Entropy **2015**, *17*, 6213-6228; doi:10.3390/e17096213

OPEN ACCESS

entropy

ISSN 1099-4300

www.mdpi.com/journal/entropy

Article

Statistical Manifolds with almost Quaternionic Structures and Quaternionic Kähler-like Statistical Submersions

Alina-Daniela Vîlcu [1,†] and Gabriel-Eduard Vîlcu [2,3,†,*]

[1] Department of Computer Science, Information Technology, Mathematics and Physics, Petroleum-Gas University of Ploieşti, Bd. Bucureşti 39, Ploieşti 100680, Romania; E-Mail: daniela.vilcu@upg-ploiesti.ro

[2] Research Center in Geometry, Topology and Algebra, Faculty of Mathematics and Computer Science, University of Bucharest, Str. Academiei 14, Sector 1, Bucharest 70109, Romania

[3] Department of Mathematical Modelling, Economic Analysis and Statistics, Petroleum-Gas University of Ploieşti, Bd. Bucureşti 39, Ploieşti 100680, Romania

† These authors contributed equally to this work.

* Author to whom correspondence should be addressed; E-Mail: gvilcu@upg-ploiesti.ro; Tel.: +40-244-575771; Fax: +40-244-575847.

Academic Editor: Kevin H. Knuth

Received: 27 June 2015 / Accepted: 1 September 2015 / Published: 7 September 2015

Abstract: In this paper we investigate statistical manifolds with almost quaternionic structures. We define the concept of quaternionic Kähler-like statistical manifold and derive the main properties of quaternionic Kähler-like statistical submersions, extending in a new setting some previous results obtained by K. Takano concerning statistical manifolds endowed with almost complex and almost contact structures. Finally, we give a nontrivial example and propose some open problems in the field for further research.

Keywords: affine connection; conjugate connection; statistical manifold; statistical submersion; almost quaternionic structure

1. Introduction

It is well known that the concept of statistical manifold arises naturally from divergencies—like Kullback–Leibler relative entropy—in statistics, information theory and related fields [1,2]. On the other hand, the notion of statistical submersion between statistical manifolds was introduced in 2001 by N. Abe and K. Hasegawa [3], the authors generalizing some basic results of B. O'Neill [4,5] concerning Riemannian submersions and geodesics. Later, K. Takano defined the concepts of Kähler-like statistical manifold and Kähler-like statistical submersion [6], Sasaki-like statistical manifold and Sasaki-like statistical submersion [7], and obtained several geometric properties. Particularly relevant examples of statistical manifolds are the exponential families, whose points are probability densities of exponential form depending on a finite number of parameters. For some important exponential families, like the multinomial distribution, the multivariate normal distribution, and the Dirichlet and von Mises–Fisher distributions, it is proved in [8] that they admit almost complex structures. Also, in [9] H. Matsuzoe and J. Inoguchi investigate the extensions of statistical structures on manifolds to their tangent bundles, proving that the tangent bundle of a flat statistical manifold has a natural almost complex statistical structure with Norden metric. Moreover, in [10] the author considers the statistical model of the multivariate normal distribution as the Riemannian manifold and constructs an interesting example of statistical submersion.

We remark that a complex version of the notion of statistical structure was also considered in [11], where the author derived a condition for the curvature of a statistical manifold to admit a kind of standard hypersurface. On the other hand, the existence of symplectic structures on statistical manifolds was investigated in [12], where the author obtained a duality relation between the Fubini–Study metric on a projective space and the Fisher metric on a statistical model on a finite set. Other interesting results concerning the geometry of statistical manifolds were recently obtained in [13–21]. In this paper, we investigate very natural kind of statistical manifold, namely those endowed with almost quaternionic structures, extending the results of K. Takano in a new setting and obtaining new curvature properties of statistical submersions. In particular, we generalize some previous results of S. Ianuş *et al.* [22] concerning Riemannian submersions between quaternionic manifolds. Recall that an almost quaternionic structure on a smooth manifold M is a 3-dimensional subbundle of $End(TM)$ which is locally spanned by an almost hypercomplex structure, *i.e.*, three almost complex structures satisfying the quaternionic identities [23]. We also note that the quaternionic structures generalize many relevant properties of 4-dimensional semi-Riemannian manifolds to higher $4n$-dimensional manifolds, some of them being relevant for mathematical physics, with important applications in string theory, solitons, theory of liquid crystals, gravity and general relativity (see [24,25] and references therein).

The present work is organized as follows. Section 2 contains definitions and basic properties of statistical manifolds and statistical submersions. In Section 3 we investigate statistical manifolds with almost quaternionic structures and introduce the concept of quaternionic Kähler-like statistical manifold. Section 4 is devoted to the study of the quaternionic Kähler-like statistical submersions. This paper ends with conclusions and several open problems in the field for further research.

2. Preliminaries

Let (M, g) be a semi-Riemannian manifold and ∇ a torsion free linear connection on M. Then ∇ is said to be compatible to g if the covariant derivative ∇g is symmetric. Moreover, the pair (∇, g) is called a statistical structure on M and the triple (M, ∇, g) is said to be a statistical manifold.

For a statistical manifold (M, ∇, g), let ∇^* be an affine connection on M such that

$$Eg(F, G) = g(\nabla_E F, G) + g(F, \nabla^*_E G), \tag{1}$$

for all $E, F, G \in \Gamma(TM)$, where $\Gamma(TM)$ denotes the set of smooth tangent vector fields on M. Then it is easy to see that the affine connection ∇^* is torsion free and $\nabla^* g$ is symmetric. This connection, ∇^*, is called *the dual connection* of ∇; the triple (M, ∇^*, g) is said to be the dual statistical manifold of (M, ∇, g); and the triple (∇, ∇^*, g) is called the dualistic structure on M [26]. We note that the concept of dual connections, whose name is motivated by the fact that $(\nabla^*)^* = \nabla$, was originally introduced by S. Amari in his seminal work [1] and later applied in various fields, like statistical physics, neural networks and information theory.

It is also easy to check that the curvature tensor R^* of ∇^* vanishes if and only if the curvature tensor R of ∇ does, and then the triple (∇, ∇^*, g) is called the dually flat structure [2]. In fact, the two curvature tensors R and R^* on M, defined with the sign convention

$$R(E, F)G = [\nabla_E, \nabla_F]G - \nabla_{[E,F]}G, \;\; R^*(E, F)G = [\nabla^*_E, \nabla^*_F]G - \nabla^*_{[E,F]}G,$$

are related by [7]

$$g(R(E, F)G, H) = -g(G, R^*(E, F)H), \tag{2}$$

for all $E, F, G, H \in \Gamma(TM)$.

We remark that the geometry of statistical manifolds simply reduces to the usual semi-Riemannian geometry when ∇ and ∇^* coincide [27]. Moreover, we note that on a statistical manifold one can define a parametric family of torsion free connections $\{\nabla^\alpha\}_{\alpha\in\mathbb{R}}$, called α-connections, by

$$\nabla^\alpha = \frac{1+\alpha}{2}\nabla + \frac{1-\alpha}{2}\nabla^*. \tag{3}$$

We remark that $\nabla^{(1)} = \nabla$, $\nabla^{(-1)} = \nabla^*$ and $\nabla^{(0)}$ is the Levi–Civita connection of the metric g. This family of α-connections has been investigated in [28], where the author obtains that ∇^α is equiaffine for any real number α, provided that (∇, ∇^*, g) is a dually flat structure, as previously noted in [29].

Let (M, g) and (M', g') be two connected semi-Riemannian manifolds of index s and s' respectively, with $0 \le s \le dimM$, $0 \le s' \le dimM'$ and $s' \le s$. A *semi-Riemannian submersion* is a smooth map $\pi : M \to M'$ which is onto and satisfies the following conditions [30]:

(i) $\pi_{*|p} : T_pM \to T_{\pi(p)}M'$ is onto for all $p \in M$;

(ii) The fibers $\pi^{-1}(p')$, $p' \in M'$, are semi-Riemannian submanifolds of M;

(iii) π_* preserves scalar products of vectors normal to fibers.

It is well known that the vectors tangent to fibers are called *vertical* and those normal to fibers are called *horizontal*. We denote by $\mathcal{V}$ the vertical distribution, by $\mathcal{H}$ the horizontal distribution and by v and h the vertical and horizontal projection. An horizontal vector field X on M is said to be *basic* if X is π-related to a vector field X' on M'. It is clear that every vector field X' on M' has a unique horizontal lift X to M and X is basic. Moreover, if X and Y are basic vector fields on M, π-related to X' and Y' on M', then we have the following properties (see [5,31]):

(i) $g(X, Y) = g'(X', Y') \circ \pi$;

(ii) $h[X, Y]$ is a basic vector field and $\pi_* h[X, Y] = [X', Y'] \circ \pi$.

Next we consider (M, ∇, g) a statistical manifold, (M', g') a semi-Riemannian manifold and let $\pi : M \to M'$ be a semi-Riemannian submersion. We denote by $\widehat{\nabla}$ and $\widehat{\nabla}^*$ the affine connections induced on fibers by the dual connections ∇ and ∇^* from M. We remark that $\widehat{\nabla}$ and $\widehat{\nabla}^*$ are well-defined, namely

$$\widehat{\nabla}_U V = v\nabla_U V,\ \widehat{\nabla}^*_U V = v\nabla^*_U V \tag{4}$$

for all $U, V \in \Gamma(\mathcal{V})$. Moreover, we can easily see that $\widehat{\nabla}$ and $\widehat{\nabla}^*$ are torsion free and conjugate to each other with respect to the induced metric on fibers. On the other hand, if we define $S := \nabla - \nabla^*$, then S is symmetric, *i.e.*, $S_E F = S_F E$, for all vector fields E, F on M, and we also find [6]:

$$\begin{aligned} 2g(\nabla_X Y, Z) =& g(S_X Y, Z) + Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ & - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]) \end{aligned} \tag{5}$$

for all $X, Y, Z \in \Gamma(\mathcal{H})$. Similarly, if ∇' and ∇'^* are affine connections on M', then we can define $S' = \nabla' - \nabla'^*$ and we have that $hS_X Y$ is basic and π-related to $S'_{X'} Y'$ if and only if $h\nabla_X Y$ (or $h\nabla^*_X Y$) is basic and π-related to $\nabla'_{X'} Y'$ (or $\nabla^*_{X'} Y'$).

Definition 1. [7] *Let* (M, ∇, g) *and* (M', ∇', g') *be two statistical manifolds. Then a semi-Riemannian submersion* $\pi : M \to M'$ *is said to be a* statistical submersion *if* $\pi_*(\nabla_X Y)_p = (\nabla'_{X'} Y')_{\pi(p)}$ *for all basic vector fields* X, Y *on* M *π-related to* X' *and* Y' *on* M'*, and* $p \in M$*.*

If $\pi : M \to M'$ is a statistical submersion, then we can define as well as in the semi-Riemannian case [32], two (1,2) tensor fields T and A on M, by the formulas:

$$T(E, F) = T_E F = h\nabla_{vE} vF + v\nabla_{vE} hF \tag{6}$$

and similarly:

$$A(E, F) = A_E F = v\nabla_{hE} hF + h\nabla_{hE} vF \tag{7}$$

for any $E, F \in \Gamma(TM)$.

We can also define, in a similar way, the tensor fields T^* and A^* on M by replacing ∇ by ∇^* in Equations (6) and (7). It is easy to check now that $T^{**} = T$ and $A^{**} = A$. Moreover, using the above Definitions one can easily prove the following result.

Lemma 1. [3,6] *T, A, T^* and A^* have the following properties:*

$$T_U V = T_V U,\ T^*_U V = T^*_V U, \tag{8}$$

$$A_X Y - A_Y X = A^*_X Y - A^*_Y X = v[X, Y], \tag{9}$$

$$A_X Y = -A^*_Y X, \tag{10}$$

$$\nabla_X Y = h\nabla_X Y + A_X Y,\ \nabla^*_X Y = h\nabla^*_X Y + A^*_X Y, \tag{11}$$

$$\nabla_U V = T_U V + \widehat{\nabla}_U V,\ \nabla^*_U V = T^*_U V + \widehat{\nabla}^*_U V, \tag{12}$$

$$\nabla_U X = h\nabla_U X + T_U X,\ \nabla^*_U X = h\nabla^*_U X + T^*_U X, \tag{13}$$

$$\nabla_X U = A_X U + v\nabla_X U,\ \nabla^*_X U = A^*_X U + v\nabla^*_X U, \tag{14}$$

$$g(T_U V, X) = -g(V, T^*_U X), \tag{15}$$

$$g(A_X Y, U) = -g(Y, A^*_X U), \tag{16}$$

for all $X, Y \in \Gamma(\mathcal{H})$ and $U, V \in \Gamma(\mathcal{V})$.

Therefore, we deduce that T (or A) vanishes identically if and only if T^* (or A^*) vanishes identically. Moreover, from (9) we deduce that if $A = 0$ then $\mathcal{H}$ is integrable. We note that if $T_U V = 0$, for all $U, V \in \Gamma(\mathcal{V})$ then π is called a statistical submersion with isometric fibers [6].

We also recall that N. Abe and K. Hasegawa [3] provided necessary and sufficient conditions for the total space of a semi-Riemannian submersion to be a statistical manifold. In particular, we note that if $\pi : M \to M'$ is a statistical submersion then any fiber is a statistical manifold (see also [6,7]).

3. Statistical Manifolds with almost Quaternionic Structures

Let M be a differentiable manifold and assume that there is a rank 3-subbundle σ of $End(TM)$ such that a local basis $\{J_1, J_2, J_3\}$ exists on sections of σ satisfying for all $\alpha \in \{1, 2, 3\}$:

$$J^2_\alpha = -\mathrm{Id}, J_\alpha J_{\alpha+1} = -J_{\alpha+1} J_\alpha = J_{\alpha+2}, \tag{17}$$

where Id denotes the identity tensor field of type (1, 1) on M and the indices are taken from $\{1, 2, 3\}$ modulo 3. Then the bundle σ is called an almost quaternionic structure on M and $\{J_1, J_2, J_3\}$ is called a canonical local basis of σ. Moreover, (M, σ) is said to be an almost quaternionic manifold [33]. It is easy to see that any almost quaternionic manifold is of dimension $4m$, $m \geq 1$.

A semi-Riemannian metric g on M is said to be adapted to the almost quaternionic structure σ if it satisfies:

$$g(J_\alpha E, J_\alpha F) = g(E, F),\ \alpha \in \{1, 2, 3\} \tag{18}$$

for all vector fields E,F on M and any canonical local basis $\{J_1, J_2, J_3\}$ of σ. Moreover, (M, σ, g) is said to be an almost Hermite quaternionic manifold [33].

Definition 2. *Let (M, g) be a semi-Riemannian manifold endowed with an almost quaternionic structure σ which has for any canonical local basis $\{J_1, J_2, J_3\}$ of σ three other tensor fields $\{J^*_1, J^*_2, J^*_3\}$ of type $(1, 1)$ on M, satisfying*

$$g(J_\alpha E, F) + g(E, J^*_\alpha F) = 0,\ \alpha \in \{1, 2, 3\} \tag{19}$$

for all vector fields E,F on M. Then (M, σ, g) is said to be an almost Hermite-like quaternionic manifold. Moreover, if (M, σ, g) is equipped with a torsion free linear connection ∇ such that ∇g is symmetric, then (M, ∇, σ, g) is said to be an almost Hermite-like quaternionic statistical manifold.

We remark that $\{J^*_1, J^*_2, J^*_3\}$ defined by (19) satisfy (17) and hence we can consider the subbundle σ^* of $End(TM)$ locally spanned by $\{J^*_1, J^*_2, J^*_3\}$. We also see that

$$(J^*_\alpha)^* = J_\alpha$$

and

$$g(J_\alpha E, J_\alpha^* F) = g(E, F),$$

for all vector fields E,F on M and $\alpha \in \{1, 2, 3\}$.

Definition 3. *Let (M, ∇, σ, g) be an almost Hermite-like quaternionic statistical manifold. Then (M, ∇, σ, g) is said to be a quaternionic Kähler-like statistical manifold if for any local basis J_1, J_2, J_3 of σ there exist three locally defined 1-forms ω_1, ω_2, ω_3 on M such that we have for all $\alpha \in \{1, 2, 3\}$:*

$$(\nabla_E J_\alpha)F = \omega_{\alpha+2}(E)J_{\alpha+1}F - \omega_{\alpha+1}(E)J_{\alpha+2}F, \tag{20}$$

for all vector fields E, F on M, where the indices are taken from $\{1, 2, 3\}$ modulo 3.

We note that if $\omega_1 = \omega_2 = \omega_3 = 0$ in (20), then (M, ∇, σ, g) is said to be a locally hyper-Kähler-like statistical manifold. Moreover, if J_1, J_2, J_3 are globally defined on M, then $(M, \nabla, J_1, J_2, J_3, g)$ is said to be a hyper-Kähler-like statistical manifold.

We remark that, if in the above definition ∇ is the Levi–Civita connection of g, then (M, ∇, σ, g), usually denoted by (M, σ, g), is called a quaternionic Kähler manifold [23,33,34].

Definition 4. *Let (M, ∇, σ, g) be a quaternionic Kähler-like statistical manifold. If the curvature tensor R with respect to ∇ satisfies*

$$\begin{aligned} R(E, F)G &= \frac{c}{4}\{g(F, G)E - g(E, G)F + \sum_{\alpha=1}^{3}[g(G, J_\alpha F)J_\alpha E - g(G, J_\alpha E)J_\alpha F] \\ &\quad + \sum_{\alpha=1}^{3}[g(E, J_\alpha F) - g(J_\alpha E, F)]J_\alpha G\}, \end{aligned} \tag{21}$$

for all vector fields E, F, G on M, where c is a real constant, then the statistical manifold (M, ∇, σ, g) is said to be of type quaternionic space form.

We remark that changing J_α for J_α^* in (21), we get the curvature tensor R^* with respect to the dual connection ∇^*. If (M, σ, g) is a quaternionic Kähler manifold satisfying (21), then M is said to be a space of constant quaternionic sectional curvature, or quaternionic space form. It is known that quaternionic space forms are locally congruent to either a quaternionic projective space $\mathbb{H}P^n(c)$ of quaternionic sectional curvature $c > 0$, a quaternionic Euclidean space $\mathbb{H}^n$ of null quaternionic sectional curvature or a quaternionic hyperbolic space $\mathbb{H}H^n(c)$ of quaternionic sectional curvature $c < 0$ [35].

Theorem 1. *(M, ∇, σ, g) is a quaternionic Kähler-like statistical manifold if and only if $(M, \nabla^*, \sigma^*, g)$ is.*

Proof. First of all, it is obvious that the triple (M, σ, g) is an almost Hermite-like quaternionic manifold if and only if (M, σ^*, g) is. Now, we take a canonical local basis $\{J_1, J_2, J_3\}$ of σ. Then, using (1) and (19) we derive for all $\alpha \in \{1, 2, 3\}$:

$$\begin{aligned} g((\nabla_G J_\alpha)E, F) &= g(\nabla_G J_\alpha E, F) - g(J_\alpha \nabla_G E, F) \\ &= -g(J_\alpha E, \nabla_G^* F) + Gg(J_\alpha E, F) + g(\nabla_G E, J_\alpha^* F) \\ &= g(E, J_\alpha^* \nabla_G^* F) + Gg(J_\alpha E, F) + Gg(E, J_\alpha^* F) - g(E, \nabla_G^* J_\alpha^* F) \\ &= -g(E, (\nabla_G^* J_\alpha^*)F), \end{aligned} \tag{22}$$

for all vector fields E, F, G on M, where the indices are taken from $\{1, 2, 3\}$ modulo 3.

On the other hand, making use of (19)and (20), we obtain

$$\begin{aligned} g((\nabla_G J_\alpha)E, F) &= \omega_{\alpha+2}(G)g(J_{\alpha+1}E, F) - \omega_{\alpha+1}(G)g(J_{\alpha+2}E, F) \\ &= -\omega_{\alpha+2}(G)g(E, J^*_{\alpha+1}F) + \omega_{\alpha+1}(G)g(E, J^*_{\alpha+2}F) \\ &= g(E, -\omega_{\alpha+2}(G)J^*_{\alpha+1}F + \omega_{\alpha+1}(G)J^*_{\alpha+2}F). \end{aligned} \tag{23}$$

From (22) and (23) we deduce

$$(\nabla^*_G J^*_\alpha)F = \omega_{\alpha+2}(G)J^*_{\alpha+1}F - \omega_{\alpha+1}(G)J^*_{\alpha+2}F,$$

for all vector fields F, G on M and for all $\alpha \in \{1, 2, 3\}$, where the indices are taken from $\{1, 2, 3\}$ modulo 3. Therefore we conclude that $(M, \nabla^*, \sigma^*, g)$ is a quaternionic Kähler-like statistical manifold. □

Corollary 1. *(M, ∇, σ, g) is a hyper-Kähler-like statistical manifold if and only if $(M, \nabla^*, \sigma^*, g)$ is.*

Proof. The assertion is clear from Theorem 1. □

Remark 1. *We note that the concepts of almost Hermite-like quaternionic manifold and quaternionic Kähler-like statistical manifold proposed in this section generalize the classical notions of almost quaternionic Hermitian manifold and quaternionic Kähler manifold* [23,33]. *In fact, an almost quaternionic Hermitian manifold is a particular case of almost Hermite-like quaternionic manifold with $J^*_\alpha = J_\alpha$, $\alpha \in \{1, 2, 3\}$, and hence with $\sigma = \sigma^*$. Similarly, any quaternionic Kähler manifold is a particular case of quaternionic Kähler-like statistical manifold, where $\nabla = \nabla^*$ is the Levi–Civita connection of the metric g.*

Example 1. *Let (M, ∇, ϕ, g) be an almost Hermite-like statistical manifold (see* [6,36] *for basic definitions and examples). Next we prove that TM can be endowed with an almost Hermite-like quaternionic statistical structure. First of all, we note that the tangent bundle TM can be equipped with the Sasaki metric G defined by*

$$G(A, B) = g(KA, KB) + g(\pi_* A, \pi_* B),$$

for all vector fields A, B on TM, where π is the natural projection of TM onto M and K is the connection map associated with the Levi–Civita connection of the metric g (see [37]*).*

We note that if $X \in \Gamma(TM)$, then there exists exactly one vector field on TM, denoted by X^h and called the horizontal lift, *and denoted X^v and called the* vertical lift *of X, such that we have for all $U \in TM$:*

$$\pi_* X^h_U = X_{\pi(U)},\ \pi_* X^v_U = 0_{\pi(U)},\ KX^h_U = 0_{\pi(U)},\ KX^v_U = X_{\pi(U)}.$$

We recall now that, according to Theorem 3 in [38]*, one can define a torsion free linear connection ∇' on TM compatible to the Sasaki metric G. Hence (TM, ∇', G) is a statistical manifold. Moreover, using the almost complex structure ϕ on M, we can also define three tensor fields J_1, J_2, J_3 on TM by the equalities:*

$$\begin{cases} J_1 X^h = X^v \\ J_1 X^v = -X^h \end{cases},$$

$$\begin{cases} J_2 X^h = (\phi X)^v \\ J_2 X^v = (\phi X)^h \end{cases},$$

$$\begin{cases} J_3 X^h = -(\phi X)^h \\ J_3 X^v = (\phi X)^v \end{cases}.$$

It is easy to see that J_1, J_2, J_3 satisfy the quaternionic identities (17) and, defining σ to be the 3-subbundle of $End(TM)$ generated by $(J_\alpha)_{\alpha=1,2,3}$, we derive immediately that (TM, ∇', σ, G) is an almost Hermite-like quaternionic statistical manifold. Moreover it can be proved that (TM, ∇', σ, G) is a hyper-Kähler-like statistical manifold if and only if (M, ∇, ϕ, g) is a flat Kähler-like statistical manifold.

4. Quaternionic Kähler-like Statistical Submersions

Definition 5. *Let (M, σ, g) and (M', σ', g') be two almost Hermite-like quaternionic manifolds. Then:*

*i. A map $f : M \to M'$ is called a $(\sigma, \sigma')-$*holomorphic map *at a point $p \in M$ if for any $J \in \sigma_p$ exists $J' \in {\sigma'}_{f(p)}$ such that $f_* \circ J = J' \circ f_*$. Moreover, we say that f is a (σ, σ')-holomorphic map if f is a (σ, σ')-holomorphic map at each point $p \in M$.*

ii. A semi-Riemannian submersion $\pi : M \to M'$ which is a (σ, σ')-holomorphic map is called an almost Hermite-like quaternionic submersion.

iii. A statistical submersion $\pi : M \to M'$ between two almost Hermite-like quaternionic statistical manifolds (M, ∇, σ, g) and $(M', \nabla', \sigma', g')$ such that π is a (σ, σ')-holomorphic map is said to be an almost Hermite-like quaternionic statistical submersion.

iv. An almost Hermite-like quaternionic statistical submersion $\pi : M \to M'$, where (M, ∇, σ, g) is a quaternionic Kähler-like statistical manifold, is called a quaternionic Kähler-like statistical submersion. *In particular, if (M, ∇, σ, g) is a (locally) hyper-Kähler-like statistical manifold, then π is called a* (locally) hyper-Kähler-like statistical submersion.

Remark 2. *We can easily check that:*

i. A map $f : M \to M'$ between two almost Hermite-like quaternionic manifolds is a (σ, σ')-holomorphic map at a point $p \in M$ if and only if for any canonical local basis $\{J_1, J_2, J_3\}$ of σ_p there exists a canonical local basis $\{J'_1, J'_2, J'_3\}$ of ${\sigma'}_{f(p)}$ such that $f_* \circ J_\alpha = J'_\alpha \circ f_*$, for $\alpha = 1, 2, 3$.

ii. A semi-Riemannian submersion $\pi : M \to M'$ between two almost Hermite-like quaternionic statistical manifolds (M, ∇, σ, g) and $(M', \nabla', \sigma', g')$ is a (σ, σ')-holomorphic map if and only if it is a (σ^*, σ'^*)-holomorphic map.

Property 1. *Let $\pi : M \to M'$ be an almost Hermite-like quaternionic statistical submersion. Then:*

i. $\mathcal{V}$ and $\mathcal{H}$ are invariant under each $J \in \sigma_p$ and $J^* \in \sigma_p$, $\forall p \in M$. Moreover, J and J^* commute with the horizontal and vertical projectors.

ii. If X is a basic vector field on M π-related to X' on M', then $J_\alpha X$ (or $J^*_\alpha X$) is a basic vector field π-related to $J'_\alpha X'$ (or $J'^*_\alpha X'$) on M', for $\alpha = 1, 2, 3$.

Proof. i. Since π is a (σ, σ')-holomorphic map, we obtain for any $V \in \Gamma(\mathcal{V})$:

$$\pi_* J_\alpha V = J'_\alpha \pi_* V = 0$$

and thus we conclude that $J_\alpha(\mathcal{V}) \subseteq \mathcal{V},\ \forall \alpha \in \{1, 2, 3\}$, where $\{J_1, J_2, J_3\}$ is a canonical local basis of σ. Similarly it follows that $J^*_\alpha(\mathcal{V}) \subseteq \mathcal{V},\ \forall \alpha \in \{1, 2, 3\}$. On the other hand, for any $X \in \Gamma(\mathcal{H})$ and $V \in \Gamma(\mathcal{V})$, we derive

$$g(J_\alpha X, V) = -g(X, J^*_\alpha V) = 0$$

and thus we conclude that $J_\alpha(\mathcal{H}) \subseteq \mathcal{H},\ \forall \alpha \in \{1, 2, 3\}$, where $\{J_1, J_2, J_3\}$ is a canonical local basis of σ. In a similar way, we obtain that $J^*_\alpha(\mathcal{H}) \subseteq \mathcal{H},\ \forall \alpha \in \{1, 2, 3\}$. The second part of the statement now follows immediately.

ii. If X is a basic vector field, then from i. $J_\alpha X$ and $J^*_\alpha X$ are horizontal vector fields. On the other hand, since π is a (σ, σ')-holomorphic map and X is π-related to X' on M' we derive that

$$\pi_* J_\alpha X = J'_\alpha \pi_* X = J'_\alpha X'$$

and similarly

$$\pi_* J^*_\alpha X = J'^*_\alpha \pi_* X = J'^*_\alpha X'$$

for $\alpha = 1, 2, 3$ and the conclusion is now clear.

□

Theorem 2. *If $\pi : M \to M'$ is an almost Hermite-like quaternionic statistical submersion, then the fibers are almost Hermite-like quaternionic statistical manifolds.*

Proof. Let $F = \pi^{-1}(p')$ be a fiber of the submersion, where $p' \in M'$. Then it is known from [3,6,7] that $(F, \widehat{\nabla}, \widehat{g} = g|_F)$ is a statistical manifold. Moreover, for any canonical local basis $\{J_1, J_2, J_3\}$ of σ, we can define

$$\widehat{J}_\alpha := J_\alpha|_F,\ \alpha = 1, 2, 3,$$

and we can consider the subbundle $\widehat{\sigma}$ locally spanned by $\{\widehat{J}_1, \widehat{J}_2, \widehat{J}_3\}$. Now it follows immediately that $(F, \widehat{\nabla}, \widehat{\sigma}, \widehat{g})$ is an almost Hermite-like quaternionic statistical manifold. □

Theorem 3. *If $\pi : M \to M'$ is a quaternionic Kähler-like statistical submersion, then $(M', \nabla', \sigma', g')$ is a quaternionic Kähler-like statistical manifold. Moreover, the fibers are also quaternionic Kähler-like statistical manifolds.*

Proof. If we take two basic vector fields X, Y on M π-related to X', Y' on M', then using Proposition 1 we derive:

$$\begin{aligned}
(\nabla'_{X'} J'_\alpha) Y' &= \nabla'_{X'}(J'_\alpha Y') - J'_\alpha(\nabla'_{X'} Y') \\
&= \nabla'_{\pi_* X}(\pi_*(J_\alpha Y)) - J'_\alpha \pi_*(\nabla_X Y) \\
&= \pi_*(\nabla_X(J_\alpha Y)) - \pi_*(J_\alpha(\nabla_X Y)) \\
&= \pi_*((\nabla_X J_\alpha) Y).
\end{aligned} \tag{24}$$

Since (M, ∇, σ, g) is a quaternionic Kähler-like statistical manifold, we have (20) and we can define 1-forms $\omega_1', \omega_2', \omega_3'$ on M' by:

$$\omega_\alpha'(X') := \omega_\alpha(X), \ \alpha = 1, 2, 3, \tag{25}$$

for any local vector field X' on M' and X a basic vector field on M such that $\pi_* X = X'$.

Next, making use of (20), (24) and (25), we obtain:

$$\begin{aligned}(\nabla'_{X'} J'_\alpha)Y' &= \pi_*(\omega_{\alpha+2}(X)J_{\alpha+1}Y - \omega_{\alpha+1}(X)J_{\alpha+2}Y) \\ &= \omega'_{\alpha+2}(X')J'_{\alpha+1}Y' - \omega'_{\alpha+1}(X')J'_{\alpha+2}Y',\end{aligned} \tag{26}$$

where the indices are taken from $\{1, 2, 3\}$ modulo 3. Therefore $(M', \nabla', \sigma', g')$ is a quaternionic Kähler-like statistical manifold.

Next, we consider $F = \pi^{-1}(p')$, $p' \in M'$, a fiber of the submersion. Then, from Theorem 2, it follows that $(F, \widehat{\nabla}, \widehat{\sigma}, \widehat{g})$ is an almost Hermite-like quaternionic statistical manifold. Using (12) we derive for all $U, V \in \Gamma(\mathcal{V})$:

$$(\nabla_U J_\alpha)V = (\widehat{\nabla}_U J_\alpha)V + (T_U J_\alpha V - J_\alpha T_U V), \ \alpha = 1, 2, 3. \tag{27}$$

On the other hand, from (20) we have

$$(\nabla_U J_\alpha)V = \omega_{\alpha+2}(U)\widehat{J}_{\alpha+1}V - \omega_{\alpha+1}(U)\widehat{J}_{\alpha+2}V, \ \alpha = 1, 2, 3. \tag{28}$$

From (27) and (28) we deduce

$$(\widehat{\nabla}_U J_\alpha)V = \omega_{\alpha+2}(U)\widehat{J}_{\alpha+1}V - \omega_{\alpha+1}(U)\widehat{J}_{\alpha+2}V, \ \alpha = 1, 2, 3 \tag{29}$$

and

$$T_U J_\alpha V = J_\alpha T_U V, \ \alpha = 1, 2, 3. \tag{30}$$

Finally, from (29) it follows that $(F, \widehat{\nabla}, \widehat{\sigma}, \widehat{g})$ is a quaternionic Kähler-like statistical manifold and the proof is now complete. □

Corollary 2. *If $\pi : M \to M'$ is a locally hyper-Kähler-like statistical submersion, then $(M', \nabla', \sigma', g')$ is a locally hyper-Kähler-like statistical manifold. Moreover, the fibers are also locally hyper-Kähler-like statistical manifolds.*

Proof. The assertion is immediate from Theorem 3. □

Theorem 4. *Let $\pi : M \to M'$ be a quaternionic Kähler-like statistical submersion. Then:*

i. $T_U V = 0$, *for all* $U, V \in \Gamma(\mathcal{V})$;

ii. $A_X Y = 0$, *for all* $X, Y \in \Gamma(\mathcal{H})$.

Proof. Since T has the symmetry property for vertical vector fields (*cf.* (8)), using (17) and (30) we derive for all $U, V \in \Gamma(\mathcal{V})$ and $\alpha = 1, 2, 3$:

$$\begin{aligned} T_U V + T_{J_\alpha U} J_\alpha V &= T_U V + J_\alpha T_{J_\alpha U} V \\ &= T_U V + J_\alpha T_V J_\alpha U \\ &= T_U V + J_\alpha^2 T_V U \\ &= T_U V - T_V U \\ &= 0. \end{aligned}$$

Therefore, we deduce

$$T_U V + T_{J_1 U} J_1 V = T_U V + T_{J_2 U} J_2 V = T_U V + T_{J_3 U} J_3 V = 0. \tag{31}$$

In particular, from (30) it follows that

$$T_{J_1 U} J_1 V = T_{J_2 U} J_2 V. \tag{32}$$

On the other hand, replacing in (31) U by $J_1 U$ and V by $J_1 V$, we derive

$$T_{J_1 U} J_1 V + T_{J_2 U} J_2 V = 0. \tag{33}$$

Now, from (32) and (33) we deduce that

$$T_{J_1 U} J_1 V = T_{J_2 U} J_2 V = 0 \tag{34}$$

and finally, from (31) and (34) we conclude that $T_U V = 0$, for all $U, V \in \Gamma(\mathcal{V})$.

Assertion ii. follows in a similar way. □

Corollary 3. *If $\pi : M \to M'$ is a quaternionic Kähler-like statistical submersion, then π has isometric fibers.*

Proof. The assertion is an obvious consequence of Theorem 4. □

Corollary 4. *If $\pi : M \to M'$ is a quaternionic Kähler-like statistical submersion, then $A^*_X Y = 0$, for all $X, Y \in \Gamma(\mathcal{H})$.*

Proof. The conclusion follows immediately from Theorem 4 and (10). □

Corollary 5. *If $\pi : M \to M'$ is a quaternionic Kähler-like statistical submersion, then the horizontal distribution is completely integrable.*

Proof. This assertion is clear from Theorem 4 and (9). □

Theorem 5. *Let $\pi : M \to M'$ be a quaternionic Kähler-like statistical submersion. If the total space of the submersion is of type quaternionic space form, then the base space of the submersion is of type quaternionic space form and each fiber is a totally geodesic submanifold of M of type quaternionic space form.*

Proof. The conclusions follow easily using the analogues of the O'Neill equations for a statistical submersion (Theorem 2.1 in [6]) and taking account of Theorem 4. $\square$

Example 2. *Let (M, ∇, σ, g) be an almost Hermite-like quaternionic statistical manifold. Then we can define a torsion free linear connection ∇' on TM such that (TM, ∇', G) is a statistical manifold [38], where G is the Sasaki metric. Next, we consider for any canonical local basis $\{J_1, J_2, J_3\}$ of σ the following tensor fields on TM, denoted by J_1', J_2', J_3':*

$$J_\alpha' X^h = (J_\alpha X)^h,\ J_\alpha' X^v = (J_\alpha X)^v,\ \alpha = 1, 2, 3.$$

Defining now the vector bundle σ' over TM generated by $\{J_1', J_2', J_3'\}$ (see [22,39]), one can easily conclude that $(TM, \nabla', \sigma', g')$ is an almost Hermite-like quaternionic statistical manifold. Moreover, we remark that

$$\pi_* J_\alpha' X^v = \pi_*(J_\alpha X)^v = 0 = J_\alpha \pi_* X^v$$

and

$$\pi_* J_\alpha' X^h = \pi_*(J_\alpha X)^h = J_\alpha X = J_\alpha \pi_* X^h.$$

Hence $\pi_* J_\alpha' = J_\alpha \pi_*$, $\alpha = 1, 2, 3$, and we conclude that the canonical projection $\pi : TM \to M$ is a (σ, σ')-holomorphic map. Therefore π is an almost Hermite-like quaternionic statistical submersion. Moreover, it follows that π is a locally hyper-Kähler-like statistical submersion if and only if (M, ∇, σ, g) is a flat locally hyper-Kähler-like statistical manifold.

5. Conclusions and Future Research

It is well known there is a deep relationship between statistics and differential geometry. A first step in this connection was given by C.R. Rao [40], who introduced a Riemannian metric in the space of probability distributions, providing a general framework for discussing problems of statistical inference, information loss and estimation, and giving an impulse to construct a geometrical theory of statistics (see, e.g., [41–48]). The most natural frame in this context is the concept of a statistical manifold [49]. As it was pointed out in [50], the statistical manifolds are naturally associated to a family of affine-metric geometries and one can obtain interesting properties relating self-parallel curves to the relative entropy. Recently, H.V. Lê [51] proved that any smooth statistical manifold can be embedded into the space of probability measures on a finite set, giving a positive answer to an open problem of S. Amari and S.L. Lauritzen (see [2]). Therefore, any smooth statistical manifold is a finite-dimensional statistical model.

In the present paper, we introduced the notions of almost Hermite-like quaternionic statistical manifold and quaternionic Kähler-like statistical submersion, obtaining several properties. We also proved that the tangent bundle of an almost Hermite-like quaternionic statistical manifold has a natural almost Hermite-like quaternionic statistical structure and showed that the canonical projection provides us a very natural example of an almost Hermite-like quaternionic statistical submersion. We believe that the concepts investigated in this work can be also studied in some new settings, namely for statistical manifolds endowed with quaternionic structures of second kind [52] (also called paraquaternionic structures [39]), Kenmotsu structures [53], 3-Sasakian structures [54], almost para-Hermitian structures [55,56] and almost para-contact structure [57,58]. We note that all these structure are of great interest not only in differential geometry, but also in various fields of science and engineering, such as string theory, integrable systems, quantum systems, statistical mechanics, motion planning, robot control and sensing, sensor networks, and digital signal processing. We look forward to studying some of these problems in detail later. Finally, we would like to note another five open problems in the field for further research.

Problem 1. *To investigate if it is possible to construct an infinite family of quaternionic Kähler-like statistical structures on the tangent bundle of an almost Hermite-like quaternionic statistical manifold. A possible answer could be obtained by deforming the almost Hermite-like quaternionic statistical structure defined in Example 2 in a similar way to* [59].

Problem 2. *To construct examples of locally hyper-Kähler-like statistical manifolds which are not hyper-Kähler ones. A possible solution could be to deform the almost Hermite-like quaternionic statistical structure from Example 1.*

Problem 3. *To investigate under what conditions the exponential families, including the well-known multinomial distribution, negative multinomial distribution, and multivariate normal distribution, admit hyper-Kähler or quaternionic Kähler structure.*

Problem 4. *To investigate the behavior of submanifolds in statistical manifolds of type quaternionic space form, as a quaternionic version of some recent results from* [13] *concerning submanifolds in statistical manifolds of constant curvature.*

Problem 5. *To define affine connections compatible with a hypercosymplectic structure* [60] *and to obtain necessary and sufficient conditions for two compatible connections to form a dualistic structure, as an extension of the results from* [61]. *Moreover, to define and investigate statistical submersions from almost Hermite-like quaternionic statistical submersions onto statistical manifolds equipped with hypercosymplectic structures.*

Acknowledgments

The second author was supported by National Research Council–Executive Agency for Higher Education Research and Innovation Funding (CNCS-UEFISCDI), project number PN-II-ID-PCE-2011-3-0118.

Author Contributions

Both authors have contributed equally to the study and preparation of the article. Both authors have read and approved the final manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

References

1. Amari, S. *Differential-Geometrical Methods in Statistics*; Lecture Notes in Statistics, Volume 28; Springer: New York, NY, USA, 1985.
2. Amari, S.; Nagaoka, H. *Method of Information Geometry*; Oxford University Press: Oxford, UK, 2000.
3. Abe, N.; Hasegawa, K. An affine submersion with horizontal distribution and its applications. *Differ. Geom. Appl.* **2001**, *14*, 235–250.
4. O'Neill, B. The fundamental equations of a submersion. *Mich. Math. J.* **1966**, *13*, 459–469.
5. O'Neill, B. Submersions and geodesics. *Duke Math. J.* **1967**, *34*, 363–373.
6. Takano, K. Statistical manifolds with almost complex structures and its statistical submersions. *Tensor N.S.* **2004**, *65*, 128–142.
7. Takano, K. Statistical manifolds with almost contact structures and its statistical submersions. *J. Geom.* **2006**, *85*, 171–187.
8. Takano, K. Exponential families admitting almost complex structures. *SUT J. Math.* **2010**, *46*, 1–21.
9. Matsuzoe, H.; Inoguchi, J. Statistical structures on tangent bundles. *Appl. Sci.* **2003**, *5*, 55–75.
10. Takano, K. Examples of the statistical submersions on the statistical model. *Tensor N.S.* **2004**, *65*, 170–178.
11. Furuhata, H. Hypersurfaces in statistical manifolds. *Differ. Geom. Appl.* **2009**, *27*, 420–429.
12. Noda, T. Symplectic structures on statistical manifolds. *J. Aust. Math. Soc.* **2011**, *90*, 371–384.
13. Aydin, M.E.; Mihai, A; Mihai, I. Some inequalities on submanifolds in manifolds of constant curvature. *Filomat* **2015**, *29*, 465–477.
14. Bensadon, J. Black-box optimization using geodesics in statistical manifolds. *Entropy* **2015**, *17*, 304–345.
15. Boyom, N.M.; Jamali, M.; Hasan Shahid, M. Multiply CR-warped product statistical submanifolds of a holomorphic stastistical space form. In Proceedings of the 2nd Conference on Geometric Science of Information, Paris, France, 28–30 October 2015, submitted.
16. Hasegawa, I.; Yamauchi, K. Conformal-projective flatness of tangent bundle with complete lift statistical structure. *Differ. Geom. Dyn. Syst.* **2008**, *10*, 148–158.
17. Min, C.; Ri, W.; Kwak, K.; An, D. Equiaffine structure and conjugate Ricci-symmetry of a statistical manifold. *Differ. Geom. Appl.* **2015**, *41*, 39–47.
18. Opozda, B. A sectional curvature for statistical structures. **2015**, arXiv:1504.01279.
19. Uohashi, K. A Hessian domain constructed with a foliation by 1-conformally flat statistical manifolds. *Int. Math. Forum* **2012**, *7*, 2363–2371.
20. Uohashi, K. Harmonic maps relative to α-connections on Hessian domains. In *Geometric Science of Information*; Nielsen, F., Barbaresco, F., Eds.; Springer: Heidelberg, Germany, 2013; pp. 745–750.
21. Zhang, J. On Monotone Embedding in Information Geometry. *Entropy* **2015**, *17*, 4485–4499.
22. Ianuş, S.; Mazzocco, R.; Vîlcu, G.E. Riemannian submersions from quaternionic manifolds. *Acta Appl. Math.* **2008**, *104*, 83–89.

23. Alekseevsky, D.V.; Marchiafava, S. Quaternionic structures on a manifold and subordinated structures. *Annali di Matematica Pura ed Applicata* **1996**, *171*, 205–273.
24. Ianuş, S.; Mazzocco, R.; Vîlcu, G.E. Harmonic maps between quaternionic Kähler manifolds. *J. Nonlinear Math. Phys.* **2008**, *15*, doi:10.2991/jnmp.2008.15.1.1.
25. Park, K.S. h-semi-slant submersions from almost quaternionic Hermitian manifolds. *Taiwan. J. Math.* **2014**, *18*, 1909–1926.
26. Uohashi, K. On α-conformal equivalence of statistical submanifolds. *J. Geom.* **2002**, *75*, 179–184.
27. Noguchi, M. Geometry of statistical manifolds. *Differ. Geom. Appl.* **1992**, *2*, 197–222.
28. Zhang, J. A note on curvature of α-connections of a statistical manifold. *Ann. Inst. Statist. Math.* **2007**, *59*, 161–170.
29. Takeuchi, J.; Amari, S. α-parallel prior and its properties. *IEEE Trans. Inf. Theory* **2005**, *51*, 1011–1023.
30. O'Neill, B. *Semi-Riemannian geometry with applications to relativity*; Academic Press: New York, NY, USA, 1983.
31. Falcitelli, M.; Ianuş, S.; Pastore, A.M. *Riemannian Submersions and Related Topics*; World Scientific Publishing: Singapore, Singapore, 2004.
32. Chen, B.-Y. *Pseudo-Riemannian Geometry, δ-Invariants and Applications*; World Scientific Publishing: Singapore, Singapore, 2011.
33. Ishihara, S. Quaternion Kählerian manifolds. *J. Differ. Geom.* **1974**, *9*, 483–500.
34. Şahin, B. Slant submanifolds of quaternion Kaehler manifolds. *Commun. Korean Math. Soc.* **2007**, *22*, 123–135.
35. Adachi, T.; Maeda, S. Some characterizations of quaternionic space forms. *Proc. Jpn. Acad. Ser. A Math. Sci.* **2000**, *76*, 168–172.
36. Takano, K. Examples of statistical manifolds with almost complex structures. *Tensor N.S.* **2008**, *69*, 58–66.
37. Dombrowski, P. On the geometry of the tangent bundle. *Journal für die reine und angewandte Mathematik* **1962**, *210*, 73–88.
38. Ianuş, I. Statistical manifolds and tangent bundles. *Univ. Politeh. Buchar. Sci. Bull. Ser. D Mech. Eng.* **1994–1995**, *56–57*, 29–34.
39. Caldarella, A. On paraquaternionic submersions between paraquaternionic Kähler manifolds. *Acta Appl. Math.* **2010**, *112*, doi:10.1007/s10440-009-9549-7.
40. Rao, C.R. Information and the accuracy attainable in the estimation of statistical parameters. *Bull. Calcutta Math. Soc.* **1945**, *37*, 81–91.
41. Amari, S. Differential geometry of curved exponential families-curvatures and information loss. *Ann. Statist.* **1982**, *10*, 357–385.
42. Barndorff-Nielsen, O.E.; Cox, D.R.; Reid, N. The role of differential geometry in statistical theory. *Int. Stat. Rev.* **1986**, *54*, 83–96.
43. Barndorff-Nielsen, O.E.; Jupp, P.E. Differential geometry, profile likelihood, L-sufficiency and composite transformation models. *Ann. Stat.* **1988**, *16*, 1009–1043.
44. Chentsov, N.N. *Statistical Decision Rules and Optimal Inferences*; Izdat. Nauka: Moscow, Russia, 1972.

45. Efron, B. Defining the curvature of a statistical problem (with applications to second order efficiency). *Ann. Stat.* **1975**, *3*, 1189–1242.
46. Kass, R.; Vos, P.W. *Geometrical Foundations of Asymptotic Inference*; Wiley: New York, NY, USA, 1997.
47. Murray, M.K.; Rice, J.W. *Differential geometry and statistics*; Monographs on Statistics and Applied Probability 48; Chapman & Hall: London, UK, 1993.
48. Vos, P.W. Fundamental equations for statistical submanifolds with applications to the Bartlett correction. *Ann. Inst. Stat. Math.* **1989**, *41*, 429–450.
49. Amari, S.; Barndorff-Nielsen, O.E.; Kass, R.E.; Lauritzen, S.L.; Rao, C.R. *Differential Geometry in Statistical Inference*; Institute of Mathematical Statistics: Beachwood, OH, USA, 1987.
50. Burdet, G.; Combe, P.; Nencka, H. Statistical manifolds: The natural affine-metric structure of probability theory. In *Mathematical Physics and Stochastic Analysis: Essays in Honour of Ludwig Streit*; World Scientific Publishing: Singapore, Singapore, 2000; pp. 138–164.
51. Lê, H.V. Statistical manifolds are statistical models. *J. Geom.* **2005**, *84*, 83–93.
52. Libermann, P. Sur les structures presque quaternioniennes de deuxième espéce. *Bull. Soc. Math. France* **1955**, *83*, 195–224.
53. Kenmotsu, K. A class of almost contact Riemannian manifolds. *Tohoku Math. J.* **1972**, *2*, 93–103.
54. Kuo, Y.Y. On almost contact 3-structure. *Tohoku Math. J.* **1970**, *22*, 325–332.
55. Gündüzalp, Y. Anti-invariant semi-Riemannian submersions from almost para-Hermitian manifolds. *J. Funct. Spaces Appl.* **2013**, *2013*, doi:10.1155/2013/720623.
56. Vaccaro, M. (Para-)Hermitian and (para-)Kähler submanifolds of a para-quaternionic Kähler manifold. *Differ. Geom. Appl.* **2012**, *30*, 347–364.
57. Gündüzalp, Y.; Şahin, B. Paracontact semi-riemannian submersions. *Turk. J. Math.* **2013**, *37*, 114–128.
58. Gündüzalp, Y.; Şahin, B. Para-contact para-complex semi-Riemannian submersions. *Bull. Malays. Math. Sci. Soc.* **2014**, *37*, 139–152.
59. Tahara, M.; Marchiafava, S.; Watanabe, Y. Quaternionic Kähler structures on the tangent bundle of a complex space form. *Rendiconti dell'Istituto di Matematica dell'Università di Trieste* **1999**, *31*, 163–175.
60. Kim, J.S.; Choi, J.; Tripathi, M.M. On generic submanifolds of manifolds equipped with a hypercosymplectic 3-structure. *Commun. Korean Math. Soc.* **2006**, *21*, 321–335.
61. Blaga, A. Dualistic structures on Kähler manifolds. *Rev. Un. Mat. Argentina* **2012**, *53*, 55–60.