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Article

The Effect of a Long-Range Correlated-Hopping Interaction on Bariev Spin Chains

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Abstract: We introduce a long-range particle and spin interaction into the standard Bariev model and show that this interaction is equivalent to a phase shift in the kinetic term of the Hamiltonian. When the particles circle around the chain and across the boundary, the accumulated phase shift acts as a twist boundary condition with respect to the normal periodic boundary condition. This boundary phase term depends on the total number of particles in the system and also the number of particles in different spin states, which relates to the spin fluctuations in the system. The model is solved exactly via a unitary transformation by the coordinate Bethe ansatz. We calculate the Bethe equations and work out the energy spectrum with varying number of particles and spins.

Keywords: Bariev model; long-range interaction; twist boundary condition; energy spectrum

1. Introduction

One dimensional (1D) (quasi-1D) systems exhibit some of the most diverse and intriguing physical phenomena seen in all of condensed matter physics, such as charge (spin) density waves, quantum wires, quantum Hall bars, Josephson junction arrays, polymers and 1D Bose-Einstein condensates. The complete description of a solid is a complex many body problem. The particles are strongly correlated and cannot be understood by removing the interactions between them or by considering the effects of interactions as a perturbation. However, for some realistic low-dimensional strongly correlated systems a proper understanding has yet to be established through the examination of simplified exactly solvable models, in which the integrability has been considered to be one of the striking properties from the points of view of physics and mathematics. The 1D Hubbard model, in which the electron hopping is strongly disturbed by the on-site Coulomb interaction, has been mainly investigated with regard of Mott-transition through its exact solution [1]. The supersymmetric $t - J$ model [2], which includes the spin fluctuations via antiferromagnetic coupling, is relevant to the description of electronic mechanisms in high-T_c superconductivity. The 1D Bariev (interacting XY) chain [3,4] is also a Hubbard-like integrable model of special interest, as it supports Cooper type hole pairs. Motivated by the inclusion of additional interactions, whether through internal impurities or external boundary fields, many works have been carried out to generalize these models for different boundary fields [5–16]. This provides a non-perturbative method to study the boundary impurity effects in one-dimensional quantum systems in condensed matter physics. Bariev model has been generalized in many ways. The Hamiltonian studied in [17] included the onsite interaction and pair hopping processes. The Bariev chains with correlated single-particle and uncorrelated pair hopping were studied in [18], but there is only one type of particle. Bariev *et al.* [19,20] have considered the situations with multi-particle hopping and interchain tunneling, respectively. However, most of the investigated systems include only the nearest neighbor interactions; the question of how to find an integrable system with long range interaction is an interesting topic.

Schulz and Shastry [21] presented a class of lattice and continuum fermion models which are exactly solvable by a pseudo-unitary transformation, leading to nontrivial and non-Fermi-liquid behavior, with an exponential dependence upon the interaction. The idea behind this approach is the finding of a basis (through a unitary transformation of the original Fock basis [22,23]) in which the model takes the form of the original Hubbard or XXZ model up to boundary twists which do not affect their solvability. Furthermore, the Schultz-Sharstry model was generalized by introducing an exponential interaction involving two spins with same orientation [24].

In this paper, we generalize the Bariev model by introducing an Schultz-Sharstry-like exponential interaction which is dependent on the spin orientations of particles in the system. We note that the applied long-range spin-dependent interaction in the hopping term can be treated as a boundary phase twist. The phase change is in turn a function of number of particles and spins. When the Aharonov-Bohm effect is added to a 1D Hubbard chain with periodic boundary conditions it contributes to an extra phase shift related to the external magnetic flux [25,26], however our model can be applied both in the situation with external magnetic field and with internal field induced by impurities or spin fluctuations. We find the charge and spin excitations in our generalized model is a function of band filling, which is similar to the model proposed by Hirsch [27] for studying the high-T_c superconductivity. The latter, however, is not integrable in 1D. By applying an unitary transformation we prove the integrability of our model. The model is solved in the framework of the coordinate Bethe ansatz [28,29]. All charge and spin momenta are determined by a set of Bethe equations. The energy spectrum is listed based on the classification of varying number of particles. These may be useful in the systems where the long-range interactions cannot be ignored by only taking account of the nearest neighbour interactions.

2. From Long-Range Interactions to a Twist Boundary Condition

To include long-range spin interactions, we introduce some coordinate dependent parameters, α, β and κ into the Hamiltonian of the standard Bariev model. The Hamiltonian of the generalized Bariev model to be studied is in the form

$$H = -t\sum_{j=1}^{L}\sum_{\sigma=\uparrow,\downarrow}\left\{c_{j+1,\sigma}^{\dagger}c_{j,\sigma}e^{i\kappa_j(\sigma)}e^{i\sum_{l=1}^{L}\left[\alpha_{j,l}(\sigma)n_{l,-\sigma}+\beta_{j,l}(\sigma)n_{l,\sigma}\right]}+h.c.\right\}\times e^{-\eta\sum_{\sigma'\neq\sigma}n_{j+\theta(\sigma-\sigma'),\sigma'}}\ , \tag{1}$$

with $c_{j,\sigma}^{\dagger}$ ($c_{j,\sigma}$) being the creation (annihilation) operator of a particle with spin σ (σ being either $\uparrow$ or $\downarrow$) located at the j_{th} site, $n_{j,\sigma} \doteq c_{j,\sigma}^{\dagger}c_{j,\sigma}$ being the number operator, and $\theta(x)$ being a step function, *i.e.*, $\theta(x) = 1$ if $x > 0$ and $\theta(x) = 0$ if $x < 0$. The anti-commutation relation is satisfied by

$$\{c_{j,\sigma}, c_{l,\sigma'}^{\dagger}\} = \delta_{j,l}\delta_{\sigma,\sigma'}\ , \quad \{c_{j,\sigma}^{\dagger}, c_{l,\sigma'}^{\dagger}\} = \{c_{j,\sigma}, c_{l,\sigma'}\} = 0. \tag{2}$$

η is a coupling constant that influences the hopping amplitude of particles. Positive and negative values of η correspond to attractive and repulsive inter-particle interactions, respectively. It is clear that the system is reduced to standard Bariev model and is integrable if α, β and κ all vanish. The exponential term of $\alpha_{j,l}$ and $\beta_{j,l}$ is a generalized Jordan-Wigner transformation which includes interactions between the particle on the j_{th} site and the occupation state of all sites on the spin chain. This can be seen clearly if we make an expansion around small α and β. If we set $\alpha_{j,l} = \beta_{j,l} = \pi$ and take the summation of l from 1 to $j-1$, the generalized Jordan-Wigner transformation will degenerate into Jordan-Wigner transformation.

For arbitrary values of α, β and κ the system described by Equation (1) is not integrable by direct coordinate Bethe ansatz because all particles in the system are coupled through the long-range interaction. So the question now turns into how to determine these free parameters but keep the integrability. For this purpose, we introduce a special unitary transformation

$$U \doteq \exp\sum_{l,m=1}^{L+1}\sum_{\mu,\nu=\uparrow,\downarrow}\left[i(\xi_{l,m}^{\mu,\nu}n_{l,\mu}n_{m,\nu}+\zeta_{l,\mu}n_{l,\mu})\right]\ , \tag{3}$$

where $\xi_{l,m}$ is the spin interaction strength between two sites and $\zeta_l \in R$ is a parameter related to the local chemical potential and magnetic field. They are all free parameters to be confirmed by specific physical models. The subscripts l and m are coordinate indices, and the superscripts μ, ν are spin indices. This is similar to choosing a different basis for the coordinate Bethe ansatz calculations. We will show later that α, β and κ can be expressed in the form of ξ and ζ. Under the transformation $c_{j,\sigma} \xrightarrow{U} Uc_{j,\sigma}U^{-1}$, the hopping term in Hamiltonian Equation (1) turns into

$$c_{j+1,\sigma}^{\dagger}c_{j,\sigma} \xrightarrow{U} c_{j+1,\sigma}^{\dagger}c_{j,\sigma}\exp\left[2i(\xi_{j+1,m}^{\sigma,\mu}-\xi_{j,m}^{\sigma,\mu})n_{m,\mu}+i(\zeta_{j+1,\sigma}-\zeta_{j,\sigma}-2\xi_{j,j+1}^{\sigma,\sigma})\right]\ , \tag{4}$$

while the particle number operator keeps unchanged.

For normal periodic boundary conditions, there will be a phase change of pkL when one particle hops from site L to $L+1=1$ ($j=L$ in Hamiltonian Equation (1)), where $k=2\pi/L$ and p is an integer. We will give up the original boundary condition of the standard Bariev model but apply new boundary conditions which can keep the integrability of model Equation (1). For the transformation Equation (3), the phase shift across the boundary is determined by the relations $\xi_{1,m} \leftrightarrow \xi_{L,m}$, $\xi_{L,1} \leftrightarrow \xi_{L,L+1}$ and $\zeta_{1,m} \leftrightarrow \zeta_{L,m}$. Without loss of generality, we can set the phase change across the boundary to be

$$\begin{aligned} \xi^{\sigma,-\sigma}_{L+1,m} &\doteq \xi^{\sigma,-\sigma}_{1,m} - \Phi_\perp(\sigma), &(5)\\ \xi^{\sigma,\sigma}_{L+1,m} &\doteq \xi^{\sigma,\sigma}_{1,m} - \Phi_\parallel(\sigma), &(6)\\ \zeta_{L+1,\sigma} - 2\xi^{\sigma,\sigma}_{L,L+1} &\doteq \zeta_{1,\sigma} - 2\xi^{\sigma,\sigma}_{L,1} - \Phi(\sigma). &(7) \end{aligned}$$

If we set

$$\begin{aligned} \alpha_{j,m}(\sigma) &\doteq 2(\xi^{\sigma,-\sigma}_{j,m} - \xi^{\sigma,-\sigma}_{j+1,m}) \,, &(8)\\ \beta_{j,m}(\sigma) &\doteq 2(\xi^{\sigma,\sigma}_{j,m} - \xi^{\sigma,\sigma}_{j+1,m}), \;\; m \in \{1, \cdots, j-1, j+2, \cdots, L\} \,, &(9)\\ \kappa_j(\sigma) &\doteq (\zeta_{j,\sigma} - \zeta_{j+1,\sigma} + 2\xi^{\sigma,\sigma}_{j,j+1}) \,, &(10) \end{aligned}$$

we can see easily that the Hamiltonian Equation (1) can reduce to the original Bariev model by the unitary transformation UHU^{-1}, up to a set of boundary conditions, and is integrable if α, β and κ all vanish. Through Equations (5)–(10), we obtain

$$\begin{aligned} \alpha_{L,m}(\sigma) &= 2(\xi^{\sigma,-\sigma}_{L,m} - \xi^{\sigma,-\sigma}_{1,m}) + \Phi_\perp(\sigma) \,, &(11)\\ \beta_{L,m}(\sigma) &= 2(\xi^{\sigma,\sigma}_{L,m} - \xi^{\sigma,\sigma}_{1,m}) + \Phi_\parallel(\sigma), \quad m \in \{2, \cdots, L-1\} \,, &(12)\\ \kappa_L(\sigma) &= (\zeta_{L,\sigma} - \zeta_{1,\sigma} + 2\xi^{\sigma,\sigma}_{L,1}) + \Phi(\sigma) \,. &(13) \end{aligned}$$

Then for the given boundary phase shift $\Phi_\perp$, $\Phi_\parallel$ and Φ the specific expressions for α, β and κ are obtained by Equations (8)–(10) with the constraints

$$\begin{aligned} \sum_{j=1}^{L} \alpha_{j,m}(\sigma) &= \Phi_\perp(\sigma) \,, &(14)\\ \sum_{\substack{j=1\\ j\neq m,m-1}}^{L} \beta_{j,m}(\sigma) + \beta_{m,m-1}(\sigma) + \beta_{m-1,m+1}(\sigma) &= \Phi_\parallel(\sigma) \,, &(15)\\ \sum_{j=1}^{L} [\kappa_j(\sigma) + \beta_{j,j}(\sigma)] &= \Phi(\sigma) \,. &(16) \end{aligned}$$

The total boundary twist is given by

$$\gamma_\sigma = \Phi(\sigma) + \Phi_\perp(\sigma) N_{-\sigma} + \Phi_\parallel(\sigma)(N_\sigma - 1) \,, \tag{17}$$

where N_σ is number of particles with spin σ. We note that the coefficient of the last term in Equation (17) is $N_\sigma - 1$ because the terms for $m = j$ and $m = j+1$ in constraint Equation (9) do not exist. This boundary condition we will call a twist boundary condition. When $\Phi_\perp$, $\Phi_\parallel$ and Φ all take a value pkL, the twist boundary condition reduces to the trivial periodic boundary condition. For any chosen twist boundary condition other than the normal periodic boundary condition, if one can find parameters α, β and κ satisfying the constraints Equations (8)–(10) and (14)–(16), the Hamiltonian Equation (1) is then solvable. The transform of Hamiltonian Equation (1) under U is

$$\begin{aligned} H \xrightarrow{U} UHU^{-1} &= -t\sum_{j=1}^{L-1}\sum_{\sigma=\uparrow,\downarrow}\left\{c^{\dagger}_{j+1,\sigma}c_{j,\sigma}+h.c.\right\}\exp\left[-\eta\sum_{\sigma'\neq\sigma}n_{j+\theta(\sigma-\sigma'),\sigma'}\right] \\ &\quad -t\sum_{\sigma=\uparrow,\downarrow}\left\{c^{\dagger}_{1,\sigma}c_{L,\sigma}\exp[i\gamma_{\sigma}]+h.c.\right\}\exp\left[-\eta\sum_{\sigma'\neq\sigma}n_{j+\theta(\sigma-\sigma'),\sigma'}\right], \end{aligned} \tag{18}$$

with the boundary term

$$c^{\dagger}_{L+1,\sigma}c_{L,\sigma}=\exp[i\gamma_{\sigma}]c^{\dagger}_{1,\sigma}c_{L,\sigma}\,. \tag{19}$$

Generally, the sites of the chain are chosen with a homogeneous distribution. So it is natural to think the effect of this boundary phase term γ_σ as an average phase shift $\delta_\sigma=\gamma_\sigma/L$ when a particle hops from one site to its neighbour sites. The corresponding Hamiltonian is then

$$H'=-t\sum_{j=1}^{L}\sum_{\sigma=\uparrow,\downarrow}\left\{e^{i\delta_{\sigma}}\tilde{c}^{\dagger}_{j+1,\sigma}\tilde{c}_{j,\sigma}+h.c.\right\}\exp\left[-\eta\sum_{\sigma'\neq\sigma}\tilde{n}_{j+\theta(\sigma-\sigma'),\sigma'}\right], \tag{20}$$

where

$$\tilde{c}^{\dagger}_{j,\sigma}=e^{-ij\delta_{\sigma}}c^{\dagger}_{j,\sigma}\,,\quad \tilde{c}_{j,\sigma}=c_{j,\sigma}e^{ij\delta_{\sigma}},\quad \tilde{n}_{j,\sigma}=\tilde{c}^{\dagger}_{j,\sigma}\tilde{c}_{j,\sigma}\,, \tag{21}$$

and the basic commutation relations are kept unchanged. δ is a function of σ, N and $N_\downarrow$, which is different from the case of a periodic chain. By comparing this with the standard Bariev model ($\alpha=\beta=\kappa=0$ in Equation (1)), one can see clearly that the introduced long-range spin interactions are equivalent to applying a twist boundary condition. However, we note that the phase shift between the neighbour sites may as well be distributed in any other way such that the sum equals γ_σ without changing any results.

3. Bethe Equations and Energy Spectrum

In the standard Bethe ansatz approach, modified for the twist boundary condition, any eigenfunction of the Hamiltonian takes a form similar to tensor products of plane waves [30]. We consider the eigenstate corresponding to N particles

$$|\Psi\rangle=\sum_{x_q=1}^{L}f(x_{q_1},\cdots,x_{q_N})\prod_{j=1}^{N}c^{\dagger}_{x_j,\sigma_j}|\Omega\rangle \tag{22}$$

in which the number of spin-down particles is $N_\downarrow$. In the region $x_{q_1}\leq\cdots\leq x_{q_N}$, the function f can be written as [31]

$$f(x_{q_1},\cdots,x_{q_N})=\epsilon_P A_{\sigma_{q_1},\cdots,\sigma_{q_N}}(k_{p_1},\cdots,k_{p_N})\times\exp\left[i\sum_{j=1}^{N}k_{p_j}x_{q_j}\right]\theta(x_{q_1}\leq\cdots\leq x_{q_N})\,. \tag{23}$$

By solving the Schrödinger equation $H|\Psi\rangle=E|\Psi\rangle$, the energy eigenvalue of the Hamiltonian Equation (18) is given by

$$E=-2t\sum_{j=1}^{N}\cos k_j\,. \tag{24}$$

We note that the form of the energy eigenvalue does not change from the standard Bariev chain. However, we will see later that the momentum k_j is now spin dependent. Two-particle scattering matrices are given by

$$\begin{aligned} S^{\alpha_1\ \alpha_2}_{\alpha_1\ \alpha_2} &= \frac{sin[(k_1-k_2)/2]}{sin[(k_1-k_2)/2+i\eta]} \quad (\alpha_1 \neq \alpha_2) \\ S^{\alpha_1\ \alpha_2}_{\alpha_2\ \alpha_1} &= \frac{sin[i\eta]}{sin[(k_1-k_2)/2+i\eta]} \exp\left[i\frac{k_1-k_2}{2} sign(\alpha_1-\alpha_2)\right] \quad (\alpha_1 \neq \alpha_2) \\ S^{\alpha_0\ \alpha_0}_{\alpha_0\ \alpha_0} &= 1 \quad (\alpha_1 = \alpha_2 = \alpha_0)\ , \end{aligned} \tag{25}$$

which are similar to the $R-$matrices of the standard 6-vertex models. The two are related via a simple gauge transformation as [32]

$$S_{12}(\lambda) = V_1(\lambda)R_{12}(\lambda)V_1^{-1}(\lambda), \quad V(\lambda) = diag\left(e^{i\lambda/4}, e^{-i\lambda/4}\right). \tag{26}$$

for $\lambda = (k_1 - k_2)$.

In general, we have

$$S_{12}(\lambda_1-\lambda_2) = V_1(\lambda_1)V_2(\lambda_2)R_{12}(\lambda_1-\lambda_2)V_1^{-1}(\lambda_1)V_2^{-1}(\lambda_2). \tag{27}$$

It is easy to show

$$\begin{aligned} & S_{12}(\lambda_1-\lambda_2)S_{13}(\lambda_1-\lambda_3)S_{23}(\lambda_2-\lambda_3) \\ =\ & V_1(\lambda_1)V_2(\lambda_2)V_3(\lambda_3)R_{12}(\lambda_1-\lambda_2)R_{13}(\lambda_1-\lambda_3)R_{23}(\lambda_2-\lambda_3)V_1^{-1}(\lambda_1)V_2^{-1}(\lambda_2)V_3^{-1}(\lambda_3), \end{aligned} \tag{28}$$

which will satisfy the Yang-Baxter equation. So the integrability of the present model is kept. One can also solve the problem by constructing the R-matrix of this model following the the techniques in [33].

The charge momentum k_j and spin momentum Λ_μ satisfy the Bethe equations

$$e^{ik_jL} = e^{-i\gamma_\uparrow}\prod_{\mu=1}^{N_\downarrow}\frac{\sin\left[\frac{(k_j-\Lambda_\mu)}{2}+\frac{i\eta}{2t}\right]}{\sin\left[\frac{(k_j-\Lambda_\mu)}{2}-\frac{i\eta}{2t}\right]}\ , \tag{29}$$

$$\prod_{\nu=1,\nu\neq\mu}^{N_\downarrow}\frac{\sin\left[\frac{(\Lambda_\mu-\Lambda_\nu)}{2}+\frac{i\eta}{t}\right]}{\sin\left[\frac{(\Lambda_\mu-\Lambda_\nu)}{2}-\frac{i\eta}{t}\right]} = e^{-i(\gamma_\downarrow-\gamma_\uparrow)}\prod_{j=1}^{N}\frac{\sin\left[\frac{(\Lambda_\mu-k_j)}{2}+\frac{i\eta}{2t}\right]}{\sin\left[\frac{(\Lambda_\mu-k_j)}{2}-\frac{i\eta}{2t}\right]}\ . \tag{30}$$

The structure of roots for these equations depends strongly on the hopping amplitude η. The interactions between particles are repulsive when $\eta > 0$, all particles with different spins cannot form a pair. In this situation all k_j must be real, which can be proved under the thermodynamical limit. $\eta < 0$ corresponds to attractive interactions in the system. Particles with different spins tend to exist in the form of cooper-pairs. The solution $k_j = \Lambda + i|\eta|$ corresponds to these bound states for charge excitations. If we choose two sets of quantum numbers I_j and J_μ and set $\theta(x;a) = \arctan\Big(\tan(x/2)\coth(a/2)\Big)$, Equations (29) and (30) then take the form

$$\begin{aligned} Lk_j &= -\gamma_\uparrow + 2\pi I'_j + \sum_{\mu=1}^{N_\downarrow}\{\pi - 2\theta(k_j-\Lambda_\mu;\eta)\} \\ &= -\gamma_\uparrow + 2\pi I_j - \sum_{\mu=1}^{N_\downarrow}\{2\theta(k_j-\Lambda_\mu;\eta)\}\ , \end{aligned} \tag{31}$$

$$\begin{aligned}\gamma_{\downarrow}-\gamma_{\uparrow} &= 2\pi J'_{\mu}-\sum_{\nu=1;\nu\neq\mu}^{N_{\downarrow}}[\pi-2\theta(\Lambda_{\mu}-\Lambda_{\nu};2\eta)]+\sum_{j=1}^{N}[\pi-2\theta(\Lambda_{\mu}-k_j;\eta)] \\ &= 2\pi J_{\mu}+\sum_{\nu=1;\nu\neq\mu}^{N_{\downarrow}}[2\theta(\Lambda_{\mu}-\Lambda_{\nu};2\eta)]-\sum_{j=1}^{N}[2\theta(\Lambda_{\mu}-k_j;\eta)]\ ,\end{aligned} \tag{32}$$

where I'_j and J'_{μ} are both common integers. The quantum numbers $I_j = I'_j + N_{\downarrow}/2$ and $J_{\mu} = J'_{\mu} + (N_{\uparrow}+1)/2$ depend upon the charge and spin property in the system. From Equations (31) and (32) we can see that I_j and J_{μ} are either integer or half-integer, according to the number of total particles and the number of spin-up (spin-down) particles. There are four cases,

- I_j and J_{μ} are both integers if N and $N_{\uparrow}$ are both odd;
- I_j and J_{μ} are both half-integers if N is odd and $N_{\uparrow}$ is even;
- I_j is a half-integer and J_{μ} is an integer if N is even and $N_{\uparrow}$ is odd;
- I_j is an integer and J_{μ} is a half-integer if N and $N_{\uparrow}$ are both even.

By taking the summation of Equations (31) and (32) over the coordinate and spin indices respectively, the momentum of the system is given as

$$P=\sum_{j=1}^{N}\left(k_j-\frac{1}{L}\gamma'\right)=\frac{2\pi}{L}\left(\sum_{j=1}^{N}I_j+\sum_{\mu=1}^{N_{\downarrow}}J_{\mu}\right)\ , \tag{33}$$

where $\gamma' = (\gamma_{\downarrow}-\gamma_{\uparrow})-(N_{\uparrow}\gamma_{\downarrow}+N_{\downarrow}\gamma_{\uparrow})/N$. In the above calculations we have used the relation $\sum_{\mu=1}^{N_{\downarrow}}\sum_{\nu=1}^{N_{\downarrow}}2\theta(\Lambda_{\mu}-\Lambda_{\nu},2\eta)=0$.

In the limiting case $\eta\longrightarrow\infty$ ($\coth(\eta)\longrightarrow 1$), we get

$$Lk_j=2\pi\left[I_j+\frac{N_{\downarrow}}{N(L+N_{\downarrow})}\sum_{l=1}^{N}(I_l-I_j)+\frac{1}{N}\sum_{\lambda=1}^{N_{\downarrow}}J_{\lambda}\right]+\gamma'\ . \tag{34}$$

Substituting it into Equation (24) we obtain the energy of the system

$$E_0(\gamma)=-2tD\cos\left[\frac{2\pi}{L}\left(\frac{1}{N}\sum_{\lambda=1}^{N_{\downarrow}}J_{\lambda}+\overline{I}+\frac{\gamma'}{2\pi}\right)\right]\ , \tag{35}$$

where $D=\sin(N\pi/L)/\sin(\pi/L)$ and $\overline{I}=(I_{min}+I_{max})/2$. We will not consider the situation where $N=L$, which occurs when the chain is half filled and D is 0. Then for arbitrary combinations of N and $N_{\downarrow}$ we have four different cases

$$\begin{aligned} E^{even\ N_{\downarrow}}_{even\ N} &= -2tD\cos\left[\tfrac{2\pi}{L}\left(\tfrac{\gamma'}{2\pi}+g+\tfrac{1}{2}+\tfrac{N_{\downarrow}}{N}h\right)\right]\ , \\ E^{odd\ N_{\downarrow}}_{even\ N} &= -2tD\cos\left[\tfrac{2\pi}{L}\left(\tfrac{\gamma'}{2\pi}+g+\tfrac{N_{\downarrow}}{N}h\right)\right]\ , \\ E^{even\ N_{\downarrow}}_{odd\ N} &= -2tD\cos\left[\tfrac{2\pi}{L}\left(\tfrac{\gamma'}{2\pi}+g+\tfrac{N_{\downarrow}}{N}h+\tfrac{N_{\downarrow}}{2N}\right)\right]\ , \\ E^{odd\ N_{\downarrow}}_{odd\ N} &= -2tD\cos\left[\tfrac{2\pi}{L}\left(\tfrac{\gamma'}{2\pi}+g+\tfrac{1}{2}+\tfrac{N_{\downarrow}}{N}h+\tfrac{N_{\downarrow}}{2N}\right)\right]\ , \end{aligned} \tag{36}$$

where g and h are any integers and also quantum numbers which describe the charge and spin excitations.

If we treat γ' as the phase shift induced by an external field, for a given N and $N_\downarrow$ the external field can only vary within a small range

$$\frac{2\mathcal{L}-1}{2N} < \frac{\gamma'}{2\pi} + \bar{I} < \frac{2\mathcal{L}+1}{2N} , \quad \mathcal{L} = -\sum_{\lambda=1}^{N_\downarrow} J_\lambda . \tag{37}$$

otherwise the spin inversion will occur.

4. Results for General η

Generally, for a finite η it is difficult to get exact solutions of the Bethe equations. In this section we try to discuss some properties of the system when $L \to \infty$. The root distribution of Bethe equations turns to a continuous density distribution, σ, in this case. We define functions

$$Z_c(k) = Lk + \gamma_\uparrow + \sum_{\mu=1}^{N_\downarrow} 2\theta(k-\Lambda_\mu;\eta) , \tag{38}$$

$$Z_s(\Lambda) = \gamma_\uparrow - \gamma_\downarrow - \sum_{\mu=1}^{N_\downarrow} 2\theta(\Lambda-\Lambda_\nu;2\eta) + \sum_{j=1}^{N} 2\theta(\Lambda-k_j;\eta) . \tag{39}$$

In the limiting case of large L Equations (38) and (39) can be expressed in the form of integrals

$$2\pi\sigma_c(k) = \lim_{L\to\infty} \frac{1}{L}\frac{d}{dk} Z_c(k), \quad 2\pi\sigma_s(\Lambda) = \lim_{L\to\infty} \frac{1}{L}\frac{d}{dk} Z_s(\Lambda) , \tag{40}$$

where

$$n = \int_{-K}^{K} \sigma_c(k)dk = \frac{N}{L}, \quad n_\downarrow = \int_{-\Lambda_0}^{\Lambda_0} \sigma_s(\Lambda)d\Lambda = \frac{N_\downarrow}{L} . \tag{41}$$

Now, we can see from Equations (31) and (32) that the relations

$$Z_c(k_j) = 2\pi I_j, \quad Z_s(\Lambda) = 2\pi J_\lambda , \tag{42}$$

$$I_{j+1} - I_j = 1, \quad J_{\mu+1} - j_\mu = 1 , \tag{43}$$

must be satisfied by roots of the Bethe equations. We are ready to obtain a set of integral equations

$$2\pi\sigma_c(k) = 1 + \int_{-\Lambda_0}^{\Lambda_0} 2\theta(k-\Lambda;\eta)\sigma_s(\Lambda)d\Lambda , \tag{44}$$

$$2\pi\sigma_s(\Lambda) = -\int_{-\Lambda_0}^{\Lambda_0} 2\theta'(\Lambda-\Lambda';2\eta)\sigma_s(\Lambda')d\Lambda' + \int_{-K}^{K} 2\theta'(\Lambda-k;\eta)\sigma_c(k)dk , \tag{45}$$

where $\theta'(x;y) = d\theta/dx = -sin(y)/4[cos(x)+cosh(y)]$.

Through a Fourier transform we obtain

$$\widetilde{\theta}'(\omega;y) = \int_{-\pi}^{\pi} \theta'(x;y)e^{-i\omega x}dx = \frac{1}{2}e^{-\omega y} . \tag{46}$$

For $\Lambda_0 = K = \pi$, we have $\sigma_c(\omega) = 1/\pi$ and $\sigma_s(\omega) = 1/2\pi$. The corresponding particle densities in Equation (41) are $n = 2$ and $n_\downarrow = 1$, which means that the numbers of two spin species are the same if there is no external field.

For a more general situation, the values of K and Λ_0 are determined by Equation (41). When the external field vanishes, the value of Λ_0 must be π. From Equations (44) and (45) we get

$$2\pi\sigma_c(k) = 1 + \sum_{\omega=0}^{\infty} \frac{e^{-ik\omega}}{1+e^{2\eta\omega}} \int_{-K}^{K} e^{i\mu\omega}\sigma_c(\mu)d\mu \,. \tag{47}$$

The numerical solutions of the Bethe Equations (29) and (30) and the corresponding eigenvalues of the Hamiltonian Equation (18) for $L = 2$ and $L = 3$ with different occupation numbers are shown in Tables 1 and 2 respectively. As mentioned before, we will only consider the cases where $N < L$. By analyzing the structure of Bethe equations, we can see that if $N_\downarrow = 0$ or $N = N_\downarrow$ the roots of the Bethe equations do not depend on η. These numerical results coincide with those obtained from the exact diagonalization of the Hamiltonian Equation (18) and the analytical results in the limiting case, $\eta \to \infty$, obtained through Equation (36). For the $L = 2$ chain, we also give the eigenvalues E with varying γ_σ, which do not change with η, as shown in Figure 1.

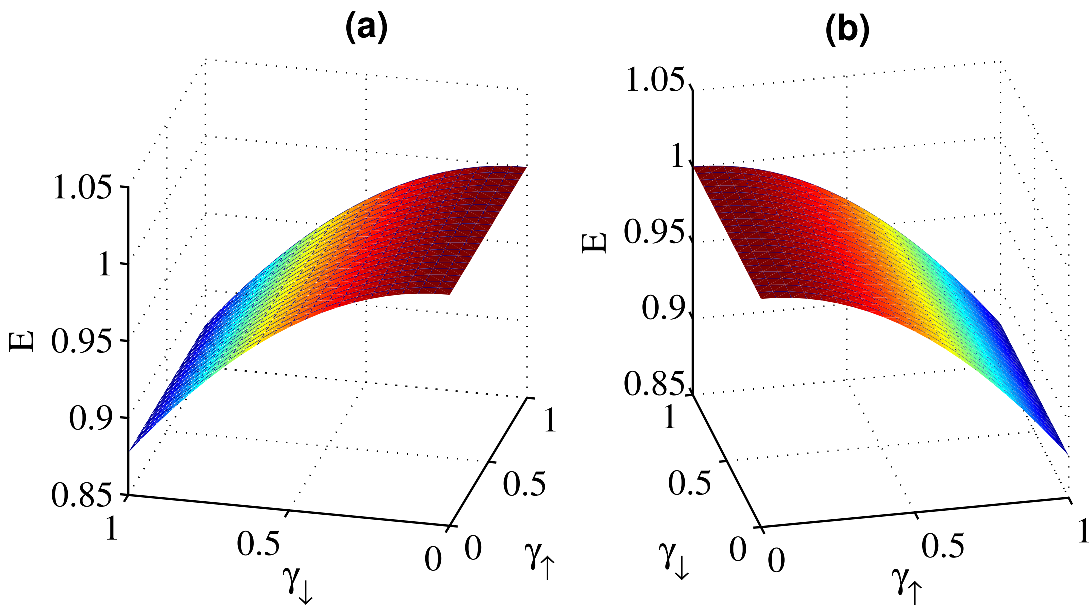

Figure 1. The eigenvalues E calculated from the exact diagonalization of the Hamiltonian Equation (18) with respect to γ_σ for $L = 2$. (**a**) $N = N_\uparrow = 1$. (**b**) $N = N_\downarrow = 1$.

Table 1. The numerical results calculated from Equations (29) and (30) for the parameters, $L = 2,\ t = \ 0.5,\ \eta = 0.3,\ \gamma_\uparrow = 0.2$, and $\gamma_\downarrow = 0.1$.

Occupation Number	k_1	Λ_1	E
$N = N_\uparrow = 1$	-0.100000	N/A	-0.995004
$N = N_\uparrow = 1$	3.041593	N/A	0.995004
$N = N_\downarrow = 1$	-0.050000	3.754988	-0.998750
$N = N_\downarrow = 1$	3.091593	0.613395	0.998750

Table 2. The numerical results calculated from Equations (29) and (30) for the parameters, $L = 3,\ t = 0.5,\ \eta = 0.3,\ \gamma_\uparrow = 0.4$, and $\gamma_\downarrow = 0.2$.

Occupation Number	k_1	k_2	Λ_1	Λ_2	E
$N = N_\uparrow = 1$	-0.066667	N/A	N/A	N/A	-0.997779
$N = N_\uparrow = 1$	2.027728	N/A	N/A	N/A	0.441197
$N = N_\uparrow = 1$	-2.161062	N/A	N/A	N/A	0.556582
$N = N_\downarrow = 1$	-0.033333	N/A	-2.834687	N/A	-0.999444
$N = N_\downarrow = 1$	2.061062	N/A	-0.740291	N/A	0.470860
$N = N_\downarrow = 1$	-2.127728	N/A	1.354104	N/A	0.528584
$N = N_\uparrow = 2$	-0.133333	1.961062	N/A	N/A	-0.610690
$N = N_\uparrow = 2$	-2.227728	-0.133333	N/A	N/A	-0.380434
$N = N_\uparrow = 2$	1.961062	-2.227728	N/A	N/A	0.991124
$N = N_\downarrow = 2$	-0.066667	2.027728	$-0.405569 + 1.839956i$	$-0.405569 - 1.839956i$	-0.556582
$N = N_\downarrow = 2$	-0.066667	-2.161062	$-2.499964 - 1.839956i$	$-2.499964 + 1.839956i$	-0.441197
$N = N_\downarrow = 2$	2.027728	-2.161062	$1.688826 + 1.839956i$	$1.688826 - 1.839956i$	0.997779

5. Conclusions

In this paper we have constructed a generalized 1D Bariev model which describes spin-1/2 particles with long-range interaction on a lattice. By employing an unitary transformation we find the Hamiltonian is equivalent to a standard Bariev Hamiltonian with twist boundary conditions. This phase twist may be used to explain the effects of an external magnetic potential and the internal fluctuations on the system. For a strong external magnetic field, spin inversions can occur in the system. By solving the Bethe equations in the limiting case where $\eta \to \infty$ we give the specific forms of energy spectrum in different system configurations with respect to total particle number and spin distributions. More general cases have been discussed but the analytical result can only be obtained in the situation where there is no external field. We also solve Bethe equations numerically and conduct the exact diagnalization of the transformed Hamitonian. The numerical results and the analytical results coincide with each other very well. To relate our model with real physical systems we need to determine the specific values of ξ, ζ and η and calculate some other physical properties of the system. This could be an interesting topic for further study.

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Author Contributions

This study was proposed by Rui-Hong Yue. The analytical results were worked out by Tao Yang, Li-Ke Cao and Kun Hao. The numerical simulations were run by Fa-Kai Wen. The manuscript was prepared by Tao Yang and Rui-Hong Yue. All authors have read and approved the final manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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