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On Extensions over Semigroups and Applications

Wen Huang, Lei Jin and Xiangdong Ye *

Department of Mathematics, University of Science and Technology of China, Hefei 230026, China; wenh@mail.ustc.edu.cn (W.H.); jinleim@mail.ustc.edu.cn (L.J.)
* Correspondence: yexd@ustc.edu.cn; Tel.: +86-551-6360-1046

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Abstract: Applying a theorem according to Rhemtulla and Formanek, we partially solve an open problem raised by Hochman with an affirmative answer. Namely, we show that if G is a countable torsion-free locally nilpotent group that acts by homeomorphisms on X, and $S \subset G$ is a subsemigroup not containing the unit of G such that $f \in \langle 1, sf : s \in S \rangle$ for every $f \in C(X)$, then (X, G) has zero topological entropy.

Keywords: extensions over semigroups; algebraic past; topological predictability; zero entropy

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1. Introduction

By a *topological dynamical system* (X, G) we mean a topological group G acts by homeomorphisms on a compact metric space X. When $G = \mathbb{Z}$, a system (X, T) is said to be *topologically predictable* or *TP*, if for every continuous function $f \in C(X)$ we have $f \in \langle 1, Tf, T^2 f, \ldots \rangle$, where $\langle \mathcal{F} \rangle \subseteq C(X)$ denotes the closed algebra generated by a family $\mathcal{F} \subseteq C(X)$. This notion was introduced by Kamiński, Siemaszko and Szymański in [1]. Moreover, Kamiński *et al.* showed that a system (X, T) is topologically predictable if and only if every factor of (X, T) is invertible, where a factor is a system (Y, S) and a continuous onto map $\pi : X \to Y$ such that $\pi \circ T = S \circ \pi$. For $\mathbb{Z}$-actions, it was shown in [2] that TP systems have zero topological entropy. Then, a natural question is whether this result also holds for general group actions with some natural modification of the definition of TP. In addition, one would like to understand what other dynamical implications TP has (for related results see [3]).

Hochman [4] examined the relation among topological entropy, invertability, and prediction in the category of topological dynamics. In particular, he studied the notion of TP for $\mathbb{Z}^d$-actions. Such an action $\{T^u\}_{u \in \mathbb{Z}^d}$ of $\mathbb{Z}^d$ by homeomorphisms on X is topologically predictable (TP) if $f \in \langle 1, T^u f : u < 0 \rangle$ for every $f \in C(X)$; here $<$ is the lexicographical ordering on $\mathbb{Z}^d$. One can also work with other orderings. In [4], the author also discussed whether this notion is independent of the generators (the lexicographic ordering certainly is not). It is not independent, because, even in dimension 1, the property TP depends on the generator, *i.e.*, TP for T does not imply it for T^{-1}. Thus, TP is a property of a group action and a given set of generators (see [3]). Moreover, Hochman ([4] Theorem 1.3) proved (in a different way) that for $\mathbb{Z}^d$-actions, TP implies zero topological entropy.

Since there is a rather complete theory of entropy, developed by Ornstein and Weiss, for actions of amenable groups on probability spaces, Hochman ([4] Problem 1.4) then asked a natural question as follows.

Problem 1. Suppose that an infinite discrete amenable group G acts by homeomorphisms on X. Let $S \subset G$ be a subsemigroup not containing the unit of G, and such that $S \cup S^{-1}$ generates G. Suppose that for every $f \in C(X)$ we have $f \in \langle 1, sf : s \in S \rangle$. Does this imply that the topological entropy $h_{top}(X, G) = 0$?

In this paper, we focus on the class of countable torsion-free locally nilpotent groups. Recall that a group is said to be *locally nilpotent* if every finitely generated subgroup of the group is nilpotent. It is clear that nilpotent groups must be locally nilpotent. Also, it is known that all nilpotent groups are amenable, and then, countable torsion-free locally nilpotent groups are all infinite discrete amenable. We will give an affirmative answer to Problem 1 for the class of countable torsion-free locally nilpotent groups. Namely, we have the following result which will be proved in Section 3.

Theorem 2. *Let G be a countable torsion-free locally nilpotent group that acts by homeomorphisms on X, and $S \subset G$ be a subsemigroup not containing the unit of G. If for every $f \in C(X)$ we have $f \in \langle 1, sf : s \in S \rangle$, then the system (X, G) has zero topological entropy.*

This paper is organized as follows. In Section 2, we introduce our main tool, the Rhemtulla-Formanek theorem and give a direct proof when G is a countable torsion-free abelian group. Finally, we prove Theorem 2 in Section 3. In addition, we give two examples in Section 4 to show the limitation of the Rhemtulla-Formanek theorem.

2. A Theorem due to Rhemtulla and Formanek

In this section, we introduce a theorem due to Rhemtulla and Formanek, which is the main tool in our paper. Let G be a group with the unit 1_G satisfying $G \setminus \{1_G\} \neq \emptyset$. Recall that G is said to be *torsion-free* if it satisfies that $g^n = 1_G$ implies $g = 1_G$ for every $g \in G$ and $n \geq 1$. A subset Φ of G is called an *algebraic past* of G if Φ is such that

$$\Phi \cdot \Phi \subset \Phi,\ \Phi \cap \Phi^{-1} = \emptyset, \text{ and } \Phi \cup \Phi^{-1} \cup \{1_G\} = G.$$

Equivalently, an algebraic past of G is a subsemigroup Φ not containing the unit of G with $\Phi \cup \Phi^{-1} \cup \{1_G\} = G$.

Ault [5] investigated a particular extension over a semigroup, and showed that any subsemigroup not containing the unit is contained in some algebraic past of the group, where the group is torsion-free and nilpotent of class two. Then Rhemtulla ([6] Theorem 4) extended it to the class of all torsion-free nilpotent groups. Later, this result was also obtained independently by Formanek [7]. Moreover, Formanek ([7] Theorem 1) proved that this result also holds when the group is torsion-free and locally nilpotent. We precisely state these results together in the following explanation.

Theorem 3 (Rhemtulla-Formanek Theorem)**.** *Let G be a torsion-free locally nilpotent group, and S be a subsemigroup of G not containing the unit. Then there exists an algebraic past Φ of G that contains S.*

The proof of Theorem 3 (see [6,7]) is not easy. We also mention that it depends on Zorn's lemma. To get a clearer idea we present a proof in the case that G is a countable torsion-free abelian group.

Proof of Theorem 3 assuming that G is a countable torsion-free abelian group. If $G = S \cup S^{-1} \cup \{1_G\}$, then by noting the fact that S is a subsemigroup of G not containing the unit 1_G, we know that S has been an algebraic past of G and thus we take $\Phi = S$ to end the proof.

Now we suppose that $G \setminus (S \cup S^{-1} \cup \{1_G\}) \neq \emptyset$. Since G is countable, we write

$$G \setminus (S \cup S^{-1} \cup \{1_G\}) = \{\varphi_n\}_{n \geq 0},$$

and set $S_0 = S$.

If $(\varphi_0^{-1})^m \notin S_0$ for any $m \geq 1$, then let S_1 be the subsemigroup generated by S_0 and $\{\varphi_0\}$, denote this by $S_1 = \langle S_0, \varphi_0 \rangle_{\text{semi}}$; otherwise, let $S_1 = S_0$.

If $(\varphi_1^{-1})^m \notin S_1$ for any $m \geq 1$, then let S_2 be the subsemigroup generated by S_1 and $\{\varphi_1\}$, denote this by $S_2 = \langle S_1, \varphi_1 \rangle_{\text{semi}}$; otherwise, let $S_2 = S_1$.

Inductively, for $n \geq 0$, we obtain S_{n+1} as follows. If $(\varphi_n^{-1})^m \notin S_n$ for any $m \geq 1$, then let S_{n+1} be the subsemigroup generated by S_n and $\{\varphi_n\}$, which is denoted by $S_{n+1} = \langle S_n, \varphi_n \rangle_{\text{semi}}$; otherwise, let $S_{n+1} = S_n$.

Finally, take $\Phi = \bigcup\limits_{n \geq 0} S_n$. Clearly, we have

$$S = S_0 \subset S_1 \subset S_2 \subset \cdots \subset S_n \subset \cdots \subset \Phi.$$

It suffices to show that Φ is an algebraic past of G.

In fact, Φ is a subsemigroup of G. This is because, if $g_1, g_2 \in \Phi$, then $g_1 \in S_{i_1}$ for some $i_1 \geq 0$ and $g_2 \in S_{i_2}$ for some $i_2 \geq 0$, and thus $g_1, g_2 \subset S_i$ by putting $i = i_1 + i_2 \geq 0$, it follows that $g_1 g_2 \in S_i \subset \Phi$ since S_i is a semigroup.

Next, we show that Φ does not contain the unit 1_G. To see this, suppose that $1_G \in \Phi$. Then $1_G \in S_j$ for some $j \geq 1$ with the smallest cardinality; that is, $1_G \in S_j$ and $1_G \notin S_{j-1}$. Such $j \geq 1$ exists because $S_0 = S$ does not contain 1_G. Since $1_G \in S_j$ and $1_G \notin S_{j-1}$, we have $S_j \neq S_{j-1}$, which implies that S_j must be equal to $\langle S_{j-1}, \varphi_{j-1} \rangle_{\text{semi}}$ according to the previous construction. Again by noting that $1_G \in S_j$, $1_G \notin S_{j-1}$, and $S_j = \langle S_{j-1}, \varphi_{j-1} \rangle_{\text{semi}}$, we have either $\varphi_{j-1}^t = 1_G$ for some $t \geq 1$ or $\varphi_{j-1}^m s = 1_G$ for some $m \geq 1$ and $s \in S_{j-1}$ since G is abelian and S_{j-1} is a semigroup. Combining this with the fact that G is torsion-free, we have that $\varphi_{j-1}^m s = 1_G$ for some $m \geq 1$ and $s \in S_{j-1}$. It then follows that $(\varphi_{j-1}^{-1})^m = s \in S_{j-1}$ for some $m \geq 1$, which implies that $S_j = S_{j-1}$, a contradiction. This shows that $1_G \notin \Phi$.

It remains to check that

$$\Phi \cup \Phi^{-1} \cup \{1_G\} = G.$$

To see this, let $h \in G$ with $h \neq 1_G$. If $h \in S \cup S^{-1}$, then $h \in \Phi \cup \Phi^{-1}$ since $S \subset \Phi$. If $h \notin S \cup S^{-1}$, then

$$h \in G \setminus (S \cup S^{-1} \cup \{1_G\}) = \{\varphi_n\}_{n \geq 0},$$

and thus $h = \varphi_n$ for some $n \geq 0$. If $(\varphi_n^{-1})^m \notin S_n$ for any $m \geq 1$, then it holds that

$$h = \varphi_n \in \langle S_n, \varphi_n \rangle_{\text{semi}} = S_{n+1} \subset \Phi.$$

Otherwise, there exists some $m \geq 1$ such that $(h^{-1})^m = (\varphi_n^{-1})^m \in S_n \subset \Phi$. Since it is clear that

$$h^{-1} \in G \setminus (S \cup S^{-1} \cup \{1_G\}) = \{\varphi_n\}_{n \geq 0},$$

we have that $h^{-1} = \varphi_k$ for some $k \geq 0$. If $(\varphi_k^{-1})^l \notin S_k$ for any $l \geq 1$, then it holds that

$$h^{-1} = \varphi_k \in \langle S_k, \varphi_k \rangle_{\text{semi}} = S_{k+1} \subset \Phi,$$

which implies that $h \in \Phi^{-1}$. Otherwise, there exists some $l \geq 1$ such that $h^l = (\varphi_k^{-1})^l \in S_k \subset \Phi$. Combining this with the fact that $(h^{-1})^m \in \Phi$, we have $1_G = \left((h^{-1})^m\right)^l (h^l)^m \in \Phi$ since Φ is a semigroup. This is a contradiction with the fact that Φ does not contain 1_G. Thus,

$$G = \Phi \cup \Phi^{-1} \cup \{1_G\}.$$

Hence, we have checked that Φ is an algebraic past of G satisfying that Φ contains S. This completes the proof. $\square$

3. Proof of Theorem 2

Applying Theorem 3 we are going to prove Theorem 2. For this purpose, we recall several necessary notions and results in the following which are introduced in [4]. Let X be a compact metric space and μ be a regular probability measure on the Borel σ-algebra of X. The entropy and the conditional entropy of finite and countable partitions are defined as usual [8–10]. For two finite or countable measurable partitions $\alpha = (A_1, A_2, \ldots)$ and $\beta = (B_1, B_2, \ldots)$ of X with finite entropy, the Rohlin metric is defined by

$$d_\mu(\alpha, \beta) = H_\mu(\alpha|\beta) + H_\mu(\beta|\alpha).$$

We say that a finite or countable partition $\alpha = (A_1, A_2, \ldots)$ is *μ-continuous* if there is a continuous function $f \in C(X)$ which is a constant μ-almost surely on each atom of A_i. Equivalently, α agrees with the partition of X into level sets of some $f \in C(X)$, up to μ-measure zero. In ([4] Proposition 3.4.) the author proved that the μ-continuous partitions are dense with respect to the Rohlin metric d_μ in the space of finite-entropy countable partitions. Now, we follow the idea in [4] and prove Theorem 2 in the following.

Proof of Theorem 2. Let G be a countable torsion-free locally nilpotent group that acts by homeomorphisms on X, and $S \subset G$ be a subsemigroup not containing the unit of G. Suppose that for every $f \in C(X)$ we have $f \in \langle 1, sf : s \in S \rangle$.

Let μ be a G-invariant Borel probability measure on X. Suppose that α is a μ-continuous partition with $H_\mu(\alpha) < \infty$, then by noting that for every $f \in C(X)$, f is measurable with respect to the σ-algebra generated by $\{sf : s \in S\}$, one shows that $H_\mu(\alpha|\alpha_S) = 0$, where $\alpha_S = \bigvee_{s \in S} s\alpha$. By Theorem 3, we can find an algebraic past Φ of G such that $S \subset \Phi$. By the Pinsker formula (see e.g., ([11] Theorem 3.1) and [12,13]), we have $h_\mu(G, \alpha) = H_\mu(\alpha|\alpha_\Phi)$, where $\alpha_\Phi = \bigvee_{g \in \Phi} g\alpha$. Since $S \subset \Phi$, we have $H_\mu(\alpha|\alpha_\Phi) \leq H_\mu(\alpha|\alpha_S)$. Thus,

$$h_\mu(G, \alpha) = H_\mu(\alpha|\alpha_\Phi) \leq H_\mu(\alpha|\alpha_S) = 0.$$

This implies that, for any μ-continuous partition α with $H_\mu(\alpha) < \infty$ we have $h_\mu(G, \alpha) = 0$. Since the μ-continuous partitions are dense with respect to the Rohlin metric d_μ in the space of all countable partitions with finite entropy, and $h_\mu(G, \beta)$ is continuous with respect to β under the Rohlin metric d_μ, we conclude that $h_\mu(G, \beta) = 0$ for every two-set measurable partition β, and hence the measure entropy $h_\mu(G) = 0$. Thus, by the variational principle [8,14,15], we get that the topological entropy $h_{top}(X, G) = 0$. This completes the proof. □

4. Examples

To demonstrate the limitation of Theorem 3, we will give two examples. Before this, we introduce a little bit notions on the theory of orderable groups.

A group G is said to be *left-orderable* if there exists a strict total ordering $<$ on its elements which is left-invariant; that is, $g < h$ implies that $kg < kh$ for all $g, h, k \in G$. If $<$ is also invariant under the right-multiplication, then we say that G is *bi-orderable*. It is not hard to see that a group G is left-orderable if and only if it contains an algebraic past. Indeed, on the one hand, for a given $<$ on G, we can take $\Phi = \{g \in G : g < 1_G\}$ as an algebraic past; on the other hand, with respect to a given algebraic past Φ, we obtain the desired linear ordering on G as follows: g_1 is less than g_2 (write $g_1 <_\Phi g_2$) if $g_2^{-1} g_1 \in \Phi$.

Clearly, a nontrivial left-orderable group must be torsion-free. In fact, if G is a left-orderable group with the unit 1_G such that $G \setminus \{1_G\} \neq \emptyset$, and Φ is an algebraic past of G, then for any $g \in G \setminus \{1_G\}$, we have either $g \in \Phi$ or $g \in \Phi^{-1}$ since $G = \Phi \cup \Phi^{-1} \cup \{1_G\}$, and hence for any $n \in \mathbb{Z}$ with $n \geq 1$, we have either $g^n \in \Phi$ or $g^n \in \Phi^{-1}$ since Φ is a semigroup, therefore $g^n \neq 1_G$

since $1_G \notin \Phi$. It is well known that an Abelian group is bi-orderable if and only if it is torsion free, non-Abelian free groups and torsion-free nilpotent groups are bi-orderable (see e.g., [16–18]).

One may ask a natural question: whether Theorem 3 holds for all those groups which are bi-orderable or left-orderable. It does not. Precisely, we give an example below, which indicates that even for a bi-orderable group G and a subsemigroup $S \subset G$ not containing the unit of G, there does not always exist an algebraic past of G that contains S.

Example 1. Let $G = \langle a, b \rangle$ be the non-Abelian free group generated by $\{a, b\}$, and S be the subsemigroup of G generated by $\{b^{-2}a^{-1}, a^{-1}, abab, ababab\}$, denote this by

$$S = \langle b^{-2}a^{-1}, a^{-1}, abab, ababab \rangle_{\text{semi}}.$$

Then G is bi-orderable since G is a non-Abelian free group. Thus, G contains algebraic pasts. Clearly, G is countable. It is easy to verify that the subsemigroup S does not contain the unit 1_G. Moreover, S generates G since $b = (a^{-1})(ab) = (a^{-1})(ababab)(abab)^{-1}$. But none of those algebraic pasts of G contains S. This is because, if there exists an algebraic past Φ of G satisfying $S \subset \Phi$, then we consider

$$\varphi = ab \in G = \Phi \cup \Phi^{-1} \cup \{1_G\}.$$

Clearly, $\varphi \neq 1_G$. If $\varphi \in \Phi$, then by noting that $S \subset \Phi$ and the fact that Φ is an algebraic past of G, we have

$$1_G = (ab)(b^{-2}a^{-1})(ab)(a^{-1}) = \varphi(b^{-2}a^{-1})\varphi(a^{-1}) \in \Phi,$$

a contradiction. If $\varphi \in \Phi^{-1}$, then $\varphi^{-1} \in \Phi$, it also follows that Φ contains 1_G since

$$1_G = (ab)^{-2}(abab) = (\varphi^{-1})^2(abab),$$

which is also a contradiction. Thus, we have checked that φ can not sit in G, this is a contradiction. So there does not exist an algebraic past of the group G that contains S. In other words, this φ above is such that $\varphi \neq 1_G$ and $1_G \in \langle S, \varphi \rangle_{\text{semi}} \cap \langle S, \varphi^{-1} \rangle_{\text{semi}}$.

Here, we also mention another example which is given at the end of [6]. We remark that in this example, the group is amenable.

Example 2. Let the group

$$G = \langle a, b; a^{-1}ba = b^{-1} \rangle.$$

Since it is metacyclic which implies metabelian, it is amenable. Thus, G is a left-orderable amenable group (see [6]). Let

$$P = \langle a^2, ba^{-2} \rangle_{\text{semi}},$$

be the semigroup generated by $\{a^2, ba^{-2}\}$. Then P is a subsemigroup of G not containing the unit 1_G. We can verify in the following that any algebraic past of G does not contain P. Suppose Φ is an algebraic past of G with $P \subset \Phi$. It is clear that $a \neq 1_G$. If $a \in \Phi^{-1}$, then $a^2 \in \Phi^{-1}$ which is a contradiction with $a^2 \in P \subset \Phi$. So we have $a \in \Phi$. This, together with $ba^{-2} \in P \subset \Phi$, implies that $ba^{-1} = (ba^{-2})a \in \Phi$ and $ba = (ba^{-2})a^3 \in \Phi$. Thus, we have

$$1_G = bb^{-1} = b(a^{-1}ba) = (ba^{-1})(ba) \in \Phi,$$

a contradiction. Hence there does not exist an algebraic past Φ of G such that Φ contains P.

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