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Logical Entropy of Fuzzy Dynamical Systems

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Abstract: Recently the logical entropy was suggested by D. Ellerman (2013) as a new information measure. The present paper deals with studying logical entropy and logical mutual information and their properties in a fuzzy probability space. In particular, chain rules for logical entropy and for logical mutual information of fuzzy partitions are established. Using the concept of logical entropy of fuzzy partition we define the logical entropy of fuzzy dynamical systems. Finally, it is proved that the logical entropy of fuzzy dynamical systems is invariant under isomorphism of fuzzy dynamical systems.

Keywords: fuzzy probability space; fuzzy partition; logical entropy; logical mutual information; fuzzy dynamical system

1. Introduction

The classical approach in information theory [1] is based on Shannon's entropy [2]. Using Shannon entropy Kolmogorov and Sinai [3,4] defined the entropy $h(T)$ of dynamical systems. Since the entropy $h(T)$ is invariant under isomorphism of dynamical systems, they received a tool for distinction of non-isomorphic dynamical systems by means of which proved the existence of non-isomorphic Bernoulli shifts. In the paper by Markechová [5] the Shannon entropy of fuzzy partitions has been defined. This concept was exploited to define the Kolmogorov-Sinai entropy h_m of fuzzy dynamical systems [6]. The obtained results generalize the corresponding results from the classical Kolmogorov theory. In [7] it was shown that h_m coincides on isomorphic fuzzy dynamical systems, hence h_m can serve as a tool for distinction of non-isomorphic fuzzy dynamical systems.

Recently the logical entropy was suggested by Ellerman [8] as a new information measure. Let $P = (p_1, \ldots, p_n) \in \Re^n$ be a probability distribution; the logical entropy of P is defined by Ellerman as the number $h(P) = \sum_{i=1}^{n} p_i(1 - p_i)$. Ellerman also defined a logical mutual information and logical conditional entropy and discussed the relation of logical entropy to Shannon's entropy. B. Tamir and E. Cohen in [9] extended the definition of logical entropy to the theory of quantum states.

The aim of this paper is to study the logical entropy in fuzzy probability spaces and fuzzy dynamical systems. The paper is organized as follows. In the next section, we give the basic definitions and some known results used in the paper and we present relevant related works. In Section 3, the logical entropy, conditional logical entropy, logical mutual information and logical conditional mutual information of fuzzy partitions of a fuzzy probability space are defined. We state and prove some of the basic properties of these measures; in particular, chain rules for logical entropy and for logical mutual information of fuzzy partitions are established. In Section 4, the logical entropy h_L of fuzzy dynamical systems is defined and studied. It is proved that the logical entropy h_L of fuzzy dynamical systems is invariant under isomorphism of fuzzy dynamical systems (Theorem 12). In this way, we obtained a new tool for distinction of non-isomorphic fuzzy dynamical systems; this result is demonstrated by Example 4. Our conclusions are given in Section 5.

2. Basic Definitions and Related Works

In this section, we recall some definitions and basic facts which will be used throughout this paper and we mention some works connected with the subject of this paper, of course, with no claim for completeness.

In the classical probability theory, an event is understood as an exactly defined phenomenon and from the mathematical point of view it is a classical set. In practice, however, we often encounter events that are described imprecisely, vaguely, so called fuzzy events. That is why various proposals for a fuzzy generalization of the notions of classical probability theory have been created. The object of our studies will be a fuzzy probability space $(\Omega,\ M, \mu)$ defined by Piasecki [10].

Definition 1. By a fuzzy probability space we mean a triplet $(\Omega,\ M, \mu)$, where Ω is a non-empty set, M is a fuzzy σ-algebra of fuzzy subsets of Ω, *i.e.*, $M \subset [0,\ 1]^{\Omega}$ such that (i) $1_\Omega \in M$; $(1/2)_\Omega \notin M$; (ii) if $a \in M$, then $a^\perp = 1_\Omega - a \in M$; (iii) if $a_n \in M,\ n = 1, 2, \ldots$, then $\cup_{n=1}^{\infty} a_n \in M$, and the mapping $\mu : M \to [0,\ \infty)$ satisfies the following conditions: (iv) $\mu(a \cup a') = 1$ for all $a \in M$; (v) if $\{a_n\}_{n=1}^{\infty} \subset M$ such that $a_i \leqslant a_j^\perp$ (point wisely) whenever $i \neq j$, then $\mu(\cup_{n=1}^{\infty} a_n) = \sum_{n=1}^{\infty} \mu(a_n)$.

The symbols $\cup_{n=1}^{\infty} a_n = \sup_n a_n$ and $\cap_{n=1}^{\infty} a_n = \inf_n a_n$ denote the fuzzy union and the fuzzy intersection of a sequence $\{a_n\}_{n=1}^{\infty} \subset M$, respectively, in the sense of Zadeh [11]. Note that operations with fuzzy sets can be introduced in various ways. A review can be found in [12] (see also [13]). Using the complementation $\perp$: $a \to a^\perp$ for every fuzzy subset $a \in M$, we see that the complementation $\perp$ satisfies two conditions: (*i*) $(a^\perp)^\perp = a$ for every $a \in M$; (*ii*) if $a \leqslant b$, then $b^\perp \leqslant a^\perp$. Therefore, M is a distributive $\sigma-$ lattice with the complementation $\perp$ for which the de Morgan laws hold: $(\cup_{n=1}^{\infty} a_n)^\perp = \cap_{n=1}^{\infty} a_n^\perp$ and $(\cap_{n=1}^{\infty} a_n)^\perp = \cup_{n=1}^{\infty} a_n^\perp$ for any sequence $\{a_n\}_{n=1}^{\infty} \subset M$. Fuzzy subsets $a,\ b$ of Ω such that $a \cap b = 0_\Omega$ are called separated fuzzy sets, fuzzy subsets $a,\ b \in M$ such that $a \leqslant b^\perp$ are called W-separated. Each fuzzy subset $a \in M$ such that $a \geqslant a^\perp$ is called a W-universum, each fuzzy subset $a \in M$ such that $a \leqslant a^\perp$ is called a W-empty set. A set from the fuzzy σ-algebra M is a fuzzy event; W-separated fuzzy events are interpreted as mutually exclusive events. A W-universum is interpreted as a certain event and a W-empty set as an impossible event. It can be proved that a fuzzy set $a \in M$ is a W-universum if and only if there exists a fuzzy set $b \in M$ such that $a = b \cup b^\perp$. The presented σ-additive fuzzy measure μ has all properties analogous to properties of a classical probability measure. We recall some of them that are used in the following.

(2.1) $\mu(a^\perp) = 1 - \mu(a)$ for every $a \in M$.

(2.2) μ is a nondecreasing function, *i.e.*, if $a,\ b \in M$ such that $a \leqslant b$, then $\mu(a) \leqslant \mu(b)$.

(2.3) $\mu(a \cup b) = \mu(a) + \mu(b) - \mu(a \cap b)$ for every $a,\ b \in M$.

(2.4) Let $b \in M$. Then $\mu(a \cap b) = \mu(a)$ for all $a \in M$ if and only if $\mu(b) = 1$.

(2.5) If $a,\ b \in M$ are W-separated, then $\mu(a \cap b) = 0$.

(2.6) If $a,\ b \in M$ such that $a \leqslant b$, then $\mu(b) = \mu(a) + \mu(a^\perp \cap b)$.

The proofs of these properties can be found in [10]. The monotonicity of fuzzy measure μ implies that this measure transforms M into the interval $[0,\ 1]$.

The above described couple $(\Omega,\ M)$ is called in the terminology of Riečan and Dvurečenskij an F-quantum space, the fuzzy measure μ is so-called F-state [14,15]. This structure has been suggested (see [14]) as an alternative mathematical model of the quantum statistical theory for the case when quantum mechanical events are described vaguely. The theory of F-quantum spaces was developed in [16–19]. According to Tamir and Cohen [9], the logical entropy could be more intuitive and useful than the Shannon entropy and also von Neumann entropy when analyzing specific quantum problems. This fact inspired us to study of logical entropy of fuzzy partitions in a fuzzy probability space.

By a fuzzy partition (of a space (Ω, M, μ)) we will understand a finite collection $\xi = \{a_1, \ldots, a_n\}$ of members of M such that $\mu(\cup_{i=1}^{n} a_i) = 1$ and $a_i \leqslant a_j^{\perp}$ whenever $i \neq j$.

We define in the set of all fuzzy partitions of a fuzzy probability space (Ω, M, μ) the relation $\prec$ in the following way: Let ξ, η be two fuzzy partitions of a fuzzy probability space (Ω, M, μ). Then $\xi \prec \eta$ iff for every $b \in \eta$ there exists $a \in \xi$ such that $b \leqslant a$. In this case, we shall say that the partition η is a refinement of the partition ξ.

Given two fuzzy partitions $\xi = \{a_1, \ldots, a_n\}$ and $\eta = \{b_1, \ldots, b_m\}$ of a fuzzy probability space (Ω, M, μ), their join $\xi \vee \eta$ is defined as the system

$$\xi \vee \eta = \left\{a_i \cap b_j;\ a_i \in \xi,\ b_j \in \eta\right\}.$$

Since $\xi \prec \xi \vee \eta$ and $\eta \prec \xi \vee \eta$, $\xi \vee \eta$ is so called common refinement of ξ and η.

Let $\xi = \{a_1, \ldots, a_n\}$ and $\eta = \{b_1, \ldots, b_m\}$ be two fuzzy partitions of a fuzzy probability space (Ω, M, μ). Then ξ and η are called statistically independent, if $\mu(a_i \cap b_j) = \mu(a_i) \cdot \mu(b_j)$, for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

If $\xi_1, \xi_2, \ldots, \xi_n$ are fuzzy partitions of a fuzzy probability space (Ω, M, μ), then we put

$$\vee_{i=1}^{n} \xi_i = \xi_1 \vee \xi_2 \vee \ldots \vee \xi_n.$$

Remark 1. A classical probability space (Ω, S, P) can be regarded as a fuzzy probability space, if we put $M = \{\chi_A;\ A \in S\}$, where χ_A is the characteristic function of a set $A \in S$, and define the mapping $\mu : M \to [0, 1]$ by $\mu(\chi_A) = P(A)$. A usual measurable partition $\{A_1, \ldots, A_n\}$ of a space (Ω, S, P) (*i.e.*, any sequence $\{A_1, \ldots, A_n\} \subset S$ such that $\cup_{i=1}^{n} A_i = \Omega$ and $A_i \cap A_j = \varnothing$ $(i \neq j)$) can be regarded as a fuzzy partition of (Ω, M, μ), if we consider $a_i = \chi_{Ai}$ instead of A_i. Namely,

$$A_i \cap A_j = \varnothing\ (i \neq j) \text{ implies } \chi_{Ai} \leqslant 1 - \chi_{Aj}\ (i \neq j),$$

and

$$\mu(\cup_{i=1}^{n} \chi_{A_i}) = \mu(\chi_{\cup_{i=1}^{n} A_i}) = P(\cup_{i=1}^{n} A_i) = P(\Omega) = 1.$$

Let us mention that a fuzzy partition can serve as a mathematical model of the random experiment whose outcomes are vaguely defined events, *i.e.*, the fuzzy events. The Shannon entropy of fuzzy partitions of a fuzzy probability space (Ω, M, μ) has been defined and studied by Markechová in [5], see also [20]. It is noted that some other conceptions of fuzzy partitions and their entropy were introduced, for example in [21–26]. While our approach is based on Zadeh's connectives, in these papers other fuzzy set operations were used.

In Section 4, we deal with fuzzy dynamical systems. The notion of fuzzy dynamical system was introduced by Markechová in [6] as follows. By a fuzzy dynamical system (Definition 6) we understand a system (Ω, M, μ, τ), where (Ω, M, μ) is any fuzzy probability space and $\tau : M \to M$ is a μ-preserving σ-homomorphism. Fuzzy dynamical systems include the dynamical systems within the meaning of the classical Kolmogorov theory (Remark 5) while allowing studying more general situations, for example, Markov's operators. Recall that a classical dynamical system is a quadruple (Ω, S, P, T), where (Ω, S, P) is a probability space and $T : \Omega \to \Omega$ is a measure preserving map, *i.e.*, $T^{-1}(A) \in S$ and $P\left(T^{-1}(A)\right) = P(A)$, whenever $A \in S$. The notion of Shannon's entropy of fuzzy partitions of a fuzzy probability space was exploited to define the Kolmogorov-Sinai entropy of fuzzy dynamical systems [6,7]. Subsequently an ergodic theory for fuzzy dynamical systems was proposed (see [27]).

Note that other approaches to a fuzzy generalization of the notion of Kolmogorov-Sinai entropy of a dynamical system can be found in [28–34]. Let us mention that while the definition of fuzzy dynamical system in this paper is based on Zadeh's connectives, in our recently published paper [28] the Lukasiewicz connectives were used to define the fuzzy set operations.

3. Logical Entropy and Logical Mutual Information of Fuzzy Partitions

Every fuzzy partition $\xi = \{a_1, \ldots, a_n\}$ of (Ω, M, μ) represents within the meaning of the classical probability theory a random experiment with a finite number of outcomes a_i, $i = 1, 2, \ldots, n$ (which are fuzzy events) with a probability distribution $p_i = \mu(a_i)$, $i = 1, 2, \ldots, n$, since $p_i \geqslant 0$ for $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} \mu(a_i) = \mu(\cup_{i=1}^{n} a_i) = 1$. For that reason, we define the logical entropy of $\xi = \{a_1, \ldots, a_n\}$ as the number

$$H_L(\xi) = \sum\nolimits_{i=1}^{n} \mu(a_i)\,(1 - \mu(a_i)). \tag{1}$$

Since $\sum_{i=1}^{n} \mu(a_i) = 1$, we can write

$$H_L(\xi) = 1 - \sum\nolimits_{i=1}^{n} (\mu(a_i))^2. \tag{2}$$

Example 1. Let $\Omega = [0, 1]$, $a : \Omega \to \Omega$, $a(\omega) = \omega$, $\omega \in \Omega$, $M = \{a, a^{\perp}, a \cup a^{\perp}, a \cap a^{\perp}, 0_\Omega, 1_\Omega\}$. If we define the mapping $\mu : M \to [0, 1]$ by the equalities $\mu(1_\Omega) = \mu(a \cup a^{\perp}) = 1$, $\mu(0_\Omega) = \mu(a \cap a^{\perp}) = 0$ and $\mu(a) = \mu(a^{\perp}) = 1/2$, then the triplet (Ω, M, μ) is a fuzzy probability space. The systems $\xi_1 = \{a, a^{\perp}\}$, $\xi_2 = \{a \cup a^{\perp}\}$, $\xi_3 = \{1_\Omega\}$ are fuzzy partitions of (Ω, M, μ) such that $\xi_3 \prec \xi_2 \prec \xi_1$. By simple calculation we get their logical entropy: $H_L(\xi_1) = 1/2$, $H_L(\xi_2) = H_L(\xi_3) = 0$. In accordance with the natural requirement, each experiment whose outcome is a certain event has zero entropy.

Some basic properties of logical entropy of fuzzy partitions are presented in the following theorems.

Theorem 1. The logical entropy H_L has the following properties:

(i) $H_L(\xi) \geqslant 0$ for every fuzzy partition ξ of a fuzzy probability space (Ω, M, μ);
(ii) if ξ, η are two fuzzy partitions of a fuzzy probability space (Ω, M, μ) such that $\xi \prec \eta$, then $H_L(\xi) \leqslant H_L(\eta)$;
(iii) $H_L(\xi) \leqslant H_L(\xi \vee \eta)$ for every fuzzy partitions ξ, η of a fuzzy probability space (Ω, M, μ).

Proof. The property (*i*) follows immediately from Equation (1).

(*ii*) Let $\xi = \{a_1, \ldots, a_n\}$, $\eta = \{b_1, \ldots, b_m\}$, $\xi \prec \eta$. Then for every $b_j \in \eta$ there exists $a_{i_0} \in \xi$ such that $b_j \leqslant a_{i_0}$. Since ξ is a system of pair wise W-separated fuzzy sets, for every $i \neq i_0$, it holds $b_j = b_j \cap a_{i_0} \leqslant a_{i_0} \leqslant a_i^{\perp}$. Hence, by the property (2.5) of fuzzy measure μ, we get

$$\mu(b_j \cap a_i) = \begin{cases} \mu(b_j), & \text{if } i = i_0; \\ 0, & \text{if } i \neq i_0. \end{cases}$$

Using this equality and the property (2.4) of fuzzy measure μ we obtain

$$\begin{aligned} \mu(b_j) \cdot (1 - \mu(b_j)) &= \sum_{i=1}^{n} \mu(a_i \cap b_j) \cdot (1 - \mu(a_i \cap b_j)) = \sum_{i=1}^{n} \mu(a_i \cap b_j) - \sum_{i=1}^{n} \left(\mu(a_i \cap b_j)\right)^2 \\ &= \mu\left(\cup_{i=1}^{n} (a_i \cap b_j)\right) - \sum_{i=1}^{n} \left(\mu(a_i \cap b_j)\right)^2 = \mu\left((\cup_{i=1}^{n} a_i) \cap b_j\right) - \sum_{i=1}^{n} \left(\mu(a_i \cap b_j)\right)^2 \\ &= \mu(b_j) - \sum_{i=1}^{n} \left(\mu(a_i \cap b_j)\right)^2. \end{aligned}$$

Therefore

$$H_L(\eta) = \sum_{j=1}^{m} \mu(b_j) \cdot (1 - \mu(b_j)) = \sum_{j=1}^{m} \mu(b_j) - \sum_{j=1}^{m} \sum_{i=1}^{n} \left(\mu(a_i \cap b_j)\right)^2 = 1 - \sum_{j=1}^{m} \sum_{i=1}^{n} \left(\mu(a_i \cap b_j)\right)^2.$$

Since

$$\sum_{j=1}^{m}\left(\mu(a_i \cap b_j)\right)^2 \leqslant \sum_{j=1}^{m}\mu(a_i \cap b_j)\sum_{j=1}^{m}\mu(a_i \cap b_j) = (\mu(a_i))^2\,,\ i = 1, 2, \ldots, n,$$

we obtain

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\mu(a_i \cap b_j)\right)^2 \leqslant \sum_{i=1}^{n}(\mu(a_i))^2.$$

This inequality implies

$$1-\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\mu(a_i \cap b_j)\right)^2 \geqslant 1-\sum_{i=1}^{n}(\mu(a_i))^2,$$

what means that

$$H_L(\eta) \geqslant H_L(\xi).$$

Since $\xi \prec \xi \vee \eta$, the inequality (*iii*) is a simple consequence of (*ii*). □

As a simple consequence of the previous theorem we obtain the following property of the logical entropy of fuzzy partitions.

Corollary 1. For any fuzzy partitions $\xi,\ \eta$ of a fuzzy probability space $(\Omega,\ M, \mu)$, it holds

$$H_L(\xi \vee \eta) \geqslant \max(H_L(\xi); H_L(\eta)).$$

Definition 2. If $\xi,\ \eta$ are two fuzzy partitions of a fuzzy probability space $(\Omega,\ M, \mu)$, then the conditional logical entropy of ξ given η is defined by the formula

$$H_L(\xi/\eta) = H_L(\xi \vee \eta) - H_L(\eta). \tag{3}$$

Remark 2. Evidently $H_L(\xi/\xi) = 0$ and from Theorem 1 it follows $H_L(\xi/\eta) \geqslant 0$.

Proposition 1. For every fuzzy partitions $\xi = \{a_1, \ldots, a_n\}$, $\eta = \{b_1, \ldots, b_m\}$ of a fuzzy probability space $(\Omega,\ M, \mu)$, it holds

$$H_L(\xi/\eta) = \sum_{j=1}^{m}\left(\mu(b_j)\right)^2 - \sum_{i=1}^{n}\sum_{j=1}^{m}\left(\mu(a_i \cap b_j)\right)^2. \tag{4}$$

Proof. By Equations (2) and (3) we get

$$\begin{aligned} H_L(\xi/\eta) = H_L(\xi \vee \eta) - H_L(\eta) &= 1-\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\mu(a_i \cap b_j)\right)^2 - 1 + \sum_{j=1}^{m}\left(\mu(b_j)\right)^2 \\ &= \sum_{j=1}^{m}\left(\mu(b_j)\right)^2 - \sum_{i=1}^{n}\sum_{j=1}^{m}\left(\mu(a_i \cap b_j)\right)^2. \quad \square \end{aligned}$$

Theorem 2. Let $\xi,\ \eta$ be two fuzzy partitions of a fuzzy probability space $(\Omega,\ M, \mu)$. Then

(*i*) $H_L(\xi/\eta) \leqslant H_L(\xi)$;
(*ii*) $H_L(\xi \vee \eta) \leqslant H_L(\xi) + H_L(\eta)$.

Proof. Let $\xi = \{a_1, \ldots, a_n\}$ and $\eta = \{b_1, \ldots, b_m\}$. Since for each $a_i \in \xi$, $i = 1, 2, \ldots, n$, we have

$$\begin{aligned} \textstyle\sum_{j=1}^{m}\mu(a_i \cap b_j)\left(\mu(b_j) - \mu(a_i \cap b_j)\right) &\leqslant \textstyle\sum_{j=1}^{m}\mu(a_i \cap b_j) \cdot \sum_{j=1}^{m}\left(\mu(b_j) - \mu(a_i \cap b_j)\right) \\ &= \mu(a_i) \cdot \left(\textstyle\sum_{j=1}^{m}\mu(b_j) - \sum_{j=1}^{m}\mu(a_i \cap b_j)\right) = \mu(a_i) \cdot (1-\mu(a_i)), \end{aligned}$$

it holds

$$H_L(\xi/\eta) = \sum_{i=1}^{n}\sum_{j=1}^{m}\mu(a_i \cap b_j)\cdot(\mu(b_j) - \mu(a_i \cap b_j)) \leqslant \sum\nolimits_{i=1}^{n}\mu(a_i)\,(1 - \mu(a_i)) = H_L(\xi).$$

This along with Equation (3) implies

$$H_L(\xi \vee \eta) = H_L(\eta) + H_L(\xi/\eta) \leqslant H_L(\eta) + H_L(\xi).$$

The proof is complete. □

Theorem 3. Let $\xi,\ \eta,\ \varsigma$ be fuzzy partitions of a fuzzy probability space $(\Omega,\ M, \mu)$. Then

$$H_L(\xi \vee \eta/\varsigma) = H_L(\xi/\varsigma) + H_L(\eta/\varsigma \vee \xi).$$

Proof. Let $\xi = \{a_1, \ldots, a_n\}$, $\eta = \{b_1, \ldots, b_m\}$, $\varsigma = \{c_1, \ldots, c_p\}$. Then by Equation (4) we get

$$\begin{aligned} &H_L(\xi/\varsigma) + H_L(\eta/\varsigma \vee \xi) \\ &= \sum_{k=1}^{p}(\mu(c_k))^2 - \sum_{i=1}^{n}\sum_{k=1}^{p}(\mu(a_i \cap c_k))^2 + \sum_{k=1}^{p}\sum_{i=1}^{n}(\mu(c_k \cap a_i))^2 - \sum_{j=1}^{m}\sum_{k=1}^{p}\sum_{i=1}^{n}\left(\mu(b_j \cap c_k \cap a_i)\right)^2 \\ &= \sum_{k=1}^{p}(\mu(c_k))^2 - \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{p}\left(\mu(a_i \cap b_j \cap c_k)\right)^2 = H_L(\xi \vee \eta/\varsigma). \quad \square \end{aligned}$$

Theorem 4. (Chain rules for logical entropy). Let $\xi_1, \xi_2, \ldots, \xi_n$ and η be fuzzy partitions of a fuzzy probability space $(\Omega,\ M, \mu)$. If we put $\xi_0 = \{1_\Omega\}$, then, for $n = 1, 2, \ldots$, the following equalities hold:

(i) $H_L(\xi_1 \vee \xi_2 \vee \ldots \vee \xi_n) = \sum_{i=1}^{n} H_L(\xi_i/ \vee_{k=0}^{i-1}\xi_k)$;

(ii) $H_L(\vee_{i=1}^{n}\xi_i/\eta) = \sum_{i=1}^{n} H_L(\xi_i/ (\vee_{k=0}^{i-1}\xi_k) \vee \eta)$.

Proof. Evidently, for any fuzzy partition ξ, we have $\xi_0 \vee \xi = \xi$, and $H_L(\xi/\xi_0) = H_L(\xi)$.

(*i*) By Equation (3) we have

$$H_L(\xi_1 \vee \xi_2) = H_L(\xi_1) + H_L(\xi_2/\xi_1).$$

For $n = 3$, using the previous equality and Theorem 3, we get

$$\begin{aligned} H_L(\xi_1 \vee \xi_2 \vee \xi_3) &= H_L(\xi_1) + H_L(\xi_2 \vee \xi_3/\xi_1) \\ &= H_L(\xi_1) + H_L(\xi_2/\xi_1) + H_L(\xi_3/\xi_2 \vee \xi_1). \end{aligned}$$

Now let us suppose that the result is true for a given $n \in N$. Then

$$\begin{aligned} &H_L(\xi_1 \vee \xi_2 \vee \ldots \vee \xi_n \vee \xi_{n+1}) \\ &= H_L(\xi_1 \vee \xi_2 \vee \ldots \vee \xi_n) + H_L(\xi_{n+1}/\xi_1 \vee \xi_2 \vee \ldots \vee \xi_n) \\ &= \sum\nolimits_{i=1}^{n} H_L(\xi_i/ \vee_{k=0}^{i-1}\xi_k) + H_L(\xi_{n+1}/\xi_1 \vee \xi_2 \vee \ldots \vee \xi_n) \\ &= \sum\nolimits_{i=1}^{n+1} H_L(\xi_i/ \vee_{k=0}^{i-1}\xi_k). \end{aligned}$$

(*ii*) For $n = 2$, using Theorem 3, we obtain

$$H_L(\xi_1 \vee \xi_2/\eta) = H_L(\xi_1/\eta) + H_L(\xi_2/\xi_1 \vee \eta) = \sum\nolimits_{i=1}^{2} H_L(\xi_i/(\vee_{k=0}^{i-1}\xi_k) \vee \eta).$$

Suppose that the result is true for a given $n \in N$. Then

$$\begin{aligned} H_L(\xi_1 \vee \xi_2 \vee \ldots \vee \xi_n \vee \xi_{n+1}/\eta) &= H_L(\vee_{i=1}^{n}\xi_i/\eta) + H_L(\xi_{n+1}/\xi_1 \vee \ldots \vee \xi_n \vee \eta) \\ &= \sum_{i=1}^{n} H_L(\xi_i/(\vee_{k=0}^{i-1}\xi_k) \vee \eta) + H_L(\xi_{n+1}/(\vee_{k=0}^{i-1}\xi_k) \vee \eta) \\ &= \sum_{i=1}^{n+1} H_L(\xi_i/(\vee_{k=0}^{i-1}\xi_k) \vee \eta). \quad \square \end{aligned}$$

Definition 3. If ξ, η are two fuzzy partitions of a fuzzy probability space (Ω, M, μ), then the logical mutual information of ξ and η is defined by the formula

$$I_L(\xi, \eta) = H_L(\xi) - H_L(\xi/\eta). \tag{5}$$

Remark 3. As a simple consequence of Equation (3) we have:

$$I_L(\xi, \eta) = H_L(\xi) + H_L(\eta) - H_L(\xi \vee \eta), \tag{6}$$

and subsequently we see that

$$I_L(\xi, \eta) = I_L(\eta, \xi) \text{ and } I_L(\xi, \xi) = H_L(\xi).$$

Corollary 2. For fuzzy partitions ξ, η of a fuzzy probability space (Ω, M, μ), it holds

$$0 \leqslant I_L(\xi, \eta) \leqslant \min(H_L(\xi);\, H_L(\eta)).$$

Proof. The result follows immediately from Equation (6) and the property (*iii*) of Theorem 1. □

Definition 4. Let ξ, η, ς be fuzzy partitions of a fuzzy probability space (Ω, M, μ). Then the logical conditional mutual information of ξ and η given ς is defined by the formula

$$I_L(\xi, \eta/\varsigma) = H_L(\xi/\varsigma) - H_L(\xi/\eta \vee \varsigma). \tag{7}$$

Theorem 5 (Chain rules for logical mutual information). Let $\xi_1, \xi_2, \ldots, \xi_n$ and η be fuzzy partitions of a fuzzy probability space (Ω, M, μ). If we put $\xi_0 = \{1_\Omega\}$, then, for $n = 1, 2, \ldots$, it holds

$$I_L(\vee_{i=1}^{n}\xi_i, \eta) = \sum_{i=1}^{n} I_L(\xi_i, \eta/\vee_{k=0}^{i-1}\xi_k).$$

Proof. By Equation (5), Theorem 4, and Equation (7), we obtain

$$\begin{aligned} I_L(\vee_{i=1}^{n}\xi_i, \eta) &= H_L(\vee_{i=1}^{n}\xi_i) - H_L(\vee_{i=1}^{n}\xi_i/\eta) \\ &= \sum_{i=1}^{n} H_L(\xi_i/\vee_{k=0}^{i-1}\xi_k) - \sum_{i=1}^{n} H_L(\xi_i/(\vee_{k=0}^{i-1}\xi_k) \vee \eta) \\ &= \sum_{i=1}^{n} \left(H_L(\xi_i/\vee_{k=0}^{i-1}\xi_k) - H_L(\xi_i/(\vee_{k=0}^{i-1}\xi_k) \vee \eta)\right) \\ & \sum_{i=1}^{n} I_L(\xi_i, \eta/\vee_{k=0}^{i-1}\xi_k). \quad \square \end{aligned}$$

Theorem 6. If fuzzy partitions ξ, η of a fuzzy probability space (Ω, M, μ) are statistically independent, then

$$I_L(\xi, \eta) = H_L(\xi) \cdot H_L(\eta).$$

Proof. Let $\xi = \{a_1, \ldots, a_n\}$, $\eta = \{b_1, \ldots, b_m\}$ be statistically independent fuzzy partitions of a fuzzy probability space $(\Omega,\ M, \mu)$. Then $\mu(a_i \cap b_j) = \mu(a_i) \cdot \mu(b_j)$, for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. By simple calculation we obtain:

$$I_L(\xi, \eta) = H_L(\xi) + H_L(\eta) - H_L(\xi \vee \eta)$$
$$= 1 - \sum_{i=1}^{n} (\mu(a_i))^2 + 1 - \sum_{j=1}^{m} \left(\mu(b_j)\right)^2 - 1 + \sum_{i=1}^{n}\sum_{j=1}^{m} \left(\mu(a_i \cap b_j)\right)^2$$
$$= 1 - \sum_{i=1}^{n} (\mu(a_i))^2 - \sum_{j=1}^{m} \left(\mu(b_j)\right)^2 + \sum_{i=1}^{n} (\mu(a_i))^2 \sum_{j=1}^{m} \left(\mu(b_j)\right)^2$$
$$= \left(1 - \sum_{i=1}^{n} (\mu(a_i))^2\right) \cdot \left(1 - \sum_{j=1}^{m} \left(\mu(b_j)\right)^2\right) = H_L(\xi) \cdot H_L(\eta). \quad \square$$

Corollary 3. If fuzzy partitions $\xi,\ \eta$ of a fuzzy probability space $(\Omega,\ M, \mu)$ are statistically independent, then

$$1 - H_L(\xi \vee \eta) = (1 - H_L(\xi)) \cdot (1 - H_L(\eta)).$$

Proof. Calculate:

$$(1 - H_L(\xi)) \cdot (1 - H_L(\eta)) = 1 - H_L(\xi) - H_L(\eta) + H_L(\xi) \cdot H_L(\eta)$$
$$= 1 - H_L(\xi) - H_L(\eta) + I_L(\xi, \eta)$$
$$= 1 - H_L(\xi \vee \eta). \quad \square$$

Definition 5. Let $\xi,\ \eta,\ \varsigma$ be fuzzy partitions of a fuzzy probability space $(\Omega,\ M, \mu)$. We say that ξ is conditionally independent to ς given η (and write $\xi \to \eta \to \varsigma$) if $I_L(\xi, \varsigma/\eta) = 0$.

Theorem 7. For fuzzy partitions $\xi,\ \eta,\ \varsigma$ of a fuzzy probability space $(\Omega,\ M, \mu)$, it holds $\xi \to \eta \to \varsigma$ if and only if $\varsigma \to \eta \to \xi$.

Proof. Let $\xi \to \eta \to \varsigma$. Then $0 = I_L(\xi, \varsigma/\eta) = H_L(\xi/\eta) - H_L(\xi/\eta \vee \varsigma)$. Therefore by Equation (3) we get:

$$H_L(\xi/\eta) = H_L(\xi/\eta \vee \varsigma) = H_L(\xi \vee \eta \vee \varsigma) - H_L(\eta \vee \varsigma).$$

Calculate:

$$I_L(\varsigma, \xi/\eta) = H_L(\varsigma/\eta) - H_L(\varsigma/\xi \vee \eta) = H_L(\varsigma \vee \eta) - H_L(\eta) - H_L(\xi \vee \eta \vee \varsigma) + H_L(\xi \vee \eta)$$
$$= H_L(\xi \vee \eta) - H_L(\eta) - H_L(\xi/\eta) = 0. \quad \square$$

Remark 4. According to Theorem 7, we may say that ξ and ς are conditionally independent given η and write $\xi \leftrightarrow \eta \leftrightarrow \varsigma$ instead of $\xi \to \eta \to \varsigma$.

Theorem 8. For fuzzy partitions $\xi,\ \eta,\ \varsigma$ of a fuzzy probability space $(\Omega,\ M, \mu)$, it holds

$$I_L(\xi, \eta \vee \varsigma) = I_L(\xi, \eta) + I_L(\xi, \varsigma/\eta) = I_L(\xi, \varsigma) + I_L(\xi, \eta/\varsigma).$$

Proof. Calculate:

$$I_L(\xi, \eta) + I_L(\xi, \varsigma/\eta) = H_L(\xi) - H_L(\xi/\eta) + H_L(\xi/\eta) - H_L(\xi/\eta \vee \varsigma)$$
$$= H_L(\xi) - H_L(\xi/\eta \vee \varsigma) = I_L(\xi, \eta \vee \varsigma).$$

The second equality is obtained in the same way. $\square$

Theorem 9. For fuzzy partitions $\xi,\ \eta,\ \varsigma$ of a fuzzy probability space $(\Omega,\ M, \mu)$ such that $\xi \to \eta \to \varsigma$, we have

(i) $\quad I_L(\xi \vee \eta, \varsigma) = I_L(\eta, \varsigma)$;

(ii) $I_L(\eta,\varsigma) = I_L(\xi,\varsigma) + I_L(\varsigma,\eta/\xi)$;
(iii) $I_L(\xi,\eta/\varsigma) \leqslant I_L(\xi,\eta)$.

Proof. (*i*) Since by the assumption $I_L(\xi,\varsigma/\eta) = 0$, using the chain rule for logical mutual information, we obtain

$$I_L(\xi \vee \eta,\varsigma) = I_L(\eta \vee \xi,\varsigma) = I_L(\eta,\varsigma) + I_L(\xi,\varsigma/\eta) = I_L(\eta,\varsigma).$$

(ii) By Theorem 8, we have $I_L(\xi \vee \eta,\varsigma) = I_L(\varsigma,\xi) + I_L(\varsigma,\eta/\xi)$. Hence using *(i)*, we can write

$$I_L(\eta,\varsigma) = I_L(\xi \vee \eta,\varsigma) = I_L(\varsigma,\xi) + I_L(\varsigma,\eta/\xi).$$

(*iii*) From (*ii*) it follows the inequality $I_L(\varsigma,\eta/\xi) \leqslant I_L(\varsigma,\eta)$. By Theorem 7 we can interchange ξ and ς. Doing so we obtain $I_L(\xi,\eta/\varsigma) \leqslant I_L(\xi,\eta)$. □

4. Logical Entropy of Fuzzy Dynamical Systems

In this section, we extend the definition of logical entropy of fuzzy partitions to fuzzy dynamical systems.

Definition 6 [6]. By a fuzzy dynamical system we mean a quadruple $(\Omega,\ M,\mu,\tau)$, where $(\Omega,\ M,\mu)$ is a fuzzy probability space and $\tau : M \to M$ is a $\mu-$ preserving $\sigma-$ homomorphism, *i.e.*, $\tau(a^{\perp}) = (\tau(a))^{\perp}$, $\tau(\cup_{n=1}^{\infty} a_n) = \cup_{n=1}^{\infty}\tau(a_n)$ and $\mu(\tau(a)) = \mu(a)$, for every $a \in M$ and any sequence $\{a_n\}_{n=1}^{\infty} \subset M$.

Let any fuzzy dynamical system $(\Omega,\ M,\ \mu,\ \tau)$ be given. Denote $\tau^2 = \tau \circ \tau$ and put $\tau^n = \tau \circ \tau^{n-1}$, $n = 1,2,\ldots$, where τ^0 is an identical mapping on M. Define $\tau^n\xi = \{\tau^n(a);\ a \in \xi\}$ for every fuzzy partition ξ of $(\Omega,\ M,\mu)$. Evidently $\tau^n\xi$ is a fuzzy partition of $(\Omega,\ M,\mu)$.

Remark 5. A classical dynamical system (Ω,S,P,T) can be regarded as a fuzzy dynamical system $(\Omega,\ M,\mu,\tau)$, if we consider a fuzzy probability space $(\Omega,\ M,\mu)$ from Remark 1 and define the mapping $\tau : M \to M$ by $\tau(\chi_A) = \chi_A \circ T = \chi_{T^{-1}(A)}, \chi_A \in M$.

Example 2. Let any fuzzy probability space $(\Omega,\ M,\mu)$ be given. Let $T : \Omega \to \Omega$ be a measure μ preserving transformation, *i.e.*, $a \in M$ implies $a \circ T \in M$ and $\mu(a \circ T) = \mu(a)$. Define the mapping $\tau : M \to M$ by the formula $\tau(a) = a \circ T$ for all $a \in M$. Then it is easy to verify that τ is a $\sigma-$ homomorphism. Moreover, $\mu(\tau(a)) = \mu(a \circ T) = \mu(a)$ for all $a \in M$. Hence τ is a $\mu-$ preserving map and the system $(\Omega,\ M,\ \mu,\ \tau)$ is a fuzzy dynamical system.

Theorem 10. Let $\xi,\ \eta$ be fuzzy partitions of a fuzzy probability space $(\Omega,\ M,\mu)$. Then, for $n = 1,2,\ldots$, the following equalities hold:

(i) $H_L(\tau^n\xi) = H_L(\xi)$;
(ii) $H_L(\tau^n\xi/\tau^n\eta) = H_L(\xi/\eta)$;
(iii) $H_L(\vee_{i=0}^{n-1}\tau^i\xi) = H_L(\xi) + \sum_{j=1}^{n-1} H_L(\xi/\vee_{i=1}^{j}\tau^i\xi)$.

Proof. Since the mapping $\tau : M \to M$ is $\mu-$ invariant, for every $a \in M$, we have $\mu(\tau^n(a)) = \mu(a)$. This fact immediately implies the equalities (*i*) and (*ii*).

We prove the assertion (*iii*) by mathematical induction. The statement is true for $n = 2$ according to Equation (3). Assume that the assertion holds for a given $n \in N$. Since by the part (*i*) of this theorem we have

$$H_L(\vee_{i=1}^{n}\tau^i\xi) = H_L(\tau(\vee_{i=0}^{n-1}\tau^i\xi)) = H_L(\vee_{i=0}^{n-1}\tau^i\xi),$$

by means of Equation (3) and the induction assumption we obtain

$$H_L(\vee_{i=0}^{n}\tau^i\xi) = H_L((\vee_{i=1}^{n}\tau^i\xi)\vee\xi) = H_L(\vee_{i=1}^{n}\tau^i\xi) + H_L(\xi/\vee_{i=1}^{n}\tau^i\xi)$$
$$= H_L(\vee_{i=0}^{n-1}\tau^i\xi) + H_L(\xi/\vee_{i=1}^{n}\tau^i\xi) = H_L(\xi) + \sum_{j=1}^{n-1} H_L(\xi/\vee_{i=1}^{j}\tau^i\xi) + H_L(\xi/\vee_{i=1}^{n}\tau^i\xi)$$
$$= H_L(\xi) + \sum_{j=1}^{n} H_L(\xi/\vee_{i=1}^{j}\tau^i\xi).$$

The proof is finished. □

In the following we define the logical entropy of fuzzy dynamical systems. The possibility of this definition is based on Proposition 2. To its proof we need the assertion of the following lemma.

Lemma 1 ([35], Theorem 4.9). Let $\{a_n\}_{n=1}^{\infty}$ be a subadditive sequence of nonnegative real numbers, *i.e.*, $a_n \geqslant 0$ and $a_{n+m} \leqslant a_n + a_m$ for every $n, m \in N$. Then $\lim_{n\to\infty}\frac{1}{n}a_n$ exists.

Proposition 2. For any fuzzy partition ξ of (Ω, M, μ), $\lim_{n\to\infty}\frac{1}{n}H_L(\vee_{i=0}^{n-1}\tau^i\xi)$ exists.

Proof. Put

$$a_n = H_L(\vee_{i=0}^{n-1}\tau^i\xi).$$

By the property (*i*) of Theorem 1, $a_n \geqslant 0$ for every $n \in N$. According to subadditivity of logical entropy (the property (*ii*) of Theorem 2) and the property (*iii*) from the previous theorem, for any $n, m \in N$, we obtain

$$a_{n+m} = H_L(\vee_{i=0}^{n+m-1}\tau^i\xi) \leqslant H_L(\vee_{i=0}^{n-1}\tau^i\xi) + H_L(\vee_{i=n}^{n+m-1}\tau^i\xi)$$
$$= a_n + H_L(\tau^n(\vee_{i=0}^{m-1}\tau^i\xi))$$
$$= a_n + H_L(\vee_{i=0}^{m-1}\tau^i\xi) = a_n + a_m.$$

This means that $\{a_n\}_{n=1}^{\infty}$ is a subadditive sequence of nonnegative real numbers, and therefore by Lemma 1, $\lim_{n\to\infty}\frac{1}{n}a_n$ exists. □

Definition 7. Let (Ω, M, μ, τ) be a fuzzy dynamical system, ξ be a fuzzy partition of (Ω, M, μ). Then we define

$$h_L(\tau,\xi) = \lim_{n\to\infty}\frac{1}{n}H_L(\vee_{i=0}^{n-1}\tau^i\xi).$$

The logical entropy of a fuzzy dynamical system (Ω, M, μ, τ) is defined by the formula

$$h_L(\tau) = \sup\{h_L(\tau,\xi)\},$$

where the supremum is taken over all fuzzy partitions ξ of (Ω, M, μ).

Remark 6. The trivial case of a fuzzy dynamical system is a quadruple (Ω, M, μ, I), where (Ω, M, μ) is any fuzzy probability space and $I: M \to M$ is an identity mapping. Since the operation $\vee$ is idempotent, for every fuzzy partition ξ of (Ω, M, μ) it holds

$$h_L(I,\xi) = \lim_{n\to\infty}\frac{1}{n}H_L(\vee_{i=0}^{n-1}I^i\xi) = \lim_{n\to\infty}\frac{1}{n}H_L(\xi) = 0.$$

The logical entropy of the fuzzy dynamical system (Ω, M, μ, I) is $h_L(I) = \sup\{h_L(I,\xi);\ \xi$ is a fuzzy partition of $(\Omega, M, \mu)\} = 0$.

Example 3. Consider the fuzzy probability space (Ω, M, μ) from Example 1. If we define a mapping $\tau: M \to M$ by the equalities $\tau(a \cup a^{\perp}) = a \cup a^{\perp}$, $\tau(1_{\Omega}) = 1_{\Omega}$, $\tau(0_{\Omega}) = 0_{\Omega}$, $\tau(a \cap a^{\perp}) = a \cap a^{\perp}$, $\tau(a) = a^{\perp}$, $\tau(a^{\perp}) = a$, then (Ω, M, μ, τ) is a fuzzy dynamical system. The systems

$\xi_1 = \{a, a^{\perp}\}$, $\xi_2 = \{a \cup a^{\perp}\}$, $\xi_3 = \{1_\Omega\}$ are fuzzy partitions of (Ω, M, μ) with $H_L(\xi_1) = 1/2$, $H_L(\xi_2) = H_L(\xi_3) = 0$. Calculate:

$$h_L(\tau, \xi_1) = \lim_{n\to\infty} \frac{1}{n} H_L(\vee_{i=0}^{n-1} \tau^i \xi_1) = \lim_{n\to\infty} \frac{1}{n} H_L(\xi_1) = 0.$$

Since $h_L(\tau, \xi_2) = h_L(\tau, \xi_3) = 0$, the logical entropy of (Ω, M, μ, τ) is the number

$$h_L(\tau) = \sup\{h_L(\tau, \xi_i);\ i = 1, 2, 3\} = 0.$$

Theorem 11. For every fuzzy partition ξ of a fuzzy probability space (Ω, M, μ) it holds

$$h_L(\tau, \xi) = h_L(\tau, \vee_{i=0}^{k} \tau^i \xi).$$

Proof. Let ξ be any fuzzy partition of a fuzzy probability space (Ω, M, μ). We get

$$\begin{aligned} h_L(\tau, \vee_{i=0}^{k} \tau^i \xi) &= \lim_{n\to\infty} \frac{1}{n} H_L(\vee_{j=0}^{n-1} \tau^j (\vee_{i=0}^{k} \tau^i \xi)) \\ &= \lim_{n\to\infty} \frac{k+n}{n} \cdot \frac{1}{k+n} H_L(\vee_{s=0}^{k+n-1} \tau^s \xi) \\ &= \lim_{n\to\infty} \frac{1}{k+n} H_L(\vee_{s=0}^{k+n-1} \tau^s \xi) = h_L(\tau, \xi). \quad \square \end{aligned}$$

The notion of isomorphism of fuzzy dynamical systems was defined in [7] as follows:

Definition 8. We say that two fuzzy dynamical systems $(\Omega_1, M_1, \mu_1, \tau_1)$, $(\Omega_2, M_2, \mu_2, \tau_2)$ are isomorphic if there exists a bijective mapping $f : M_1 \to M_2$ satisfying the following conditions:

(i) f preserves the operations, *i.e.*, $f(\cup_{n=1}^{\infty} a_n) = \cup_{n=1}^{\infty} f(a_n)$, $f(a^{\perp}) = 1_{\Omega_2} - f(a)$, for any sequence $\{a_n\}_{n=1}^{\infty} \subset M_1$ and for every $a \in M_1$.

(ii) The diagram f $\begin{array}{ccc} M_1 & \xrightarrow{\tau_1} & M_1 \\ \downarrow & & \downarrow f \\ M_2 & \xrightarrow[\tau_2]{} & M_2 \end{array}$ is commutative, *i.e.*, $f(\tau_1(a)) = \tau_2(f(a))$, for every $a \in M_1$.

(iii) $\mu_1(a) = \mu_2(f(a))$ for every $a \in M_1$.

Remark 7. It is easy to see that, for every $b_1, b_2 \in M_2$, $f^{-1}(b_1 \cap b_2) = f^{-1}(b_1) \cap f^{-1}(b_2)$. Namely, because f is bijective, for every $b_1, b_2 \in M_2$, there exist $a_1, a_2 \in M_1$ such that $f^{-1}(b_1) = a_1$, $f^{-1}(b_2) = a_2$, and we have

$$f^{-1}(b_1 \cap b_2) = f^{-1}(f(a_1) \cap f(a_2)) = f^{-1}(f(a_1 \cap a_2)) = a_1 \cap a_2 = f^{-1}(b_1) \cap f^{-1}(b_2).$$

In an analogous way, we get that for every $b_1, b_2 \in M_2$, $f^{-1}(b_1 \cup b_2) = f^{-1}(b_1) \cup f^{-1}(b_2)$ and for every $b \in M_2$

$$\left(f^{-1}(b)\right)^{\perp} = f^{-1}(b^{\perp}) \text{ and } \mu_2(b) = \mu_1(f^{-1}(b)).$$

In the following theorem we prove that the logical entropy of fuzzy dynamical systems is invariant under isomorphism.

Theorem 12. If fuzzy dynamical systems $(\Omega_1, M_1, \mu_1, \tau_1)$, $(\Omega_2, M_2, \mu_2, \tau_2)$ are isomorphic, then

$$h_L(\tau_1) = h_L(\tau_2).$$

Proof. Let a mapping $f : M_1 \to M_2$ represents an isomorphism of systems $(\Omega_1, M_1, \mu_1, \tau_1)$, $(\Omega_2, M_2, \mu_2, \tau_2)$. Let $\xi = \{a_1, \ldots, a_n\}$ be a fuzzy partition of a fuzzy probability space (Ω_1, M_1, μ_1). Put

$$f(\xi) = \{f(a_1), f(a_2), \ldots, f(a_n)\}.$$

Since

$$\mu_2(\cup_{i=1}^n f(a_i)) = \mu_2(f(\cup_{i=1}^n a_i)) = \mu_1(\cup_{i=1}^n a_i) = 1$$

and

$$f(a_i) \cap (f(a_j))^{\perp} = f(a_i) \cap f(a_j^{\perp}) = f(a_i \cap a_j^{\perp}) = f(a_i), \text{ whenever } i \neq j,$$

the system $f\,(\xi)$ is a fuzzy partition of a fuzzy probability space $(\Omega_2,\ M_2,\ \mu_2)$. Moreover,

$$H_L(f\,(\xi)) = 1 - \sum_{i=1}^n (\mu_2(f(a_i)))^2 \ = 1 - \sum_{i=1}^n (\mu_1(a_i))^2 \ = H_L(\xi)$$

and

$$\begin{aligned} h_L(\tau_2, f\,(\xi)) &= \lim_{n\to\infty}\frac{1}{n}H_L(\vee_{i=0}^{n-1}\tau_2^i f(\xi)) \\ = \lim_{n\to\infty}\frac{1}{n}H_L(\vee_{i=0}^{n-1} f(\tau_1^i\xi)) &= \lim_{n\to\infty}\frac{1}{n}H_L(f(\vee_{i=0}^{n-1}\tau_1^i\xi)) \\ &= \lim_{n\to\infty}\frac{1}{n}H_L(\vee_{i=0}^{n-1}\tau_1^i\xi) = h_L(\tau_1,\xi). \end{aligned}$$

Therefore
$\{h_L(\tau_1,\xi);\ \xi \text{ is a fuzzy partition of } (\Omega_1,\ M_1,\ \mu_1)\} \subset \{h_L(\tau_2,\eta);\ \eta \text{ is a fuzzy partition of } (\Omega_2,\ M_2,\ \mu_2)\}$ and consequently

$$h_L(\tau_1) = \sup\{h_L(\tau_1,\xi)\} \leqslant \sup\{h_L(\tau_2,\eta)\} = h_L(\tau_2),$$

where the supremum on the left side of the inequality is taken over all fuzzy partitions ξ of $(\Omega_1,\ M_1,\ \mu_1)$ and the supremum on the right side of the inequality is taken over all fuzzy partitions η of $(\Omega_2,\ M_2,\ \mu_2)$.

Let us prove the opposite inequality. Let $\eta = \{b_1, \ldots, b_m\}$ be a fuzzy partition of a fuzzy probability space $(\Omega_2,\ M_2,\ \mu_2)$. Then the system $f^{-1}(\eta) = \left\{f^{-1}(b_1), \ldots, f^{-1}(b_m)\right\}$ is a fuzzy partition of a fuzzy probability space $(\Omega_1,\ M_1,\ \mu_1)$. Indeed, according to the previous remark we have

$$\mu_1(\cup_{i=1}^m f^{-1}(b_i)) = \mu_1(f^{-1}(\cup_{i=1}^m b_i)) = \mu_2(\cup_{i=1}^m b_i) = 1$$

and

$$f^{-1}(b_i) \cap (f^{-1}(b_j))^{\perp} = f^{-1}(b_i) \cap f^{-1}(b_j^{\perp}) = f^{-1}(b_i \cap b_j^{\perp}) = f^{-1}(b_i), \text{ whenever } i \neq j.$$

Calculate:

$$H_L(f^{-1}\,(\eta)) = 1 - \sum_{i=1}^m \left(\mu_1(f^{-1}(b_i))\right)^2 \ = 1 - \sum_{i=1}^m (\mu_2(b_i))^2 \ = H_L(\eta)$$

and

$$\begin{aligned} h_L(\tau_1, f^{-1}\,(\eta)) &= \lim_{n\to\infty}\frac{1}{n}H_L(\vee_{i=0}^{n-1}\tau_1^i(f^{-1}(\eta))) \\ = \lim_{n\to\infty}\frac{1}{n}H_L(\vee_{i=0}^{n-1} f^{-1}(\tau_2^i\eta)) &= \lim_{n\to\infty}\frac{1}{n}H_L(f^{-1}(\vee_{i=0}^{n-1}\tau_2^i\eta)) \\ &= \lim_{n\to\infty}\frac{1}{n}H_L(\vee_{i=0}^{n-1}\tau_2^i\,\eta) = h_L(\tau_2,\eta). \end{aligned}$$

Hence
$\{h_L(\tau_2,\eta);\ \eta \text{ is a fuzzy partition of } (\Omega_2,\ M_2,\ \mu_2)\} \subset \{h_L(\tau_1,\xi);\ \xi \text{ is a fuzzy partition of } (\Omega_1,\ M_1,\ \mu_1)\}$ and consequently

$$h_L(\tau_2) = \sup\{h_L(\tau_2,\eta)\} \leqslant \sup\{h_L(\tau_1,\xi)\} = h_L(\tau_1),$$

where the supremum on the left side of the inequality is taken over all fuzzy partitions η of $(\Omega_2,\ M_2,\ \mu_2)$ and the supremum on the right side of the inequality is taken over all fuzzy partitions ξ of $(\Omega_1,\ M_1,\ \mu_1)$.

Because $h_L(\tau_1) \leqslant h_L(\tau_2)$ and $h_L(\tau_2) \leqslant h_L(\tau_1)$, the proof is complete. □

Remark 8. From Theorem 12 it follows that if $h_L(\tau_1) \neq h_L(\tau_2)$, then the corresponding fuzzy dynamical systems $(\Omega_1,\ M_1,\ \mu_1,\ \tau_1)$, $(\Omega_2,\ M_2,\ \mu_2,\ \tau_2)$ are non-isomorphic. Thus, the logical entropy distinguishes non-isomorphic fuzzy dynamical systems. We illustrate this result by the following example.

Example 4. Consider the probability space $(\Omega,\ S, P)$, where Ω is the unit interval $[0,\ 1]$, S is the $\sigma-$algebra of all Borel subsets of $[0,\ 1]$, and $P : S \to [0,\ 1]$ is the Lebesgue measure, *i.e.*, $P([x, y]) = y - x$ for any $x, y \in [0,\ 1]$, $x < y$. Now we can construct a fuzzy probability space $(\Omega,\ M, \mu)$, where $M = \{\chi_A;\ A \in S\}$, and the mapping $\mu : M \to [0,\ 1]$ is defined by $\mu(\chi_A) = P(A)$. Let $c \in (0, 1)$, and $T_c : [0,\ 1] \to [0,\ 1]$ is defined by the formula $T_c(x) = x + c \pmod 1$. Let us consider the fuzzy dynamical system $(\Omega,\ M,\ \mu,\ \tau_c)$, where the mapping $\tau_c : M \to M$ is defined by $\tau_c(\chi_A) = \chi_A \circ T_c = \chi_{T_c^{-1}(A)}$ for any $\chi_A \in M$. The logical entropy distinguishes non-isomorphic fuzzy dynamical systems $(\Omega,\ M,\ \mu,\ \tau_c)$ for different c. Namely, $h_L(\tau_c) = 0$, if $c = 1/2$, but $h_L(\tau_c) > 0$ for $c = 1 - \sqrt{2}$.

5. Conclusions

In this paper, we introduced the notion of logical entropy of fuzzy partition of a given fuzzy probability space. The proposed measure can be used (in addition to the Shannon entropy of fuzzy partition) as a measure of information of experiment whose outcomes are fuzzy events. We also defined the notions of logical conditional entropy, logical mutual information and logical conditional mutual information of fuzzy partitions. We proved basic properties of the suggested measures. Subsequently the concept of logical entropy of fuzzy partitions was exploited to define the logical entropy of fuzzy dynamical systems. Finally, it was shown that isomorphic fuzzy dynamical systems have the same logical entropy. In this way, we obtained a new tool for distinction of non-isomorphic fuzzy dynamical systems. This result is demonstrated in Example 4.

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