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Article

Delayed-Compensation Algorithm for Second-Order Leader-Following Consensus Seeking under Communication Delay

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Abstract: In this paper, the leader-following consensus algorithm, which is accompanied with compensations related to neighboring agents' delayed states, is constructed for second-order multi-agent systems with communication delay. Using frequency-domain analysis, delay-independent and delay-dependent consensus conditions are obtained for second-order agents respectively to converge to the dynamical leader's states asymptotically. Simulation illustrates the correctness of the results.

Keywords: second-order multi-agent systems; communication delay; leader-following consensus problem; delayed-compensation algorithm

1. Introduction

Coordination control mechanism of distributed multiple autonomous agents exists commonly in social, nature and engineering applications. As one of the hottest issues in multi-agent systems, consensus problem, which requires the several autonomous agents reach a state agreement in a distributed cooperative manner, has stimulated more and more research interests recently in various research fields, such as physics, artificial intelligence, automatic control, *etc*.

Communication delays related to information transmission cannot be neglected in multi-agent network, and consensus algorithms with communication delays are usually divided into synchronously-coupled and asynchronously-coupled forms. In synchronously-coupled form, each agent introduce self-delays, which equal the corresponding communication delays, in the coordination control part. In asynchronously-coupled form, each agent uses its current state, or its delayed state with the delay different from the corresponding communication delay to compare with its delayed neighboring agents' states.

Convergence of synchronously-coupled consensus algorithm depends on communication delay strictly for multi-agent systems under fixed [1–5] or switched topologies [6–13]. To our great delight, the first-order and second-order agents with stationary consensus algorithms in the asynchronously-coupled form can be robust to arbitrary communication delays by choosing proper control parameters [14–22]. However, under dynamical consensus algorithm composed of the position and velocity consensus coordination parts, asynchronously-coupled form just drives the second-order agents to achieve a stationary consensus, and the consensus convergence is strictly dependent on the communication delay [23,24].

To make second-order agents with dynamical consensus algorithm in asynchronously-coupled form get the original dynamical consensus, delayed-compensation consensus algorithms have been proposed in [25–28]. Liu *et al.* designed the delay-dependent compensations based on the information of global desired objective for each agent, and obtained delay-independent consensus condition for second-order multi-agent systems [25–27]. Liu and Liu [28] constructed compensations for normal asynchronously-coupled consensus algorithm based on neighboring agents' delayed states, and gained delay-dependent sufficient conditions for second-order multi-agent systems with communication delay under a general digraph. The results in [28] have demonstrated that asynchronously-coupled consensus algorithm accompanied with delay-dependent compensations can tolerate higher communication delay than synchronously-coupled consensus algorithm.

In this paper, leader-following consensus algorithm composed of the position and velocity consensus coordination parts is constructed to make second-order agents converge to a dynamical leader. The leader-following algorithm in asynchronously-coupled form is accompanied with compensations on neighboring agents' delayed states, but does not require each agent to get leader's states. Using frequency-domain analysis, we obtain delay-independent sufficient condition for second-order agents to converge to the dynamical leader's states asymptotically. Then, delay-dependent sufficient conditions are obtained for the multi-agent systems with communication delay under general digraph based on linear fractional transformation and small-gain theorem.

2. Problem Formulation

2.1. Agent's Dynamics

Investigate the second-order dynamic agents:

$$\begin{aligned} \dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= u_i(t), i = 1, 2, \cdots, n, \end{aligned} \tag{1}$$

where $x_i \in R$, $v_i \in R$, and $u_i \in R$ denote the position, velocity and acceleration, respectively, of the agent i. In addition, the leader's dynamics are modelled as

$$\dot{x}_0(t) \quad = \quad v_0, \tag{2}$$

where $x_0 \in R$ denotes the position, and $v_0 \in R$ is a constant denoting the desired velocity for all agents (1). As a typical control method, leader-following consensus algorithms have been extensively adopted to solve the stationary and dynamical consensus problems for multi-agent systems, and abundant results have been achieved in the past decade [5,6,25,26,29–32]. Under the leader-following coordination control structure, the agents' states are required to converge to the leader's states asymptotically, *i.e.*, $\lim_{t\to\infty} x_i(t) = x_0(t)$, $\lim_{t\to\infty} v_i(t) = v_0$.

2.2. Interconnection Topology

Interconnection topology of multi-agent systems is usually described as a digraph. Agents can be considered as the nodes of a digraph, while the information flow between neighboring agents is regarded as a directed edge between the neighboring nodes of the digraph.

A weighted digraph $G = (V, E, A)$ of order n is composed of a set of vertices $V = \{1, \cdots, n\}$, a set of edges $E \subseteq V \times V$ and a weighted adjacency matrix $A = [a_{ij}] \in R^{n\times n}$ with $a_{ij} \geq 0$. The node indexes belong to a finite index set $\mathcal{I} = \{1, \cdots, n\}$. A directed edge from the node i to the node j of the diagraph G is denoted by $e_{ij} = (i, j) \in E$. We assume that the adjacency elements associated with the edges of the digraph are positive, *i.e.*, $a_{ij} > 0 \Leftrightarrow e_{ij} \in E$. Moreover, we assume $a_{ii} = 0$ for all $i \in \mathcal{I}$. The set of neighbors of node i is denoted by $N_i = \{j \in V : (i, j) \in E\}$. The Laplacian matrix of the weighted digraph G is defined as $L = D - A = [l_{ij}] \in R^{n\times n}$, where $D = \mathrm{diag}\{\sum_{j=1}^{n} a_{ij}, i \in \mathcal{I}\}$ is the degree matrix of G.

In the digraph, if there is a path from one node i to another node j, then j is said to be *reachable* from i. If a node is reachable from every other node in the digraph, then we say it *globally reachable*.

2.3. Delayed-Compensation Consensus Algorithm

For second-order agents (1), we take following consensus algorithm similar to that in [5,25]

$$u_i(t) = \kappa\Big(\sum_{j\in N_i} a_{ij}((x_j(t) - x_i(t)) + \gamma(v_j(t) - v_i(t))) + b_i((x_0(t) - x_i(t)) + \gamma(v_0 - v_i(t)))\Big), \tag{3}$$

where $\kappa > 0$, $\gamma > 0$, N_i denotes the neighbors of agent i, $a_{ij} > 0$ is adjacency element of A in the digraph $G = (V, E, A)$, and b_i denotes the linking weight from agent i to the leader (2). Assume that $b_i > 0$ if there is a directed edge from agent i to the leader; otherwise, $b_i = 0$. In the rest of the paper, the notation $B = \mathrm{diag}\{b_i, i \in \mathcal{I}\}$ is used.

With non-negligible communication delay, the algorithm (3) in asynchronously-coupled form becomes

$$\begin{aligned} u_i(t) \quad = \quad & \kappa\Big(\sum_{j\in N_i} a_{ij}((x_j(t-\tau) - x_i(t)) + \gamma(v_j(t-\tau) - v_i(t))) \\ & + b_i((x_0(t-\tau) - x_i(t)) + \gamma(v_0 - v_i(t)))\Big), \end{aligned} \tag{4}$$

where $\tau > 0$ is the communication delay between neighboring agents.

Remark 1. *Used in the stationary consensus problem of multi-agent systems with communication delays [4,22], asynchronously-coupled consensus algorithm has shown its delay-independent consensus convergence property. With regard to the dynamical consensus problem of second-order or high-order multi-agent systems, nevertheless, asynchronously-coupled consensus algorithm changes the final consensus behavior and cannot achieve the original dynamical consensus convergence [24,28]. In the same way, it is evident that the algorithm (4) cannot drive second-order agents (1) to converge to the dynamical leader's states asymptotically. In this case, several consensus algorithms with compensations related to delayed states of neighboring agents [28], desired target [25] and dynamical leader [26,27] were added to consensus algorithm in order to remain the original control objective. The compensation constructed in this paper is similar to that in [28], but we just add compensation to position coordination part:*

$$\begin{aligned} u_i &= \kappa(\sum_{j\in N_i} a_{ij}((x_j(t-\tau)-x_i(t)-(x_j(t-(m+1)\tau)-x_j(t-m\tau)))+\gamma(v_j(t-\tau)-v_i(t))) \\ &\quad +b_i((x_0(t-\tau)-x_i(t)-(x_0(t-(m+1)\tau)-x_0(t-m\tau)))+\gamma(v_0-v_i(t)))), \end{aligned} \tag{5}$$

where $-(x_j(t-(m+1)\tau)-x_j(t-m\tau))$ is compensation, and $m \geq 1$ is an integer.

Remark 2. *The compensation's form in (5) is more general than that in [28], and we can choose the integer m arbitrarily. Analogous to the delayed-compensation consensus algorithm in [25–28], the new algorithm (5) also requires each agent to know exact value of communication delay. Particularly, the compensations in algorithm (5) is different from the delayed-compensation leader-following consensus algorithm in [26,27], since our proposed algorithm does not require each agent to get the leader's states.*

Then, the closed-loop form of second-order dynamic agents (1) with (5) is formulated as

$$\begin{aligned} \dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= \kappa(\sum_{j\in N_i} a_{ij}((x_j(t-\tau)-x_i(t)-(x_j(t-(m+1)\tau)-x_j(t-m\tau)))+\gamma(v_j(t-\tau)-v_i(t))) \\ &\quad +b_i((x_0(t-\tau)-x_i(t)-(x_0(t-(m+1)\tau)-x_0(t-m\tau)))+\gamma(v_0-v_i(t)))), i\in\mathcal{I}. \end{aligned} \tag{6}$$

For leader-following structure, we come to the following property from Lemma 2 in [25] and Lemma 3 in [29].

Lemma 1. *Assume that the interconnection topology graph of n agents together with the leader in system (6) has the leader as a globally reachable node. Then, the matrix $L+B$ has no zero eigenvalues, and $D+B > \mu I$ where $\mu > 0$, and L is the Laplacian matrix of the interconnection topology of n agents without leader.*

Define $\bar{x}_i = x_i - x_0, \bar{v}_i = v_i - v_0, i \in \mathcal{I}$, and we get

$$\begin{aligned} \dot{\bar{x}}(t) &= \bar{v}(t), \\ \dot{\bar{v}}(t) &= \kappa(A\bar{x}(t-\tau)-D\bar{x}(t)-A\bar{x}(t-(m+1)\tau)+A\bar{x}(t-m\tau)+\gamma A\bar{v}(t-\tau)-\gamma D\bar{v}(t) \\ &\quad -B\bar{x}(t)-\gamma B\bar{v}(t)), \end{aligned} \tag{7}$$

where $\bar{x} = [\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n]^T$ and $\bar{v} = [\bar{v}_1, \bar{v}_2, \cdots, \bar{v}_n]^T$. Taking Laplace transform of system (7) yields the characteristic equation on $\bar{x}(t)$ as follows

$$\det(s^2 I + \kappa(-A\mathrm{e}^{-\tau s} + A\mathrm{e}^{-(m+1)\tau s} - A\mathrm{e}^{-m\tau s} - \gamma s A\mathrm{e}^{-\tau s} + (D+B)(1+\gamma s))) = 0. \tag{8}$$

3. Main Results

3.1. Delay-Independent Consensus Criterion

Theorem 1. *Assume that the interconnection topology of n agents and a leader in system (6) has the leader as a globally reachable node. Then, all the agents in system (6) asymptotically converge to the leader's state, if*

$$\bar{\sigma}(\frac{\kappa(3 + \mathrm{j}(2+\gamma\omega))A}{-\omega^2 I + \kappa(1+\mathrm{j}\gamma\omega)(D+B)}) < 1 \tag{9}$$

hold for $\omega \in R$, where $\bar{\sigma}(\cdot)$ denotes the largest singular value of matrix.

Before proving Theorem 1, we introduce the following lemma firstly.

Lemma 2. *If $|h_1| < |h_2|$, $\bar{\sigma}(h_1 W) < \bar{\sigma}(h_2 W)$, where $W \in C^{n\times n}$, $h_1 \in C$, $h_2 \in C$, and C denotes the complex number.*

Lemma 2 can be easily proved according to the definition of matrix's singular value.

Now, we present the proof of Theorem 1.

The characteristic Equation (8) can be rewritten as

$$\det(s^2 I + \kappa(D+B)(1+\gamma s) + \kappa A\mathrm{e}^{-\tau s}(-1 + \mathrm{e}^{-m\tau s} - \mathrm{e}^{-(m-1)\tau s} - \gamma s)) = 0. \tag{10}$$

From Lemma 1, we get $D + B > \mu I$ for some $\mu > 0$, so $s^2 I + \kappa(D+B)(1+\gamma s)$ just has its zeros on the open left half complex plane. According to the generalized Nyquist stability criterion, the roots of characteristic Equation (10) all lie on the open left half complex plane, as long as the eigenloci of $M_1(\mathrm{j}\omega) = \frac{\kappa A\mathrm{e}^{-\mathrm{j}\omega\tau}(-1+\mathrm{e}^{-\mathrm{j}m\omega\tau} - \mathrm{e}^{-\mathrm{j}(m-1)\omega\tau} - \mathrm{j}\gamma\omega)}{-\omega^2 I + \kappa(D+B)(1+\mathrm{j}\gamma\omega)}$, *i.e.*, $\lambda(M_1(\mathrm{j}\omega))$, does not enclose the point $(-1, \mathrm{j}0)$ for $\omega \in R$, where $\lambda(\cdot)$ denotes the matrix eigenvalue.

Since

$$\begin{aligned} & |-1 + \mathrm{e}^{-\mathrm{j}m\tau\omega} - \mathrm{e}^{-\mathrm{j}(m-1)\tau\omega} - \mathrm{j}\gamma\omega| \\ = \; & |-1 + \cos(m\tau\omega) - \cos((m-1)\tau\omega) + \mathrm{j}(-\sin(m\tau\omega) + \sin((m-1)\tau\omega) - \gamma\omega)| \\ < \; & |3 + \mathrm{j}(2+\gamma\omega)| \end{aligned}$$

hold for $\omega \in (0, \infty)$, we obtain from Lemma 2 and the condition (9) that

$$\begin{aligned} \rho(M_1(\mathrm{j}\omega)) & \leq \bar{\sigma}(M_1(\mathrm{j}\omega)) \\ & = \bar{\sigma}(\frac{\kappa A(-1 + \mathrm{e}^{-\mathrm{j}m\tau\omega} - \mathrm{e}^{-\mathrm{j}(m-1)\tau\omega} - \mathrm{j}\gamma\omega)}{-\omega^2 I + \kappa(D+B)(1+\mathrm{j}\gamma\omega)}) \\ & < \bar{\sigma}(\frac{\kappa(3 + \mathrm{j}(2+\gamma\omega))A}{-\omega^2 I + \kappa(1+\mathrm{j}\gamma\omega)(D+B)}) \\ & < 1 \end{aligned} \tag{11}$$

holds for $\omega \in (0, \infty)$, where $\rho(\cdot)$ denotes matrix spectral radius.

Hence, $\lambda(M_1(\mathrm{j}\omega))$ does not enclose the point $(-1, \mathrm{j}0)$ for all $\omega \in R$, *i.e.*, the roots of the characteristic Equation (10) all lie on the open left half of the complex plane. Therefore, the closed-loop system (7) is asymptotically stable, and the agents in (6) converge to the leader's states asymptotically. Theorem 1 is proved.

Remark 3. *The result in Theorem 1 demonstrates that the delay robustness of delayed-compensation leader-following algorithm (5) is much better than that of synchronously-coupled consensus algorithm, since convergence of synchronously-coupled consensus algorithm always depends on communication delay strictly from existing works* [2,3,7]. *However, the delay-independent sufficient condition (9) is effective for second-order multi-agent systems with proper interconnection topology and coupling weights, but not for all interconnection topologies that have the leader as a globally reachable node.*

Remark 4. *Assume that the interconnection topology of n agents and a leader in system (6) has the leader as a globally reachable node, and each agent just has one direct path to reach the leader. For simplicity, we can assume that the direct edge from agent i to j satisfies $i > j$, and we get*

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ a_{32} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ a_{n2} & a_{n3} & \cdots & 0 \end{bmatrix}.$$

Then, we can obtain

$$\begin{aligned} & \rho(M_1(\mathrm{j}\omega)) \\ = \; & \rho(\mathrm{diag}\{\frac{\kappa(-1+\mathrm{e}^{-\mathrm{j}m\tau\omega}-\mathrm{e}^{-\mathrm{j}(m-1)\tau\omega}-\mathrm{j}\gamma\omega)}{-\omega^2+\kappa(d_i+b_i)(1+\mathrm{j}\gamma\omega)}\}A) \\ = \; & 0 < 1, \end{aligned}$$

since the eigenvalues of A are all equivalent to zero. Therefore, all the agents in system (6) asymptotically converge to the leader's states, i.e., our proposed algorithm (5) can tolerate arbitrary communication delay without regard to the coupling weights and the control parameters.

From Theorem 1, if interconnection topology has more directed edges from agents to the leader and less links between agents themselves, and the coupling weights between agents and leader are higher, the agents can achieve delay-independent asymptotic consensus more easily.

Example 1. *Investigate a multi-agent network with four agents and a leader given by (6), and the interconnection topology is shown in Figure 1. It is obvious that the leader is the globally reachable node. The weights of the directed edges are: $a_{12} = 1.5, a_{24} = 0.3, a_{32} = 1.4, a_{41} = 0.9, a_{42} = 0.6, b_1 = 5, b_2 = 8$, and the control parameters are chosen as $\kappa = 1.5$ and $\gamma = 0.8$. Besides, we take $m = 3$ for the compensations in (6). With given coupling weights and control parameters, two largest singular values $\bar{\sigma}_1 = \bar{\sigma}(\frac{\kappa A(-1+\mathrm{e}^{-\mathrm{j}m\tau\omega}-\mathrm{e}^{-\mathrm{j}(m-1)\tau\omega}-\mathrm{j}\gamma\omega)}{-\omega^2 I+\kappa(D+B)(1+\mathrm{j}\gamma\omega)})$ and $\bar{\sigma}_2 = \bar{\sigma}(\frac{\kappa(3+\mathrm{j}(2+\gamma\omega))A}{-\omega^2 I+\kappa(1+\mathrm{j}\gamma\omega)(D+B)})$ in (11) are shown in Figure 2. From the condition (9) in Theorem 1, the agents converge to the leader's states without any relationship with the communication delay (see, Figure 3).*

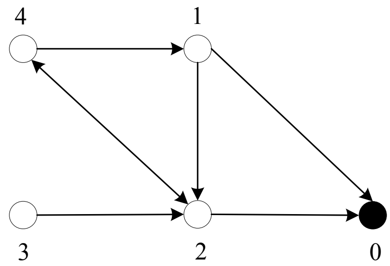

Figure 1. Topology 1: Network of four agents and a leader.

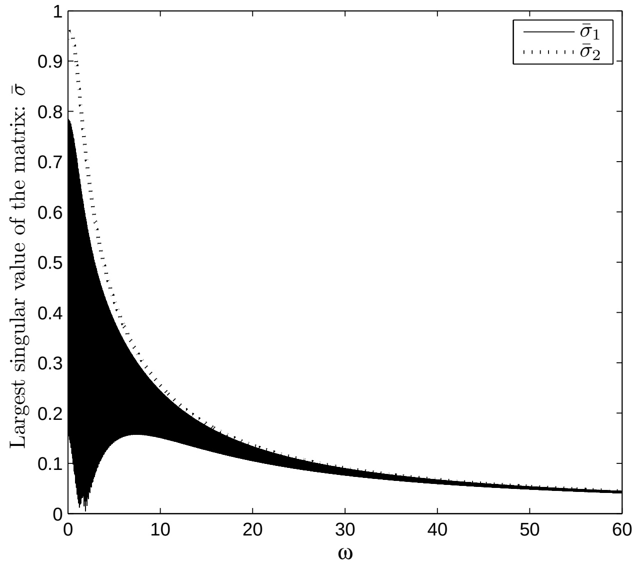

Figure 2. Largest singular value with $b_1 = 5$ and $b_2 = 8$.

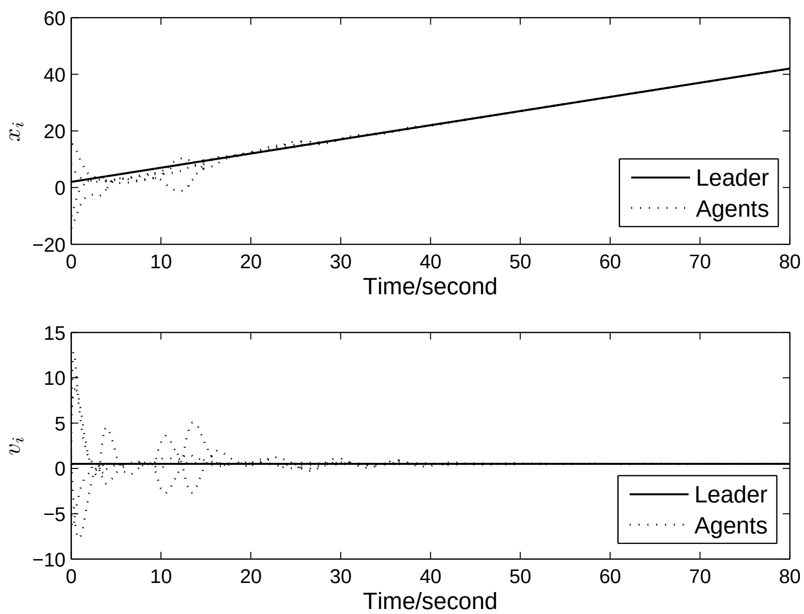

Figure 3. Delay-independent consensus convergence.

However, the condition (9) does not hold for any coupling weights and interconnection structure. In topology 1, if the coupling weights from agent 1 and agent 2 to the leader are chosen as $b_1 = 0.5$ and $b_2 = 0.8$, and $\bar{\sigma}_1$ and $\bar{\sigma}_2$ are shown in Figure 4. Obviously, the condition (9) does not hold, and we find that the communication delay that the system (6) can bear now is $\tau < 1.46$(s) through numerical simulation. Moreover, if the interconnection topology of four agents and a leader is described in Figure 5, and we choose the same coupling weights and control parameters as above. Then, it is found that the condition (9) can not hold with arbitrary coupling weight b_2.

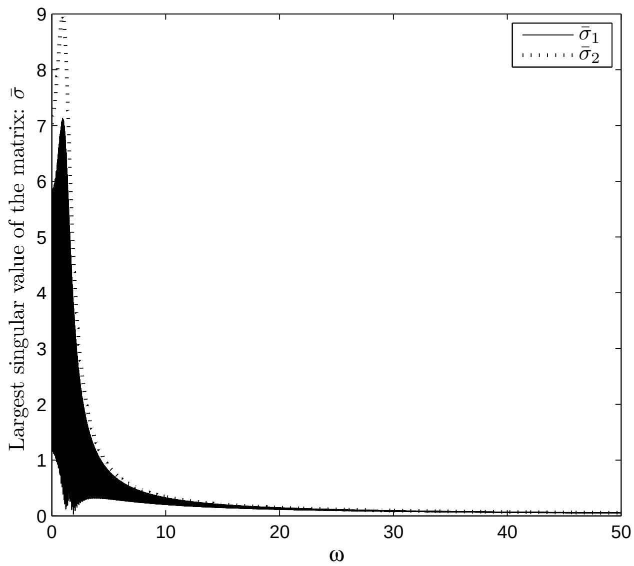

Figure 4. Largest singular values with $b_1 = 0.5$ and $b_2 = 0.8$.

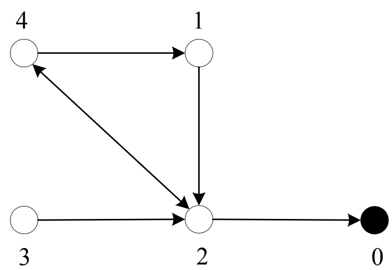

Figure 5. Topology 2: Network of four agents and a leader.

3.2. Delay-Dependent Consensus Criterion

Theorem 2. *Assume that the agents in system (6) without communication delay converges to the leader's states asymptotically. Let*

$$M_2(s) = \frac{(\frac{s^2}{1+\gamma s}(L+B)^{-1}A)G(s)}{I+G(s)}, \tag{12}$$

where $G(s) = \frac{\kappa(1+\gamma s)}{s^2}(L+B)$*. Then, all the agents in system (6) converge to the leader's states asymptotically, if*

$$\tau(m\tau+\gamma)\bar{\sigma}(M_2(j\omega)) < 1 \tag{13}$$

i.e.,

$$\tau < \frac{-\gamma + \sqrt{\gamma^2 + \frac{4m}{\bar{\sigma}(M_2(j\omega))}}}{2m} \tag{14}$$

holds for $\omega \in R$.

Proof. Under the assumption that the agents without communication delay converge to the leader's states asymptotically, the roots of following equation

$$\det(s^2 I + \kappa(1+\gamma s)(L+B)) = 0$$

all lie on the open left half complex plane, so we obtain $\mathrm{rank}(L+B) = n$.

Rewrite the Equation (8) as follows

$$\det(s^2 I + \kappa(1+\gamma s)(L+B)(I + \frac{1}{1+\gamma s}(L+B)^{-1}A((1-e^{-s\tau})(1-e^{-m\tau s}) + \gamma s(1-e^{-\tau s}))) = 0. \tag{15}$$

Let $p(s) = \det(s^2 I + \kappa(1+\gamma s)(L+B)(I + \frac{1}{1+\gamma s}(L+B)^{-1}A((1-e^{-s\tau})(1-e^{-m\tau s}) + \gamma s(1-e^{-\tau s})))$. When $s = 0$, $p(0) = \det(0^2 I + \kappa(1+\gamma 0)(L+B)(I + \frac{1}{1+\gamma 0}(L+B)^{-1}A((1-e^{-0\tau})(1-e^{-m\tau 0}) + \gamma 0(1-e^{-\tau 0}))) = \det(\kappa(L+B)) \neq 0$, *i.e.*, $p(s)$ has no zeros at $s = 0$.

When $s \neq 0$, the Equation (15) equals

$$\det(I + \frac{\kappa(1+\gamma s)}{s^2}(L+B)(I + \frac{1}{1+\gamma s}(L+B)^{-1}A((1-e^{-s\tau})(1-e^{-m\tau s}) + \gamma s(1-e^{-\tau s}))) = 0. \tag{16}$$

The feedback diagram corresponding to the Equation (16) is illustrated in Figure 6. Using the linear fractional transformation, the diagram in Figure 6 can be equivalently transformed into the form shown in Figure 7, where $M_2(s)$ is defined in Equation (12). The characteristic equation of the closed-loop system in Figure 7 is

$$\det(I + (\frac{1-e^{-m\tau s}}{s} + \gamma)\frac{1-e^{-\tau s}}{s}M_2(s)) = 0. \tag{17}$$

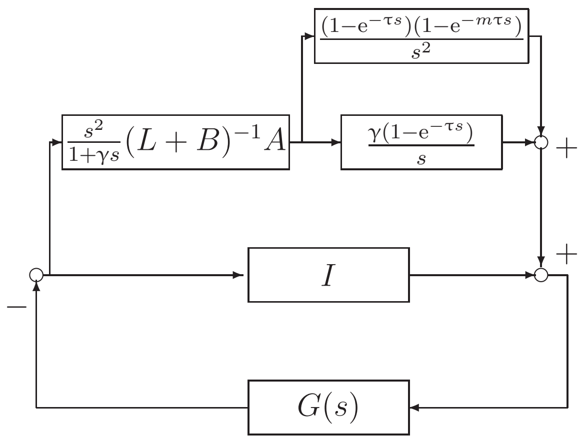

Figure 6. The diagram of the Equation (16).

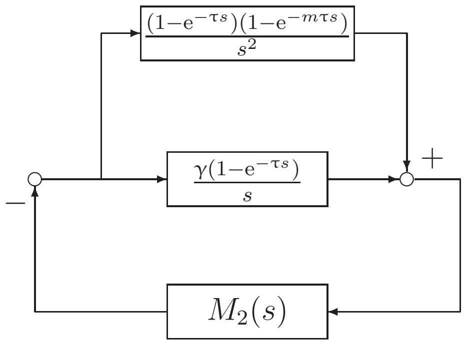

Figure 7. Equivalent transformation of Figure 6.

Obviously, $\frac{(1-e^{-m\tau s})}{s^2}$, $\frac{(e^{-\tau s}-1)}{s}$ and $M_2(s)$ both have no poles in the open right half complex plane from the assumption in Theorem 2.

By computing, we obtain

$$\begin{aligned}\rho\Big(\Big(\frac{1-e^{-jm\tau\omega}}{j\omega}+\gamma\Big)\frac{1-e^{-j\tau\omega}}{j\omega}M_2(j\omega)\Big) &\le \bar{\sigma}\Big(\Big(\frac{1-e^{-jm\tau\omega}}{j\omega}+\gamma\Big)\frac{1-e^{-j\tau\omega}}{j\omega}M_2(j\omega)\Big)\\ &\le \Big(\Big|\frac{1-e^{-jm\tau\omega}}{j\omega}\Big|+\gamma\Big)\Big|\frac{1-e^{-j\tau\omega}}{j\omega}\Big|\bar{\sigma}(M_2(j\omega)).\end{aligned}$$

Because $\max_{\omega\in[0,+\infty)}|\frac{1-e^{-j\tau\omega}}{j\omega}| < \tau$, from the condition (13),

$$\begin{aligned}\rho\Big(\Big(\frac{1-e^{-jm\tau\omega}}{j\omega}+\gamma\Big)\frac{1-e^{-j\tau\omega}}{j\omega}M_2(j\omega)\Big) &\le \tau(m\tau+\gamma)\bar{\sigma}(M_2(j\omega))\\ &< 1\end{aligned}$$

holds for all $\omega \in R$.

Based on small gain theorem, therefore, $\det(I + (\frac{1-e^{-m\tau s}}{s}+\gamma)\frac{1-e^{-\tau s}}{s}M_2(s))$ is non-singular for $Re(s) \ge 0$, *i.e.*, the roots of the characteristic Equation (17) all lie on the open left half complex plane. Hence, the agents in system (6) converge to the leader's states asymptotically. □

Remark 5. *For the interconnection topology that has the leader as a globally reachable node, the condition (14) in Theorem 2 always gives a criterion to calculate the delay bound for the multi-agent systems (6), but does not have special requirements on topology structure, coupling weights and control parameters.*

Remark 6. *Obviously, the positive integer m in the compensations has different impacts on the conditions (9) and (14) respectively. The condition (9) is independent of m, while the largest delay value in (14) decreases when m increases.*

Remark 7. *For n agents (6), delay-independent condition (9) can be applied firstly. If (9) does not hold, then, we use the delay-dependent condition (14) to obtain an upper bound of the delay value.*

Example 2. *Consider a multi-agent system composed of four agents and a leader given by (6). For simplicity, we choose the interconnection topology 2 in Figure 5. In addition, coupling weights are $a_{12} = 1.5, a_{24} = 0.3, a_{32} = 1.4, a_{41} = 0.9, a_{42} = 0.6, b_2 = 0.8$, the control parameters are $\kappa =$*

$1.5, \gamma = 0.8$, *and* $m = 2$. *Example 1 has shown that delay-independent condition (9) in Theorem 1 does not hold for the agents under topology 2 with these coupling weights and control parameters. According to the condition (14) in Theorem 2, we get that the agents in system (6) can reach the leader's states asymptotically if* $\tau < 0.2461$*(s) (see, Figure 8). To our delight, the leader-following consensus algorithm (5) can tolerate distinct largest communication delay with different* m*, for example,* $\tau_{\max} = 0.705(s)$ *with* $m = 1$*,* $\tau_{\max} = 0.71(s)$ *with* $m = 2$*,* $\tau_{\max} = 1.38$*(s) with* $m = 3$*, and* $\tau_{\max} = 0.98(s)$ *with* $m = 4$*. By choosing proper* m*, therefore, we can get an algorithm with best delay robustness. Moreover, the largest communication delay the synchronously-coupled consensus algorithm can bear in this case is* $\tau_{\max} = 0.32(s)$*, which is much less than our proposed algorithm (5).*

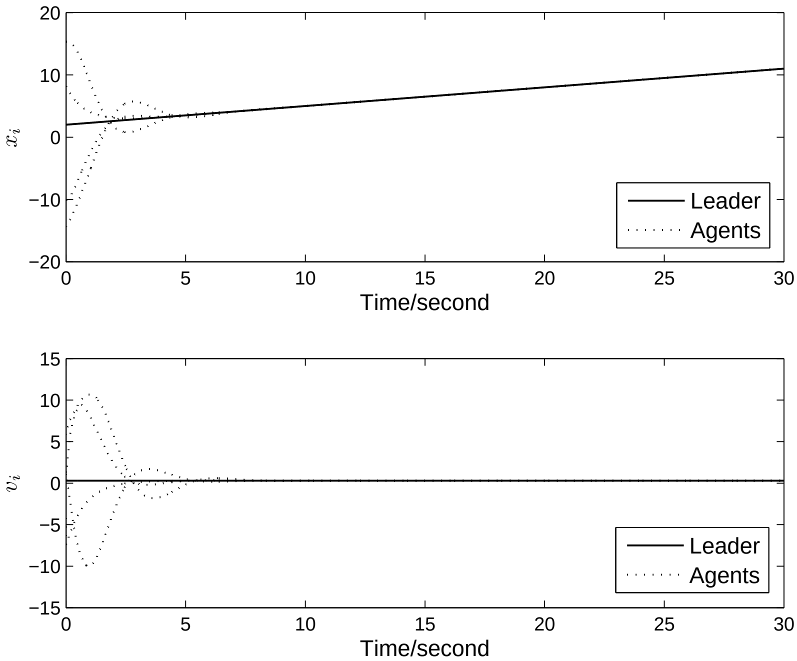

Figure 8. Delay-independent consensus convergence.

4. Conclusions

In this paper, we propose a delayed-compensation leader-following consensus algorithm for second-order multi-agent systems to track a dynamical leader asymptotically under communication delay. Our proposed algorithm includes the position and velocity consensus coordination parts in asynchronously-coupled form, and the compensations related to the delayed neighboring agents' states are just added into the position coordination part. Based on frequency-domain analysis, we obtain delay-independent and delay-dependent sufficient conditions, respectively, for the second-order agents converging to the dynamical leader's states asymptotically. Although the delay-independent condition does not hold for the agents with arbitrary interconnection structure and coupling weights, it sufficiently proves that delayed-compensation algorithm in asynchronously-coupled form can tolerate much higher communication delay than synchronously-coupled algorithm. The delay-dependent sufficient condition provides a criterion to calculate an upper bound of communication delay, and it is very interesting that choosing proper delay value of the compensations can yield different delay robustness for our proposed algorithm. However, the results herein are just for the second-order multi-agent systems with static interconnection topology and identical communication input delay, and the consensus conditions are just sufficient and a little conservative. Hence, our future work will focus on proposing some proper analysis methods to obtain less conservative consensus conditions for the second-order multi-agent systems under switching topologies and distinct communication delays.

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Author Contributions

Both authors contributed to the initial motivation of the work, to the research and numerical experiments and to the writing. Both authors have read and approved the final manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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