet become a mature discipline. This is due to the majority of researchers who adopt the inductive logic instead of the deductive logic typical of mature scientific sectors. The deductive approach was inaugurated by Gnedenko in the reliability domain. We mean to continue Gnedenko's work and we use the Boltzmann-like entropy to pursue this objective. This paper condenses the papers published in the past decade which illustrate the calculus of the Boltzmann-like entropy. It is demonstrated how the every result complies with the deductive logic and are consistent with Gnedenko's achievements.

Keywords: stochastic systems; reliability theory; reversibility; Boltzmann-like entropy; deductive logic; mature discipline

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1. Introduction

The reliability of machineries and the mortality of individuals are topics of great interest for the scientific community and common people as well. *Reliability theory* is an abstract approach aimed to gain theoretical insights into engineering and biology. Presently, the vast majority of researchers make conclusions about populations based on information extracted from random samples; in short, authors follow statistical *inductive logic*.

A mature discipline instead complies with *deductive logic*, that is to say theorists derive the results from principles and axioms using theorems. Gnedenko was the first to take this course in the reliability domain [1]. He assumes that the system S is a Markov chain and from this assumption concludes that the probability of good functioning without failure is the *general exponential function*:

$$P(t) = e^{-\int_0^t \lambda(t)dt} \tag{1}$$

where the hazard function $\lambda(t)$ determines the reliability of the system in each instant:

$$\lambda(t) = -P'(t)/P(t) \tag{2}$$

Gnedenko demonstrates that the probability distribution (Equation (1)) comes from the conditional probability typical of Markov chains. Equation (1) originates from the operations that a system executes one after the other, and the following formal logic statement summarizes Gnedenko's inference:

$$\text{Chained Units} \Rightarrow \text{General Exponential Function} \tag{3}$$

The hazard (or mortality) function $\lambda(t)$ is the key element in Equation (1) in that $\lambda(t)$ tunes up the general exponential function (1). Some authors hold that the hazard function is characterized by three phases: a new system has the decreasing hazard rate in the early part of lifetime where it is undergoing burn-in and debugging of machines. This period is followed by an interval when failures are due to causes resulting in a constant failure rate. The last period of life is one in which the system is experiencing the most severe wear out and thus has an increasing failure rate. However significant evidence contradicts this tripartite form of $\lambda(t)$ which authors usually call *bathtub curve*. For instance, researchers show the irregular degeneracy of electronic circuits [2]. The hazard rate presents humps so evident that Wong [3] labels this: "*roller coaster distribution*". In biology $\lambda(t)$ has very differing trends [4]. For example the mortality function of *Hidra magnipapilata* is constant throughout the entire lifetime. Several experts notice the discrepancy between empirical data and the bathtub model, and negate any validity to it. Ascher [5] claims that "the bathtub curve is merely a statement of apparent plausibility which has never been validated". More recently Kececioglu and Sun [6], Zairi [7], and Klutke with others [8] share this skeptical judgment.

One cannot deny that the hazard rate has not yet determined in a rigorous manner, and in our opinion we should proceed with the deductive logic inaugurated by Gnedenko in the reliability domain. This is the objective of the present mathematical work.

We begin with a preliminary theoretical inquiry

2. A Lesson from Thermodynamics

The second law of thermodynamics claims that the entropy of an isolated system will increase as the system goes forward in time. This entails—in a way—that physical objects have an inherent tendency towards disorder, and a general predisposition towards decay. Such a wide-spreading process of annihilation hints an intriguing parallel with the decadence of biological and artificial systems to us. The issues of reliability theory are not far away from some issues inquired by thermodynamics and this closeness suggests us to introduce the entropy function for the study of reliable/reparable systems.

We mean to detail the Markovian model and assume that the continuous stochastic system S has *m states* which are mutually exclusive:

$$S = (A_1 \ OR \ A_2 \ OR \ .... \ OR \ A_m), \ m > 0 \tag{4}$$

Each state is equipped with a set of *sub-states* or *components* or *parts* which work together toward the same purpose. Formally, the generic state A_i (i = 1, 2, … m) is equipped with n sub-states:

$$A_i = (A_{i1} \ AND \ A_{i2} \ AND \ \dots \ AND \ A_{in}), \ n > 0 \tag{5}$$

We consider that the states of the stochastic system S can be more or less reversible [9], and mean to calculate the reversibility property using the *Boltzmann-like entropy* H_i where P_i is the probability of A_i:

$$H_i = H(A_i) = \ln (P_i) \tag{6}$$

The proof is in [10].

We confine our attention to:

- The *functioning state* A_f and the *reliability entropy* H_f;
- The *recovery state* A_r and the *recovery entropy* H_r.

The meanings of H_f and H_r can be described as follows:

When the functioning state is irreversible, the system S works steadily. In particular, the more A_f is irreversible, the more H_f is high and S is capable of working and reliable. On the other hand, when H_f is low, S often abandons A_f in the physical reality. The system switches to A_r since S fails and is unreliable. The recovery entropy calculates the irreversibility of the recovery state, this implies that the more H_r is high, the more A_r is stable and in practice S is hard to be repaired and/or cured in the world. In sum H_r *expresses the aptitude of S to work or to live without failures; the entropy* H_r *illustrates the disposition of S toward reparation or restoration to health*.

As an application, suppose a and b are two devices in series with probability of good functioning: $P_f(a) = 10^{-200}$, $P_f(b) = 10^{-150}$. We can calculate the probability of the overall system and later capability of good working of S with the entropy:

$$P_f(S) = [P_f(a) \cdot P_f(b)] = 10^{-350} \tag{7}$$

$$H_f(S) = \log[P_f(S)] = \log (10^{-350}) = -805.9 \tag{8}$$

The Boltzmann-like entropy is additive and one can follow this way with the same result:

$$\begin{aligned} H_f(S) = [H_f(a) + H_f(b)] = \log[P_f(a)] + \log[P_f(b)] = \log[P_f(a) \cdot P_f(b)] = \\ = \log [10^{-200} \cdot 10^{-150}] = = \log (10^{-350}) = -805.9 \end{aligned} \tag{9}$$

As second case, suppose a device degrades during the interval (t_1, t_2); and the probability of good functioning are the following: $P_f(t_1) = 10^{-10}$, $P_f(t_2) = 10^{-200}$. The entropies $H_f(t_1) = \log(10^{-10}) = -23.0$; and $H_f(t_2) = -460.5$ qualify the irreversibility of the device and one obtains how much the capability of good functioning has sloped down:

$$\Delta H_f = H_f(t_2) - H_f(t_1) = -460.5 - (-23.0) = -437.5 \tag{10}$$

3. Basic Assumption

Real events are multi-fold. Mechanical, electrical, thermal, chemical and other material effects interfere in the physical reality. The generic component A_{ig} ($g = 1, 2, \ldots n$) involves a series of collateral physical mechanisms that run in parallel A_{ig}. Universal experience brings evidence how side effects change A_{ig}. Parallel interferences work by time passing and at last impede the correct functioning to A_{ig}. Thus we can establish a general property for the system components:

$$\text{The part } A_{ig} \text{ degenerates as time goes by} \tag{11}$$

For example, Carnot defines a model for the heat engine that includes two bodies at temperature T_1 and T_2 ($T_1 \neq T_2$), the gas A_{ig} does the mechanical work via cycles of contractions and expansions. The mounting disorder of the molecules—qualified by the thermodynamic entropy—results in the decreasing performances of A_{ig}. More unwanted side effects—e.g. the attrition amongst the gears and the heat dispersion—impact on other components and progressively harm the effectiveness of the heat engine.

4. Simple Degeneration of Systems

We detail Equation (11) and establish *the regular degeneration of components*. The reliability entropy of A_{ig} decreases linearly as time goes by:

$$H_{fg} = H_{fg}(t) = -\,c_g\, t.\ c_g > 0 \tag{12}$$

From hypothesis Equation (12) one can prove that the probability of good functioning P_f follows *the exponential law* with constant hazard rate:

$$P_f = P_f(t) = e^{-c\,t},\ c > 0 \tag{13}$$

$$\lambda(t) = c \tag{14}$$

The proof may be found in [11].

5. Complex Degeneration of Systems

When assumption Equation (12) comes true over a certain period of time, the components A_{f1}, A_{f2}, …, A_{fn} worsen to the extent that they set up a *cascade effect* [11]. The cascade effect consists of the generic part A_{ig} that spoils one or more close components while the system proceeds to run. A cascade effect can be linear or otherwise compound. In the first stage we assume the component A_{ig} harms the close part A_{ik} and this in turn damages another one and so on:

$$\text{The cascade effect is linear} \tag{15}$$

Suppose the linear cascade effect occurs while principle Equation (12) is still true of necessity, one can prove that the probability of good functioning is the *exponential-power function*:

$$P_f = P_f(t) = \mathrm{b}e^{-\mathrm{a}t^n}\text{, a, b} > 1 \tag{16}$$

The hazard function is a *power of time*:

$$\lambda(t) = \mathrm{a}t^{n-1} \tag{17}$$

In the second stage we suppose that the component A_{ig} damages the components all around:

The cascade effect is compound (18)

This hypothesis—alternative to linear waterfall effect—yields that the probability of functioning is *the exponential-exponential function* and the hazard rate is *exponential of time*:

$$P_f = P_f(t) = \mathrm{g}\,e^{-\mathrm{d}e^t}\text{, g, d} > 1 \tag{19}$$

$$\lambda(t) = \mathrm{d}e^t \tag{20}$$

The proofs of Equations (16) and (19) may be found in [12].

6. Conclusive Remarks

(A) The present paper adopts the Boltzmann-like entropy and develops the ensuing logical inferences:

Regular degeneration of system's components ⇒ Exponential Function
Regular degeneration + linear cascade effect ⇒ Exponential-Power Function (21)
Regular degeneration + composite cascade effect ⇒ Exponential-Exponential Function

Chaining implies a true dependency between chained operations, and Gnedenko derives the general exponential function (1) from the Markovian dependency. Gnedenko's work and the present work are consistent despite the different mathematical techniques in use. In particular:

- *Assumptions* of statements (21): We model *S* by mean of Equations (4) and (5) that are Markovian chains. The regular degeneration of A_{ig} and the cascade effects make explicit some special behaviors of chained operations.
- *Conclusions* of statements (21): Mathematical results (13), (16) and (19) are special cases of function (1).

(B) The present approach adopts the deductive logic and Equations (14), (17) and (20) have been related to precise causes and not to precise periods of system lifetime. In other words, the function $\lambda(t)$ can be a constant, it can follow the power or exponential distributions in any interval of the system life. Each result in (21) has been obtained from precise hypotheses, and those hypotheses may come true during the system juvenile period, the maturity and the senescence alike.

(C) Authors recognize that sometimes the organs of appliances and biological beings degenerate at constant rate—in accord to Equation (14)—during the middle age. Several machines have linear structures and the probability of good functioning follows the Weibull distribution during ageing that corresponds to Equation (17). The body of animals and humans appear rather intricate and during ageing $\lambda(t)$ follow the Gompertz distribution in agreement with Equation (20). In conclusion, on one hand the present frame does not hold that the bathtub curve is the standard form of $\lambda(t)$ in accordance with empirical evidence. On the other hand the theoretical results obtained here do not exclude that a special system can take after the bathtub curve. The bathtub curve is a concept that may be used for describing particular forms of hazard functions

(D) The Boltzmann entropy plays a fundamental role on the theoretical plane as it clarifies why systems follow the second law of thermodynamics; instead it is not so common in engineering calculations. The Boltzmann-like entropy has the same virtues and limits of the Boltzmann entropy. It helps us to pass from studying "how" a system declines, to studying "why" a system declines, though the use of the Boltzmann-like entropy in applications is not so manageable and we mean to improve the present thread of research.

In closing, we mean to highlight how the Boltzmann-like entropy sustains a promising approach for developing a deductive theory of aging integrating mathematical methods with engineering notions and specific biological knowledge.

Author Contributions

P. Rocchi conceived the present theoretical frame; G. Capacci contributed to write the paper. Both authors have read and approved the final manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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