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Article

Implications of Non-Differentiable Entropy on a Space-Time Manifold

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Abstract: Assuming that the motions of a complex system structural units take place on continuous, but non-differentiable curves of a space-time manifold, the scale relativity model with arbitrary constant fractal dimension (the hydrodynamic and wave function versions) is built. For non-differentiability through stochastic processes of the Markov type, the non-differentiable entropy concept on a space-time manifold in the hydrodynamic version and its correspondence with motion variables (energy, momentum, *etc.*) are established. Moreover, for the same non-differentiability type, through a scale resolution dependence of a fundamental length and wave function independence with respect to the proper time, a non-differentiable Klein–Gordon-type equation in the wave function version is obtained. For a phase-amplitude functional dependence on the wave function, the non-differentiable spontaneous symmetry breaking mechanism implies pattern generation in the form of Cooper non-differentiable-type pairs, while its non-differentiable topology implies some fractal logic elements (fractal bit, fractal gates, *etc.*).

Keywords: non-differentiable entropy; fractal bit; space-time manifold; space-time scale relativity theory

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1. Introduction

Analyzing the motion of a particle on a fractal curve [1–4], we observe a big discrepancy between the space coordinates and the temporal one (considered as the affine parameter of motion curve), the latter being non-fractal. This discrepancy also has an immediate abnormal consequence: the particle travels on an infinite length curve in a finite time span, and so, it has an infinite velocity. In order to eliminate this contradiction, we will assume that the temporal coordinate of the fractal curve is also a fractal one. Thus, most elements of the non-relativistic approach of scale relativity theory with arbitrary constant fractal dimension, as described in [5–11], remain valid, but the time differential element dt is now replaced by the proper time differential element $d\tau$. In this way, not only the space, but the entire space-time continuum is considered to be non-differentiable and, therefore, fractal. In a such frame, we shall extend in the present paper the results from [12] by introducing the concept of relativistic non-differentiable entropy (non-differentiable entropy on a space-time manifold). Some of its characteristics and implications are also given.

The paper is organized as follows: in Section 2, some consequences of non-differentiability are presented; in Section 3, the relativistic non-differentiable motion operator is introduced; in Section 4, relativistic non-differentiable geodesics and the non-differentiable Klein–Gordon-type equation are given; in Section 5, relativistic non-differentiable entropy and its correspondence with relativistic motion variables are studied; in Section 6, non-differentiable spontaneous symmetry breaking and pattern generation are investigated; and in Section 7, non-differentiable topology and logic are considered.

2. Consequences of Non-Differentiability on a Space-Time Manifold

Let us suppose that in a Minkowski-type space-time, the motions of structural units (complex system components [13–18]) take place on continuous, but non-differentiable curves (in particular, fractal curves). The non-differentiability of motion curves implies the following:

(i) Any continuous, but non-differentiable curve is explicitly scale dependent (which will be referred to as $\delta\tau$). In other words, its length tends to infinity when its proper time interval, $\Delta\tau$, tends to zero. Then, a continuous, but non-differentiable space-time is fractal in Mandelbrot's sense [4].

(ii) The differential proper time reflection invariance of any field variable is broken. For example, the proper time derivative of four-coordinate field X^μ can be written two-fold:

$$\left[\frac{dX^\mu}{d\tau}\right]_+ = \lim_{\Delta\tau\to 0_+} \frac{X^\mu(\tau+\Delta\tau) - X^\mu(\tau)}{\Delta\tau}$$

$$\left[\frac{dX^\mu}{d\tau}\right]_- = \lim_{\Delta\tau\to 0_-} \frac{X^\mu(\tau) - X^\mu(\tau-\Delta\tau)}{\Delta\tau} \tag{1}$$

These relations are equivalent in the differentiable case, $\Delta\tau \to -\Delta\tau$. In the non-differentiable case, the previous definitions fail, since the limits $\Delta\tau \to 0_\pm$ are no longer defined. In the approach of the non-differentiable model, the physical phenomena are related to the behavior of the function during the "zoom" operation on the proper time resolution $\delta\tau$. Then, by means of the substitution principle, $\delta\tau$ will be identified with the differential element $d\tau$, *i.e.*, $\delta\tau \equiv d\tau$ and will be considered as independent variables. Thus, every standard field $Q(\tau)$ is replaced by the non-differentiable field $Q(\tau, d\tau)$ explicitly dependent on the proper time resolution interval whose derivative is undefined only in the unobservable limit, $\Delta\tau \to 0$. As a consequence, two derivatives of every non-differentiable field as explicit functions of τ and $d\tau$ will be defined. For example, the two derivatives of the four-coordinate field $X^\mu(\tau, \Delta\tau)$ take the following form:

$$\frac{d_+X^\mu}{d\tau} = \lim_{\Delta\tau \to 0_+} \frac{X^\mu(\tau + \Delta\tau, \Delta\tau) - X^\mu(\tau, \Delta\tau)}{\Delta\tau}$$

$$\frac{d_-X^\mu}{d\tau} = \lim_{\Delta\tau \to 0_-} \frac{X^\mu(\tau, \Delta\tau) - X^\mu(\tau - \Delta\tau, \Delta\tau)}{\Delta\tau} \tag{2}$$

The sign $+$ corresponds to the forward physical process and the sign $-$ to the backwards one.

(iii) The differential of four-coordinate field $dX^\mu(\tau, \Delta\tau)$ can be expressed as the sum of two differentials, one not scale dependent (differentiable part $d_\pm x^\mu(\tau)$) and the other scale dependent (non-differentiable part $d_\pm \xi^\mu(\tau, d\tau)$), *i.e.*,

$$d_\pm X^\mu(\tau, \Delta\tau) = d_\pm x^\mu(\tau) + d_\pm \xi^\mu(\tau, d\tau); \tag{3}$$

(iv) The non-differentiable part of the four-coordinate field satisfies the non-differentiable equation:

$$d_\pm \xi^\mu(\tau, d\tau) = \lambda^\mu_\pm (d\tau)^{1/D_F} \tag{4}$$

where $\lambda^\mu_\pm$ are constant coefficients whose statistical significance will be given in what follows;

(v) The differential proper time reflection invariance is recovered by combining the derivatives $d_+/d\tau$ and $d_-/d\tau$ in the non-differentiable operator:

$$\frac{\hat{d}}{d\tau} = \frac{1}{2}\left(\frac{d_+ + d_-}{d\tau}\right) - \frac{i}{2}\left(\frac{d_+ - d_-}{d\tau}\right) \tag{5}$$

This specific procedure is called, according to [19], "extension in complex by differentiability". Applying now the non-differentiable operator to the four-coordinate field yields the complex velocity:

$$\begin{aligned} \hat{V}^\mu &= \frac{\hat{d}X^\mu}{d\tau} = \frac{1}{2}\left(\frac{d_+X^\mu + d_-X^\mu}{d\tau}\right) - \frac{i}{2}\left(\frac{d_+X^\mu - d_-X^\mu}{d\tau}\right) = \\ &= \frac{1}{2}\left(\frac{d_+x^\mu + d_-x^\mu}{d\tau} + \frac{d_+\xi^\mu + d_-\xi^\mu}{d\tau}\right) - \frac{i}{2}\left(\frac{d_+x^\mu - d_-x^\mu}{d\tau} + \frac{d_+\xi^\mu - d_-\xi^\mu}{d\tau}\right) = V^\mu - iU^\mu \end{aligned} \tag{6}$$

with:

$$V^\mu = \frac{1}{2}(v^\mu_+ + v^\mu_-), U^\mu = \frac{1}{2}(v^\mu_+ - v^\mu_-), v^\mu_+ = \frac{d_+x^\mu + d_+\xi^\mu}{d\tau}, v^\mu_- = \frac{d_-x^\mu + d_-\xi^\mu}{d\tau} \tag{7}$$

The real part V^μ is differentiable and scale resolution independent, while the imaginary one U^μ is non-differentiable and scale resolution dependent.

(vi) There can be found an infinite number of non-differentiable curves (geodesics) relating any pair of its points. This is true at all scales. The structural unit is substituted with the geodesics themselves, so that any measurement is interpreted as a selection of the geodesics by the measuring device. The infinity of geodesics in the bundle, their non-differentiability and the two values of the derivative imply a generalized statistical fluid-like description (non-differentiable fluid). Then, the average values of the fluid variables must be considered in the previously mentioned sense, so the average of $d_\pm X^\mu$ is:

$$\langle d_\pm X^\mu \rangle \equiv d_\pm x^\mu \tag{8}$$

with:

$$\langle d_{\pm}\xi^{\mu}\rangle \equiv 0 \tag{9}$$

3. Non-Differentiable Motion Operator on a Space-Time Manifold

Let us now consider that the movement curves (continuous and non-differentiable) are immersed in the space-time and that X^{μ} are the four-coordinates of a point on the curve. We also consider a variable field $Q(X^{\mu},\tau)$ and the following Taylor expansion, up to the second order:

$$d_{\pm}Q(X^{\mu},\tau) = \partial_{\tau}Qd\tau + \partial_{\mu}Qd_{\pm}X^{\mu} + \frac{1}{2}\partial_{\mu}\partial_{\nu}Qd_{\pm}X^{\mu}d_{\pm}X^{\nu} \tag{10}$$

The relations (9) are valid in any point of the space-time manifold and more for the points "X^{μ}" on the non-differentiable curve, which we have selected in Relation (9).

From here, forward and backward average values of (9) become:

$$\langle d_{\pm}Q(X^{\mu},\tau)\rangle = \langle\partial_{\tau}Qd\tau\rangle + \langle\partial_{\mu}Qd_{\pm}X^{\mu}\rangle + \frac{1}{2}\langle\partial_{\mu}\partial_{\nu}Qd_{\pm}X^{\mu}d_{\pm}X^{\nu}\rangle \tag{11}$$

We make the following stipulations: the average values of the variables field $Q(X^{\mu},\tau)$ and its derivatives coincide with themselves, and the differentials $d_{\pm}X^{\mu}$ and $d\tau$ are independent. Therefore, the average of their products coincide with the product of their averages. In these conditions, (11) takes the form:

$$d_{\pm}Q(X^{\mu},\tau) = \partial_{\tau}Qd\tau + \partial_{\mu}Q\langle d_{\pm}X^{\mu}\rangle + \frac{1}{2}\partial_{\mu}\partial_{\nu}Q\langle d_{\pm}X^{\mu}d_{\pm}X^{\nu}\rangle \tag{12}$$

or, using (3), (8) and (9)

$$d_{\pm}Q(X^{\mu},\tau) = \partial_{\tau}Qd\tau + \partial_{\mu}Qd_{\pm}x^{\mu} + \frac{1}{2}\partial_{\mu}\partial_{\nu}Q(d_{\pm}x^{\mu}d_{\pm}x^{\nu} + \langle d_{\pm}\xi^{\mu}d_{\pm}\xi^{\nu}\rangle) \tag{13}$$

Even the average values of the four-non-differentiable coordinate $d_{\pm}\xi^{\mu}$ is null for the higher order of the four-non-differentiable coordinate average, the situation can be different. Let us focus now on the mean $\langle d_{\pm}\xi^{\mu}d_{\pm}\xi^{\nu}\rangle$. Using (4), we can write:

$$\langle d_{\pm}\xi^{\mu}d_{\pm}\xi^{\nu}\rangle = \pm\lambda^{\mu}_{\pm}\lambda^{\nu}_{\pm}(d\tau)^{(2/D_F)-1}d\tau \tag{14}$$

where we accepted the following: the sign $+$ corresponds to $d\tau > 0$, while the sign $-$ corresponds to $d\tau < 0$.

Then, (13) takes the form:

$$\begin{aligned} d_{\pm}Q(X^{\mu},\tau) &= \partial_{\tau}Qd\tau + \partial_{\mu}Qd_{\pm}x^{\mu} + \frac{1}{2}\partial_{\mu}\partial_{\nu}Qd_{\pm}x^{\mu}d_{\pm}x^{\nu}\pm \\ &\pm\frac{1}{2}\partial_{\mu}\partial_{\nu}Q[\lambda^{\mu}_{\pm}\lambda^{\nu}_{\pm}(d\tau)^{(2/D_F)-1}d\tau] \end{aligned} \tag{15}$$

If we divide by $d\tau$ and neglect the terms that contain differential factors, using the method from [5–11], we obtain:

$$\frac{d_{\pm}Q(X^{\mu},\tau)}{d\tau} = \partial_{\tau}Q + \nu^{\mu}_{\pm}\partial_{\mu}Q \pm \frac{1}{2}\lambda^{\mu}_{\pm}\lambda^{\nu}_{\pm}(d\tau)^{(2/D_F)-1}\partial_{\mu}\partial_{\nu}Q \tag{16}$$

These relations also allow us to define the operators:

$$\frac{d_{\pm}}{d\tau}=\partial_{\tau}+\nu^{\mu}_{\pm}\partial_{\mu}\pm\frac{1}{2}\lambda^{\mu}_{\pm}\lambda^{\nu}_{\pm}(d\tau)^{(2/D_F)-1}\partial_{\mu}\partial_{\nu} \tag{17}$$

Under these circumstances, let us calculate $\hat{d}/d\tau$. Taking into account (5), (6) and (17), we obtain:

$$\begin{aligned}\frac{\hat{d}Q}{d\tau}=\ &\frac{1}{2}\left[\left(\frac{d_{+}Q+d_{-}Q}{d\tau}\right)-i\left(\frac{d_{+}Q-d_{-}Q}{d\tau}\right)\right]=\\ &=\frac{1}{2}\left\{\left[\partial_{\tau}Q+\nu^{\mu}_{+}\partial_{\mu}Q+\frac{1}{2}\lambda^{\mu}_{+}\lambda^{\nu}_{+}(d\tau)^{(2/D_F)-1}\partial_{\mu}\partial_{\nu}Q\right]+\right.\\ &+\left.\left[\partial_{\tau}Q+\nu^{\mu}_{-}\partial_{\mu}Q-\frac{1}{2}\lambda^{\mu}_{-}\lambda^{\nu}_{-}(d\tau)^{(2/D_F)-1}\partial_{\mu}\partial_{\nu}Q\right]\right\}-\\ &-\frac{i}{2}\left\{\left[\partial_{\tau}Q+\nu^{\mu}_{+}\partial_{\mu}Q+\frac{1}{2}\lambda^{\mu}_{+}\lambda^{\nu}_{+}(d\tau)^{(2/D_F)-1}\partial_{\mu}\partial_{\nu}Q\right]-\right.\\ &-\left.\left[\partial_{\tau}Q+\nu^{\mu}_{-}\partial_{\mu}Q-\frac{1}{2}\lambda^{\mu}_{-}\lambda^{\nu}_{-}(d\tau)^{(2/D_F)-1}\partial_{\mu}\partial_{\nu}Q\right]\right\}=\\ &=\partial_{\tau}Q+\left(\frac{\nu^{\mu}_{+}+\nu^{\mu}_{-}}{2}-i\frac{\nu^{\mu}_{+}-\nu^{\mu}_{-}}{2}\right)\partial_{\mu}Q+\\ &+\frac{1}{4}(d\tau)^{(2/D_F)-1}[(\lambda^{\mu}_{+}\lambda^{\nu}_{+}-\lambda^{\mu}_{-}\lambda^{\nu}_{-})-i(\lambda^{\mu}_{+}\lambda^{\nu}_{+}+\lambda^{\mu}_{-}\lambda^{\nu}_{-})]\partial_{\mu}\partial_{\nu}Q=\\ &=\partial_{\tau}Q+\hat{V}^{\mu}\partial_{\mu}Q+\frac{1}{4}(d\tau)^{(2/D_F)-1}D^{\mu\nu}\partial_{\mu}\partial_{\nu}Q\end{aligned} \tag{18}$$

where:

$$\begin{aligned}D^{\mu\nu}&=d^{\mu\nu}-i\overline{d}^{\mu\nu}\\ d^{\mu\nu}&=\lambda^{\mu}_{+}\lambda^{\nu}_{+}-\lambda^{\mu}_{-}\lambda^{\nu}_{-},\overline{d}^{\mu\nu}=\lambda^{\mu}_{+}\lambda^{\nu}_{+}+\lambda^{\mu}_{-}\lambda^{\nu}_{-}\end{aligned} \tag{19}$$

The relation also allows us to define the non-differentiable operator:

$$\frac{\hat{d}}{d\tau}=\partial_{\tau}+\hat{V}^{\mu}\partial_{\mu}+\frac{1}{4}(d\tau)^{(2/D_F)-1}D^{\mu\nu}\partial_{\mu}\partial_{\nu} \tag{20}$$

If the non-differentiability of the motion curve is realized through a Markov-type stochastic process [1,2,4]:

$$\lambda^{\mu}_{+}\lambda^{\nu}_{+}=\lambda^{\mu}_{-}\lambda^{\nu}_{-}=-\lambda\eta^{\mu\nu} \tag{21}$$

where $\eta^{\mu\nu}$ is the Minkowski metric, then the non-differentiable operator takes the form:

$$\frac{\hat{d}}{d\tau}=\partial_{\tau}+\hat{V}^{\mu}\partial_{\mu}+i\frac{\lambda}{2}(d\tau)^{(2/D_F)-1}\partial_{\mu}\partial^{\mu} \tag{22}$$

4. Non-Differentiable Geodesics on a Space-Time Manifold: Non-Differentiable Klein–Gordon-Type Equation

We now apply the principle of scale covariance [1,2] (physics laws are simultaneously invariant, both with respect to the four-coordinate transformations and with respect to scale transformations) and postulate that the passage from differentiable mechanics to the non-differentiable mechanics, which is considered here, can be implemented by replacing the standard time derivative $d/d\tau$ by the non-differentiable operator $\hat{d}/d\tau$. This operator plays the role of a "covariant derivative operator", namely it is used to write the fundamental equations of dynamics under the same form as in the classical and differentiable case. Thus, applying the operator (20) to the complex velocity (6), the geodesics equation is obtained:

$$\frac{\hat{d}\hat{V}^\mu}{d\tau} = \partial_\tau \hat{V}^\mu + \hat{V}^\nu \partial_\nu \hat{V}^\mu + \frac{1}{4}(d\tau)^{(2/D_F)-1} D^{\alpha\beta}\partial_\alpha\partial_\beta \hat{V}^\mu \equiv 0 \tag{23}$$

or, using (6), through separation of motions on scales resolutions (separation of the real part from the imaginary one):

$$\begin{aligned}\frac{\hat{d}V^\mu}{d\tau} &= \partial_\tau V^\mu + V^\nu\partial_\nu V^\mu - U^\nu \partial_\nu U^\mu + \frac{1}{4}(d\tau)^{(2/D_F)-1} d^{\alpha\beta}\partial_\alpha\partial_\beta V^\mu - \\ &\quad -\frac{1}{4}(d\tau)^{(2/D_F)-1}\overline{d}^{\alpha\beta}\partial_\alpha\partial_\beta U^\mu = 0 \\ \frac{\hat{d}U^\mu}{d\tau} &= \partial_\tau U^\mu + V^\nu\partial_\nu U^\mu + U^\nu \partial_\nu V^\mu + \frac{1}{4}(d\tau)^{(2/D_F)-1} d^{\alpha\beta}\partial_\alpha\partial_\beta U^\mu + \\ &\quad +\frac{1}{4}(d\tau)^{(2/D_F)-1}\overline{d}^{\alpha\beta}\partial_\alpha\partial_\beta V^\mu = 0\end{aligned} \tag{24}$$

For motions on non-differentiable curves through a Markov-type stochastic process [1,2,4], the geodesics equation takes the form:

$$\frac{\hat{d}\hat{V}^\mu}{d\tau} = \partial_\tau \hat{V}^\mu + \hat{V}^\nu \partial_\nu \hat{V}^\mu + i\frac{\lambda}{2}(d\tau)^{(2/D_F)-1}\partial^\nu\partial_\nu \hat{V}^\mu = 0 \tag{25}$$

or, through scale resolution separation:

$$\begin{aligned}\frac{\hat{d}V^\mu}{d\tau} &= \partial_\tau V^\mu + V^\nu\partial_\nu V^\mu - \left(U^\mu - \frac{\lambda}{2}(d\tau)^{(2/D_F)-1}\partial^\nu\right)\partial_\nu U^\mu = 0 \\ \frac{\hat{d}U^\mu}{d\tau} &= \partial_\tau U^\mu + V^\nu\partial_\nu U^\mu + \left(U^\mu - \frac{\lambda}{2}(d\tau)^{(2/D_F)-1}\partial^\nu\right)\partial_\nu V^\mu = 0\end{aligned} \tag{26}$$

Let us choose $\hat{V}^\mu$ in terms of the wave function Ψ:

$$\hat{V}_\alpha = i\lambda(d\tau)^{(2/D_F)-1}\partial_\alpha \ln\Psi \tag{27}$$

Then, the geodesics Equation (25) becomes:

$$\begin{aligned}\frac{\hat{d}\hat{V}_\alpha}{d\tau} &= i\lambda(d\tau)^{(2/D_F)-1}\partial_\tau\partial_\alpha\ln\Psi + \Big[i\lambda(d\tau)^{(2/D_F)-1}\partial^\mu\ln\Psi + \\ &\quad + \frac{i\lambda}{2}(d\tau)^{(2/D_F)-1}\partial^\mu\Big]\partial_\mu\partial_\alpha\left[i\lambda(d\tau)^{(2/D_F)-1}\ln\Psi\right] = 0\end{aligned} \tag{28}$$

Since:

$$\begin{aligned}&\partial_\alpha(\partial_\mu\ln\Psi\partial^\mu\ln\Psi) = 2\partial^\mu\ln\Psi\partial_\alpha\partial_\mu\ln\Psi \\ &\partial_\alpha\partial_\mu\partial^\mu\ln\Psi = \partial^\mu\partial_\mu\partial_\alpha\ln\Psi \\ &\partial_\alpha(\partial_\mu\ln\Psi\partial^\mu\ln\Psi + \partial_\mu\partial^\mu\ln\Psi) = \partial_\alpha\left(\frac{\partial_\mu\partial^\mu\Psi}{\Psi}\right)\end{aligned} \tag{29}$$

Equation (28) becomes:

$$i\lambda(d\tau)^{(2/D_F)-1}\partial_\tau\partial_\alpha\ln\Psi + \lambda^2(d\tau)^{(4/D_F)-2}\partial_\alpha\left(\frac{\partial_\mu\partial^\mu\Psi}{\Psi}\right) = 0 \tag{30}$$

By integrating the above relation up to an arbitrary phase factor, which may be set constant by a suitable choice of the phase of Ψ, we obtain:

$$\lambda^2(d\tau)^{(4/D_F)-2}\partial_\mu\partial^\mu\Psi + i\lambda(d\tau)^{(2/D_F)-1}\partial_\tau\Psi + \overline{\omega}^2\Psi = 0 \tag{31}$$

with $\overline{\omega}$ an integration constant. This constant is a critical velocity imposed by means of scale resolution. From here, if the wave function is independent of τ, $\partial_\tau\Psi = 0$, the relation (31) becomes a non-differentiable Klein–Gordon-type equation:

$$\partial_\mu\partial^\mu\Psi + \frac{1}{\overline{\Lambda}^2}\Psi = 0 \tag{32}$$

with:

$$\overline{\Lambda} = \overline{\Lambda}_0(d\tau)^{(2/D_F)-1}, \overline{\Lambda}_0 = \frac{\lambda}{\omega} \tag{33}$$

From (33), a scale resolution dependence of the fundamental length $\overline{\Lambda}$ results [1,2,4], where $\overline{\Lambda}_0$ is the fundamental unscaled length. For motions on Peano curves, $D_F = 2$, at scale $\overline{\Lambda} = \lambda/\omega = \hbar/m_0c$, $\lambda = \hbar/m_0$, $\overline{\omega} \equiv c$ [1,2], with $\hbar$ the reduced Planck constant, m_0 the rest mass of the particle and c the vacuum velocity light, (33) takes the usual form of Klein–Gordon equation [20]:

$$\partial_\mu\partial^\mu\Psi + (m_0c/\hbar)^2\Psi = 0.$$

5. Non-Differentiable Entropy on a Space-Time manifold and Its Implications

Using the explicit form of the wave function, $\Psi = \sqrt{\rho}e^{iS}$, where $\sqrt{\rho}$ is an amplitude and S is a phase, the expression of U_α becomes:

$$U_\alpha = -\lambda\partial_\alpha \ln\sqrt{\rho} \tag{34}$$

Thus, it results:

$$\left[U_\mu - \frac{\lambda}{2}(d\tau)^{(2/D_F)-1}\partial_\mu\right]\partial^\mu U_\alpha = \lambda^2(d\tau)^{(4/D_F)-2}\left(\partial^\mu \ln\sqrt{\rho}\partial_\mu\partial_\alpha \ln\sqrt{\rho} + \frac{1}{2}\partial^\mu\partial_\mu\partial_\alpha \ln\sqrt{\rho}\right) \tag{35}$$

Since the identities from (29) also work in variable $\ln\sqrt{\rho}$, Equation (35) becomes:

$$\begin{aligned}\left[U_\mu - \frac{\lambda}{2}(d\tau)^{(2/D_F)-1}\partial_\mu\right]\partial^\mu U_\alpha &= \frac{\lambda^2}{2}(d\tau)^{(4/D_F)-2}\partial_\alpha(\partial^\mu \ln\sqrt{\rho}\partial_\mu \ln\sqrt{\rho} + \partial^\mu\partial_\mu \ln\sqrt{\rho}) = \\ &= \frac{1}{2}\lambda^2(d\tau)^{(4/D_F)-2}\partial_\alpha\left(\frac{\partial^\mu\partial_\mu\sqrt{\rho}}{\sqrt{\rho}}\right)\end{aligned} \tag{36}$$

which implies through the specific non-differentiable potential:

$$Q = \frac{\lambda^2}{2}(d\tau)^{(4/D_F)-2}\frac{\partial^\mu\partial_\mu\sqrt{\rho}}{\sqrt{\rho}} = \frac{\lambda^2}{2}(d\tau)^{(4/D_F)-2}(U^\mu U_\mu - \lambda\partial^\mu U_\mu) \tag{37}$$

the specific non-differentiable force:

$$F_\alpha = \frac{\lambda^2}{2}(d\tau)^{(4/D_F)-2}\partial_\alpha\left(\frac{\partial^\mu\partial_\mu\sqrt{\rho}}{\sqrt{\rho}}\right) = \left[U_\mu - \frac{\lambda}{2}(d\tau)^{(2/D_F-1)}\partial_\mu\right]\partial^\mu U_\alpha. \tag{38}$$

Thus, the first Equation (26) takes the form:

$$\frac{\hat{d}V_\alpha}{d\tau} = \partial_\tau V_\alpha + V^\mu \partial_\mu V_\alpha = \frac{\lambda^2}{2}(d\tau)^{(4/D_F)-2}\partial_\alpha\left(\frac{\partial^\mu \partial_\mu \sqrt{\rho}}{\sqrt{\rho}}\right) \tag{39}$$

Since:

$$V_\alpha = \lambda(d\tau)^{(2/D_F)-1}\partial_\alpha S \tag{40}$$

which implies:

$$V^\nu \partial_\nu V_\alpha = V^\nu \partial_\alpha V_\nu \tag{41}$$

Relation (39) becomes:

$$\frac{\hat{d}V_\alpha}{d\tau} = \partial_\tau V_\alpha + V^\nu \partial_\alpha V_\nu - \frac{\lambda^2}{2}(d\tau)^{(4/D_F)-2}\partial_\alpha\left(\frac{\partial^\nu \partial_\nu \sqrt{\rho}}{\sqrt{\rho}}\right) = 0 \tag{42}$$

and more, for $\partial_\tau V_\alpha = 0$

$$\partial_\alpha\left[V^\nu V_\nu - \lambda^2(d\tau)^{(4/D_F)-2}\left(\frac{\partial^\nu \partial_\nu \sqrt{\rho}}{\sqrt{\rho}}\right)\right] = 0. \tag{43}$$

Now, by a suitable choice of the constant integration and knowing that [1,2]:

$$V^\mu V_\mu = (E^2 - \mathbf{p}^2 c^2)/m_0^2 c^2$$

we obtain the energy expression in the form:

$$E = \pm\left[m_0^2 c^4 + \mathbf{p}^2 c^2 + (m_0 c\lambda)^2 (d\tau)^{(4/D_F)-2}\frac{\Box\sqrt{\rho}}{\sqrt{\rho}}\right]^{1/2}. \tag{44}$$

In Relation (44), E is the non-differentiable energy of the structural unit, $\mathbf{p}$ is the non-differentiable momentum of the structural unit, $p_0 = m_0 c$ is the non-differentiable rest momentum of the structural unit and $\Box$ is the d'Alembert operator. The standard result (relativistic energy) is obtained from Equation (44) assuming motions on Peano curves of the structural units and constant non-differentiable state density of the structural units.

Let us define now the logarithmic function:

$$\overline{S}(X^\mu, V_\mu, d\tau) = \ln\rho(X^\mu, V_\mu, d\tau) \tag{45}$$

that will be called later non-differentiable entropy on a space-time manifold (relativistic non-differentiable entropy). This characterizes the disorder degree of a non-differentiable system on a space-time manifold. In the classical case, Equation (44) takes the standard form:

$$\frac{\mathbf{V}_D^2}{2} + Q = \text{const.} \tag{46}$$

with the specific fractal potential:

$$Q = -2\lambda^2(d\tau)^{(2/D_F)-1}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} = -\frac{\mathbf{V}_F^2}{2} - \lambda(d\tau)^{(2/D_F)-1}\nabla\cdot\mathbf{V}_F \tag{47}$$

From here, we can find all results correlated with non-differentiable entropy on a space-manifold [12]. What is really important is the fact that from Equation (44) written in the form:

$$\lambda^2(d\tau)^{(4/D_F)-2}\frac{\Box\exp(\overline{S}/2)}{\exp(\overline{S}/2)}=\frac{E^2-\mathbf{p}^2c^2}{m_0^2c^2} \tag{48}$$

it results that the relativistic non-differentiable entropy depends on the motion state of complex system structural unit through the total non-differentiable energy E, the internal energy m_0c^2 and the non-differentiable momentum $\mathbf{p}$.

6. Non-Differentiable Spontaneous Symmetry Breaking: Pattern Generation

Let us admit that between the phase S and the amplitude $\sqrt{\rho}$ of the wave function Ψ, there exists the functional dependence $S=S(\sqrt{\rho})$. Then, from (34) and (40), the first approximation of Equation (48) (which implies $\mathbf{p}^2=\text{const.}\rho\equiv a\rho,(E/c)^2-(m_0c)^2=\text{const.}\equiv b$) becomes:

$$\lambda^2(d\tau)^{(4/D_F)-2}(\Delta\sqrt{\rho}-c^{-2}\partial_{tt}\sqrt{\rho})=a\rho\sqrt{\rho}-b\sqrt{\rho}. \tag{49}$$

Moreover, with the substitutions:

$$\begin{aligned}&\left(k^2-\frac{\omega^2}{c^2}\right)^{-1/2}\frac{(2b)^{1/2}}{\lambda}(d\tau)^{1-(2/D_F)}(k_xx+k_yy+k_zz-\omega t)=\xi\\&\mathbf{k}^2=k_x^2+k_y^2+k_z^2,\\&\sqrt{\rho}=\left(\frac{b}{a}\right)^{1/2}\cdot f\end{aligned} \tag{50}$$

Equation (49) takes the form:

$$\partial_{\xi\xi}f=f^3-f \tag{51}$$

Equation (51) can be also obtained by means of the non-differentiable variational principle $\delta\int Ld\overline{\tau}=0$, with $d\overline{\tau}$ the non-differentiable elementary volume applied to the non-differentiable Lagrangian density:

$$L=\frac{1}{2}(\partial_\xi f)^2-\wp(f) \tag{52}$$

with the "potential":

$$\wp(f)=\left(\frac{f^4}{4}\right)-\left(\frac{f^2}{2}\right) \tag{53}$$

The equation $\partial_{\xi\xi}f=0$ has the solutions $f_F^{(1)}=0,f_F^{(2,3)}=\pm1$. By calculating the second derivative with respect to ξ of the "potential" (53) and substituting the values $f^{(1,2,3)}$ into the result of this differentiation, we find $\wp_{\xi\xi}(0)=-1,\wp_{\xi\xi}(\pm1)=2>0$, *i.e.*, the solution $f_F^{(2,3)}=\pm1$ is associated with the minimum "energy". Hence, the model under consideration has a double-degenerated non-differentiable vacuum state (for details for the standard case, see [21]).

From (52) results both the "energy",

$$\varepsilon(f)=\int_{-\infty}^{+\infty}d\xi\left[\frac{1}{2}(\partial_\xi f)^2+\wp(f)\right] \tag{54}$$

and the "energy" relative to the non-differentiable vacuum:

$$\varepsilon(f) - \varepsilon(f_F) = \int_{-\infty}^{+\infty} d\xi \left[\frac{1}{2}(\partial_\xi f)^2 + \frac{1}{4}(f^2 - 1)^2\right] \tag{55}$$

Since all terms in (55) are positive and in view of the infinite limits of integration, the finiteness of the "energy" implies that at $\xi \to \pm\infty$:

$$\partial_\xi f = 0, \frac{1}{4}(f^2 - 1)^2 = 0 \tag{56}$$

From this, it follows that at $\xi \to \pm\infty$, the function $f(\xi)$ tends to its non-differentiable vacuum value $f_F^{(2,3)} \to \pm 1$. In order to find the explicit form of the solution of (52), we multiply it by $\partial_\xi f$ and subsequently over ξ. This yields:

$$\frac{1}{2}(\partial_\xi f)^2 = -\frac{f^2}{2} + \frac{f^4}{4} + \frac{1}{2} f_0 \tag{57}$$

where f_0 is a non-differentiable integration constant. From this, we have:

$$\xi - \xi^0 = \int_0^f \frac{df}{\sqrt{\frac{f^4}{4} - \frac{f^2}{2} + \frac{1}{2} f_0}} \tag{58}$$

where ξ^0 is the other non-differentiable integration constant. To this solution, it corresponds, for an arbitrary f_0, an infinite value of the "energy" $\varepsilon(f)$. To obtain the solution with finite "energy", we make use of the boundary conditions $f_F^{(2,3)} = \pm 1$. From (57) it results that $f_0 = 1/2$. Replacing this value of f_0 into (58), the solution $f_k(\xi)$ of Equation (57) with a finite "energy" is:

$$f_k(\xi) = f(\xi - \xi^0) = \tanh\left[\frac{1}{\sqrt{2}}(\xi - \xi^0)\right] \tag{59}$$

This is called the non-differentiable kink solution (the reader can refer to [22,23] for details concerning kink-type standard solutions).

Combining (55) with the expression $f_F^{(2)} = 1$ and the expression for f_k, we obtain the "energy" of the non-differentiable kink relative to the non-differentiable vacuum:

$$\varepsilon(f_k) - \varepsilon(f_F) = \frac{2\sqrt{3}}{3} \tag{60}$$

The non-differentiable kink solution is obtained by a non-differentiable spontaneous symmetry breaking (the non-differentiable vacuum state is not invariant with respect to the non-differentiable group of transformations, which makes invariant Equation (51), while the non-differentiable Lagrangian density is invariant with respect to the same group). This corresponds to a non-differentiable pattern in the form of the Cooper-type non-differentiable pair (particularly, the superconducting pair (Cooper pair) from superconductivity of Type I) [24].

7. Non-Differentiable Topology and Logic

A non-differentiable topological method can be applied because the admissible number of non-differentiable kinks is determined by the non-differentiable topological properties of the non-differentiable symmetry group of Equation (51). In this context, the following problems must be solved:

(i) The number of admissible non-differentiable kink solutions determined by the non-differentiable topological properties of the Equation (51);

(ii) The non-differentiable topological charge.

The non-differentiable kink solution can be obtained as a mapping of a non-differentiable zero-sphere S^{F0}, taken at infinity onto the non-differentiable vacuum manifold of the model induced by means of Equation (51). The non-differentiable homotopy group for this model is $\Pi_{F0}(Z_{F0}) = Z_{F2}$, *i.e.*, the model gives rise to two solutions: a constant solution and the non-differentiable kink solution. Details on an usual homotopic mapping are given in [24].

The non-differentiable topological charge is:

$$q_F = \frac{1}{2}\int_{-\infty}^{+\infty} j(\xi)d\xi = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{df}{d\xi}d\xi = \frac{1}{2}[f(+\infty) - f(-\infty)] \tag{61}$$

The non-differentiable vacuum solution (absence of spatial gradients) and the non-differentiable kink solution can be characterized by attributing the $q_F = 0$ and $q_F = 1$, respectively (the result is obtained by an adequate normalization of f). Since (51) is a non-differentiable Ginzburg–Landau equation type [23], it follows that $q_F = 0$, and the non-differentiable vacuum solution describes the behavior of the non-differentiable system in the absence of self-structuring, *i.e.*, its non-differentiable ground states, while $q_F = 1$ and the non-differentiable kink solution describes the behavior of the non-differentiable system in the presence of self-structuring, *i.e.*, of the Cooper-type non-differentiable pair generation.

Now, for these values of the non-differentiable topological charge, one can associate the fractal bit, that is a physical system that can exist in two distinct states (an unstructured state or non-differentiable vacuum and a structured one or of the Cooper-type non-differentiable pair). These states are used in order to represent $0(d\tau)$ and $1(d\tau)$, that is a single binary fractal digit. The only possible operations (non-differentiable (fractal) gates) that are compatible with such systems are the non-differentiable (fractal) IDENTITY

$$0(d\tau) \to 0(d\tau), 1(d\tau) \to 1(d\tau)$$

and the non-differentiable (fractal) NOT(FNOT):

$$0(d\tau) \to 1(d\tau), 1(d\tau) \to 0(d\tau)$$

All of these constitute the fundaments of a non-differentiable (fractal) logic.

8. Concluding Remarks

The main conclusions of the present paper are the following:

(i) Assuming that in a Minkowski-type space-time the motions of structural units take place on continuous, but non-differentiable curves, a scale relativity theory with an arbitrary constant fractal dimension is built.

(ii) Non-differentiable geodesics on a space-time manifold and its diverse variants (the hydrodynamics one and in the wave function) are obtained. Particularly, if the wave function is independent of the motion curve affine parameter and if we consider a scale resolution dependence on a fundamental length, then the non-differentiable geodesics imply a non-differentiable Klein–Gordon-type equation. In this last situation, the standard result (Klein–Gordon equation) is obtained for motions on Peano curves at the Compton scale.

(iii) The concept of non-differentiable entropy on a space-time manifold (relativistic non-differentiable entropy) is introduced. It is proven that its three-dimensional projection is dependent (through total energy, internal energy and impulse) on the structural unit motion state. In such a context, the Klein–Gordon equation corresponds to a particular case of geodesics that is independent of its proper time, precisely those for which the motion takes place on Peano curves at the Compton scale and constant non-differentiable entropy.

(iv) Admitting that there exists a functional dependence between the phase and the amplitude of the wave function, in accordance with Consequence (vi) of Section 2, the non-differentiable fluid is self-structuring through a spontaneous symmetry breaking-type mechanism. Cooper-type non-differentiable pairs result, which confer superconductibility-type properties to the non-differentiable fluid. Moreover, the motions on such geodesics imply maximum entropy.

(v) Since the admissible number of non-differentiable kinks is determined by the non-differentiable topological properties of the symmetry group induced by Equation (11), a non-differentiable topological method can be applied. Then, some elements of a fractal logic, such as fractal bits, fractal gates (fractal IDENTITY, fractal NOT), *etc*., are obtained.

(vi) Such a formalism can be applied to complex systems in biology, precisely in problems related to fertility: the coupling between ovule and spermatozoon. The efficient interaction between one sperm and one oocyte leading to fertilization relies on specific informational energy exchange events. As a consequence, shortly after fertilization, before the first mitotic cell division, a developmental transition process commences, in order to trigger cell modeling from each gamete to a complex, multipotent zygote [25]. Genetically-encrypted data allow for various pathways to be upregulated, among which autophagy is one of the main players. During this cellular process, paternal mitochondria are selectively destructed in fertilized eggs [26]. The rationale for this event relies on evolutionary conservation strategies to prevent both the transmission of paternal mitochondrial DNA to the offspring and the establishment of heteroplasmy [27]. This is necessary, as, during fertilization, sperms compete with each other to reach and fertilize the oocyte and, in doing so, consume a great amount of energy produced by mitochondria via oxidative phosphorylation, generating reactive oxygen species (ROS), which could irreparably damage the integrity of mitochondrial DNA. This may be the main explanation why mitochondrial DNA mutates at a faster rate than nuclear DNA [26]. There are studies sustaining the avoidance of ROS-dependent mutation as an evolutionary pressure underlying maternal mitochondrial inheritance and the developmental origin of the female germ line [28]. Most interestingly, it has been postulated that the maintenance of any gene within a bioenergetic organelle may be the result of natural selection with a selective advantage for the individual organelle in its ability to respond to changes in the redox state of its bioenergetic membrane and to regulate the synthesis of proteins in the electron transport chain by means of gene expression [28]. Therefore, in complex systems, complex bioinformatic systems are required to ensure long-term species conservation.

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Author Contributions

In this paper, Maricel Agop provided the original idea and constructed its framework. Together with Alina Gavriluţ, the detailed calculation was conducted, and they were responsible for drafting and revising the whole paper. Gavril Ştefan devoted efforts to some valuable comments on revising the paper. Bogdan Doroftei devoted efforts to revising the paper. All authors have read and approved the final manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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