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# Better Heisenberg Limits, Coherence Bounds, and Energy-Time Tradeoffs via Quantum Rényi Information

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**Abstract:** An uncertainty relation for the Rényi entropies of conjugate quantum observables is used to obtain a strong Heisenberg limit of the form  $\text{RMSE} \geq f(\alpha)/((N) + \frac{1}{2})$ , bounding the root mean square error of any estimate of a random optical phase shift in terms of average photon number, where  $f(\alpha)$  is maximised for non-Shannon entropies. Related simple yet strong uncertainty relations linking phase uncertainty to the photon number distribution, such as  $\Delta\Phi \geq \max_n p_n$ , are also obtained. These results are significantly strengthened via upper and lower bounds on the Rényi mutual information of quantum communication channels, related to asymmetry and convolution, and applied to the estimation (with prior information) of unitary shift parameters such as rotation angle and time, and to obtain strong bounds on measures of coherence. Sharper Rényi entropic uncertainty relations are also obtained, including time-energy uncertainty relations for Hamiltonians with discrete spectra. In the latter case almost-periodic Rényi entropies are introduced for nonperiodic systems.

**Keywords:** uncertainty relations; Rényi entropy; Heisenberg limit; quantum metrology; asymmetry; coherence; time observables



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## 1. Introduction

Quantum mechanics places fundamental limits on the information which can be gained in various contexts, ranging from the accuracy to which the phase shift of an optical probe state can be estimated to the secure key rate that can be obtained from a cryptographic protocol. Such limits are often formulated via uncertainty relations that restrict, for example, the degree to which values of two observables can be jointly specified, or the degree to which both an intended party and an eavesdropper can access quantum information [1].

Entropic uncertainty relations place particularly strong restrictions, and underlie the main themes of this paper. One has, for example, the number-phase uncertainty relation [2,3]

$$H(N|\rho) + H(\Phi|\rho) \geq \log 2\pi + H(\rho), \quad (1)$$

for the number and canonical phase observables of an optical mode,  $N$  and  $\Phi$ . Here  $H(A|\rho) = -\sum_a p(a|\rho) \log p(a|\rho)$  is the Shannon entropy of an observable  $A$  with probability distribution  $p(a|\rho)$ , for a state described by density operator  $\rho$ , and  $H(\rho) = -\text{tr}[\rho \log \rho]$  denotes the von Neumann entropy of the state. The choice of logarithm base is left open throughout, corresponding to a choice of units, e.g., to bits for base 2 and nats for base  $e$ . It follows that the number and phase uncertainties of any quantum state, as quantified by their Shannon entropies, cannot both be arbitrarily small.

Entropic uncertainty relations have useful counterparts in quantum metrology. For example, if a random phase shift  $\Theta$  of an optical probe state  $\rho$  is estimated via some measurement  $\Theta_{\text{est}}$ , then it follows from uncertainty relation (1) that the error in the estimate,  $\Theta_{\text{est}} - \Theta$ , is strongly constrained by the tradeoff relation [4]

$$H(N|\rho) + H(\Theta_{\text{est}} - \Theta|\rho) \geq \log 2\pi + H(\rho). \quad (2)$$

This relation applies to arbitrary estimates, rather than to a particular phase observable  $\Phi$ , and further implies that the root-mean-square error (RMSE) of the estimate is bounded by [4–6]

$$\text{RMSE} := \langle (\Theta_{\text{est}} - \Theta)^2 \rangle^{1/2} \geq \frac{\sqrt{2\pi/e^3}}{\langle N \rangle + 1}, \tag{3}$$

where  $\langle N \rangle = \text{tr}[\rho N]$  denotes the average photon number of the probe state. This is a strong form of the well-known Heisenberg limit in quantum metrology, which states that the phase error can asymptotically scale no better than  $\langle N \rangle^{-1}$ , where such limits cannot be obtained via quantum Fisher information methods without additional assumptions [7] (see also Section 3.1). The above bounds are valid for both linear and nonlinear phase shifts, can be further strengthened to take into account any prior information about the phase shift  $\Theta$ , and in many cases far outperform bounds based on quantum Fisher information [4] (see also Section 2).

While the above results arise via properties of standard Shannon and von Neumann entropies, it is known that various elements of quantum information theory can be generalised to the family of Rényi entropies,  $H_\alpha(A|\rho)$  and  $H_\alpha(\rho)$ , and associated Rényi relative entropies  $D_\alpha(\rho||\sigma)$  [1]. These quantities are labelled by a real index,  $\alpha \geq 0$ , and reduce to the standard entropies and relative entropy for  $\alpha = 1$ . One has, for example, the Rényi uncertainty relation [8,9]

$$H_\alpha(N|\rho) + H_\beta(\Phi|\rho) \geq \log 2\pi, \quad \frac{1}{\alpha} + \frac{1}{\beta} = 2, \tag{4}$$

analogous to Equation (1). Several questions then immediately arise. Are such generalisations to Rényi entropies advantageous? Why are the uncertainties of  $N$  and  $\Phi$  characterised by two different Rényi entropies,  $H_\alpha$  and  $H_\beta$ , in Equation (4)? Furthermore, why is there no term depending on the degree of purity of the state, analogous to  $H(\rho)$  in Equation (1)?

Several positive answers to the first question above are known, in contexts such as mutually unbiased bases [8], quantum cryptography [10], and quantum steering [11]. An aim of this paper is to demonstrate further unambiguous advantages of Rényi entropic uncertainty relation (4), in the context of quantum metrology. For example, it will be shown in Section 2 to lead to a generalised Heisenberg limit of the form

$$\text{RMSE} \geq \frac{f(\alpha)}{\langle N \rangle + \frac{1}{2}} \tag{5}$$

for random phase shifts, where the function  $f(\alpha)$  is maximised for the choice  $\alpha \approx 0.7471$ . This choice not only improves on the denominator in Equation (3) (corresponding to  $\alpha = 1$ ), but also improves on the numerator, by around 4%, with the result being independent of Rényi entropies and any interpretation thereof. Further entropic bounds on the RMSE are obtained in Section 2, as well as related simple yet strong uncertainty relations for number and canonical phase observables, such as

$$\Delta\Phi \geq \max_n p(n|\rho). \tag{6}$$

A second aim of the paper is to further strengthen uncertainty relations and metrology bounds such as Equations (2)–(6), achieved in Section 3 via finding upper and lower bounds for the classical Rényi mutual information of quantum communication channels [12–15], which also shed light on the second and third questions above. The upper bounds are based on the notion of Rényi asymmetry [16], recently applied to energy-time uncertainty relations for conditional Rényi entropies by Coles et al. [17]. The lower bounds relate to the convolution of the prior and error distributions. For example, the number-phase uncertainty relation

$$A_\alpha^N(\rho) + H_\alpha(\Phi|\rho) \geq \log 2\pi, \quad \alpha \geq \frac{1}{2}, \tag{7}$$

is obtained in Section 3, which generalises Equation (1) for Shannon entropies, and strengthens Equation (4) for Rényi entropies to take the degree of purity of the state into account. Here  $A_\alpha^N(\rho)$  denotes the associated Rényi asymmetry, which may be interpreted as quantifying the intrinsically ‘quantum’ uncertainty of  $N$ , and satisfies a duality property for pure states that underpins the relationship between the indexes  $\alpha$  and  $\beta$  in Equation (4).

The results in Section 3 hold for the general case of unitary displacements generated by an operator with a discrete spectrum (such as  $N$ ). Applications to strong upper and lower bounds for several measures of coherence [18,19], the estimation of rotation angles, and energy-time metrology and uncertainty relations, are briefly discussed in Section 4. In the latter case, almost-periodic Rényi entropies are introduced for the time uncertainties of non-periodic systems, analogously to the case of standard entropies [20]. Conclusions are given in Section 5, and proof technicalities are largely deferred to appendices.

## 2. Metrology Bounds, Heisenberg Limit and Uncertainty Relations via Rényi Entropies

In this section, an analogue of metrology relation (2) is derived for Rényi entropies, via uncertainty relation (4). The improved Heisenberg limit (5) follows as a consequence, as well as several simple uncertainty relations for number and phase, including Equation (6). Stronger versions of these results will be obtained in Section 3.

### 2.1. Definition of Rényi Entropies and Rényi Lengths

To proceed, several definitions are necessary. First, the photon number of an optical mode is described by a Hermitian operator  $N$  having eigenstates  $\{|n\rangle\}$ ,  $n = 0, 1, 2, \dots$ , with associated probability distribution  $p(n|\rho) = \langle n|\rho|n\rangle$  for a state described by density operator  $\rho$ . A phase shift  $\theta$  of the field is correspondingly described by  $\rho_\theta = e^{-iN\theta}\rho e^{iN\theta}$ .

Second, the canonically conjugate phase observable  $\Phi$  is described by the positive-operator-valued measure (POVM)  $\{|\phi\rangle\langle\phi|\}$ , with  $\phi$  ranging over the unit circle and

$$|\phi\rangle := \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} e^{-in\phi} |n\rangle, \tag{8}$$

and associated canonical phase probability density  $p(\phi|\rho) = \langle\phi|\rho|\phi\rangle$  [21,22]. It is straightforward to check that this density is translated under phase shifts, i.e.,  $p(\phi|\rho_\theta) = p(\phi - \theta|\rho)$ .

Third, the classical Rényi entropies of  $N$  and  $\Phi$  are defined by [1]

$$H_\alpha(N|\rho) := \frac{1}{1-\alpha} \log \sum_{n=0}^{\infty} p(n|\rho)^\alpha, \quad H_\alpha(\Phi|\rho) := \frac{1}{1-\alpha} \log \int d\phi p(\phi|\rho)^\alpha, \tag{9}$$

for  $\alpha \in [0, \infty)$ . These reduce to the standard Shannon entropies in the limit  $\alpha \rightarrow 1$  (using, e.g.,  $\lim_{\alpha \rightarrow 1} [g(\alpha) - g(1)] / [\alpha - 1] = g'(1)$  for  $g(\alpha) = \log \sum_n p(n|\rho)^\alpha$ ). They provide measures of uncertainty that are small for highly peaked distributions and large for spread-out distributions. In particular,  $H_\alpha(N) = 0$  and  $H_\alpha(\Phi|\rho) = \log 2\pi$  for any number state  $\rho = |n\rangle\langle n|$ . Direct measures of uncertainty are given by the associated Rényi lengths

$$L_\alpha(N|\rho) := \left[ \sum_{n=0}^{\infty} p(n|\rho)^\alpha \right]^{\frac{1}{1-\alpha}}, \quad L_\alpha(\Phi|\rho) := \left[ \int d\phi p(\phi|\rho)^\alpha \right]^{\frac{1}{1-\alpha}}, \tag{10}$$

which quantify the effective spreads of  $N$  and  $\Phi$  over the nonnegative integers and the unit circle, respectively, [23]. Note that uncertainty relation (4) can be rewritten in the form

$$L_\alpha(N|\rho)L_\beta(\Phi|\rho) \geq 2\pi, \quad \frac{1}{\alpha} + \frac{1}{\beta} = 2 \tag{11}$$

for these spreads, akin to the usual Heisenberg uncertainty relation.

### 2.2. Entropic Tradeoff Relation for Phase Estimation

If some estimate  $\theta_{\text{est}}$  is made of a phase shift  $\theta$  applied to a probe state, then the estimation error,  $\theta_{\text{err}} = \theta_{\text{est}} - \theta$ , will have a highly-peaked probability density for a good estimate, and a spread-out probability density for a poor estimate. Hence, the quality of the estimate can be quantified in terms of the Rényi entropy of  $p(\theta_{\text{err}})$ . The following theorem imposes a tradeoff between the quality of any estimate of a completely unknown phase shift and the number entropy of the probe state.

**Theorem 1.** For any estimate  $\Theta_{\text{est}}$  of a uniformly random phase shift  $\Theta$  applied to a probe state  $\rho$ , the estimation error  $\Theta_{\text{est}} - \Theta$  satisfies the tradeoff relation

$$H_\alpha(\Theta_{\text{est}} - \Theta|\rho) + H_\beta(N|\rho) \geq \log 2\pi, \quad \frac{1}{\alpha} + \frac{1}{\beta} = 2. \tag{12}$$

Note that the condition on  $\alpha$  and  $\beta$  implies  $\alpha, \beta \geq \frac{1}{2}$ . For the case of Shannon entropies, i.e.,  $\alpha = \beta = 1$ , this result has been previously obtained via entropic uncertainty relation (1) [5]. A similar method is used in Appendix A to prove the general result of the theorem via entropic uncertainty relation (4). It is worth emphasising that, unlike uncertainty relation (4), Theorem 1 applies to *any* estimate of the random phase shift, including the canonical phase measurement  $\Phi$  as a special case (for this case Equation (4) is recovered).

Theorem 1 implies that no phase-shift information can be gained via a probe state  $|n\rangle\langle n|$ , as expected since number eigenstates are insensitive to phase shifts. In particular, the number entropy  $H_\beta(N|\rho)$  vanishes for any index  $\beta$  and so the error entropy in Equation (12) must reach its maximum value of  $\log 2\pi$ , which is only possible if the error has a uniform probability density, i.e.,  $p(\theta_{\text{err}}) = 1/(2\pi)$ .

Conversely, Theorem 1 connects informative estimates with probe states that have a high number entropy. For example, if a canonical phase measurement is used to estimate a random phase shift of the pure probe state  $|\psi\rangle = (n_{\text{max}} + 1)^{-1/2}(|0\rangle + |1\rangle + \dots + |n_{\text{max}}\rangle)$ , the error distribution may be calculated, using Equation (A2) of Appendix A for  $\Theta_{\text{est}} \equiv \Phi$ , as

$$p(\theta_{\text{err}}) = \langle \theta_{\text{err}} | \rho | \theta_{\text{err}} \rangle = |\langle \theta_{\text{err}} | \psi \rangle|^2 = \frac{1}{2\pi(n_{\text{max}} + 1)} \left| \sum_{n=0}^{n_{\text{max}}} e^{in\theta_{\text{err}}} \right|^2, \tag{13}$$

and the Shannon entropy of the error then follows via Equation (69) of Reference [24] as

$$\begin{aligned} H(\Theta_{\text{est}} - \Theta|\rho) &= \log 2\pi + \log(n_{\text{max}} + 1) + 2[1 - 1^{-1} - 2^{-1} - \dots - (n_{\text{max}} + 1)^{-1}] \log e \\ &\leq \log 2\pi - \log(n_{\text{max}} + 1) + 2(1 - \gamma) \log e, \end{aligned} \tag{14}$$

where  $\gamma \approx 0.5772$  is Euler’s constant. Hence such a probe state leads to an arbitrarily low uncertainty for the error as  $n_{\text{max}}$  increases. Moreover, Theorem 1 implies that this estimate is near-optimal, under the constraint of at most  $n_{\text{max}}$  photons, in the sense that the error entropy is within  $2(1 - \gamma) \log e \approx 1.2$  bits of the minimum possible,  $\log 2\pi - \log(n_{\text{max}} + 1)$ , allowed by Equation (12) under this constraint.

This last result strongly contrasts with Fisher information methods, which suggest that the best possible single-mode probe state, under the constraint of at most  $n_{\text{max}}$  photons, is the simple superposition state  $2^{-1/2}(|0\rangle + |n_{\text{max}}\rangle)$  [25]. However, it follows from Theorem 1 that this probe state cannot be optimal for the estimation of a random phase shift. In particular, noting that  $H_\beta(N|\rho) = \log 2$  for this case, Equations (10) and (12) give

$$\log L_\alpha(\Theta_{\text{est}} - \Theta|\rho) = H_\alpha(\Theta_{\text{est}} - \Theta|\rho) \geq \log \pi \tag{15}$$

for any value of  $n_{\max}$ , in stark contrast to Equation (14). Indeed, choosing  $\alpha = 2$  gives

$$\oint d\theta_{\text{err}} \left[ p(\theta_{\text{err}}) - \frac{1}{2\pi} \right]^2 = L_2(\Theta_{\text{est}} - \Theta|\rho)^{-1} - \frac{1}{\pi} + \frac{1}{4\pi^2} \leq \frac{1}{4\pi^2}, \tag{16}$$

implying that  $p(\theta_{\text{err}})$  cannot be too different from a uniform distribution. Hence the simple superposition state has a poor performance in comparison to the probe state in Equation (14), for the case of uniformly random estimates. A more direct comparison with Fisher information bounds is made in the following subsection, and the difference explained in Section 3.1.

2.3. Lower Bounds for RMSE, a Strong Heisenberg Limit, and Number-Phase Uncertainty Relations

Equation (10) and Theorem 1 imply that the Rényi length of the error for any estimate of a random phase shift has the lower bound  $L_\alpha(\Theta_{\text{est}} - \Theta|\rho) \geq 2\pi/L_\beta(N|\rho)$ . However, a more familiar length measure for characterising the performance of an estimation scheme is the root-mean-square error (RMSE) of the estimate, given by  $\text{RMSE} = \langle (\Theta_{\text{est}} - \Theta)^2 \rangle^{1/2}$ . Note that, in contrast to the case of entropies and Rényi lengths, a well-known ambiguity arises:  $\theta_{\text{err}}^2 = (\theta_{\text{est}} - \theta)^2$  is not a periodic function, and hence evaluation of the RMSE depends on the choice of a phase reference interval for the error  $\theta_{\text{err}}$ . Fortunately this is easily resolved: a perfect estimate corresponds to a zero error, and hence the reference interval centred on zero, i.e.,  $\theta_{\text{err}} \in [-\pi, \pi)$ , will be used.

The following theorem gives three strong lower bounds for the RMSE, where the third has the form of a generalised Heisenberg limit as discussed in Section 1. A corollary to this theorem, further below, gives corresponding preparation uncertainty relations for number and phase.

**Theorem 2.** For any estimate  $\Theta_{\text{est}}$  of a uniformly random phase shift  $\Theta$  applied to a probe state  $\rho$ , the root-mean-square error  $\text{RMSE} = \langle (\Theta_{\text{est}} - \Theta)^2 \rangle^{1/2}$  has the lower bounds

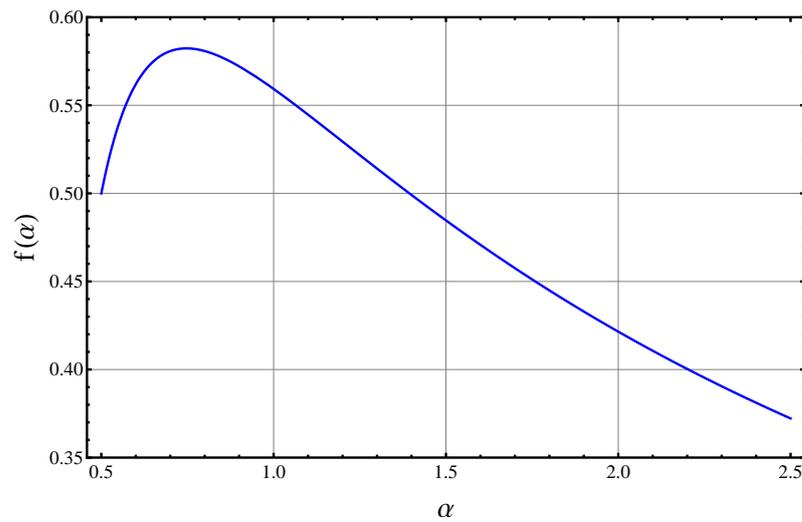
$$\text{RMSE} \geq \frac{\pi}{\sqrt{3} L_{1/2}(N|\rho)}, \quad \text{RMSE} \geq \max_n p(n|\rho), \quad \text{RMSE} \geq \frac{f_{\max}}{\langle N \rangle + \frac{1}{2}}, \tag{17}$$

where  $L_{1/2}(N|\rho) = [\sum_n \sqrt{p(n|\rho)}]^2$  is a Rényi length as defined in Equation (10), and  $f_{\max} \approx 0.5823$  denotes the maximum value of the function

$$f(\alpha) := \begin{cases} 2\alpha^{-1} \left(\frac{\pi}{3\alpha-1}\right)^{\frac{1}{2}} \left(\frac{3\alpha-1}{2}\right)^{\frac{1}{1-\alpha}} \frac{(1-\alpha)^{\frac{1}{2}} \Gamma(\frac{1}{1-\alpha})}{\Gamma(\frac{1}{1-\alpha}-\frac{1}{2})}, & \frac{1}{2} \leq \alpha \leq 1, \\ 2\alpha^{-1} \left(\frac{\pi}{3\alpha-1}\right)^{\frac{1}{2}} \left(\frac{3\alpha-1}{2}\right)^{\frac{1}{1-\alpha}} \frac{(\alpha-1)^{\frac{1}{2}} \Gamma(\frac{\alpha}{\alpha-1}+\frac{1}{2})}{\Gamma(\frac{\alpha}{\alpha-1})}, & \alpha \geq 1, \end{cases} \tag{18}$$

which is achieved for the choice  $\alpha \approx 0.7471$ .

In Equation (18),  $\Gamma(x)$  denotes the Gamma function and the value of  $f(1)$  is defined by taking the limit  $\alpha \rightarrow 1$  in either expression and using  $\lim_{x \rightarrow 0} (1 - 3x/2)^{1/x} = e^{-3/2}$  and  $\lim_{x \rightarrow \infty} x^{1/2} \Gamma(x) / \Gamma(x + \frac{1}{2}) = 1$ , to obtain  $f(1) = \sqrt{2\pi}/e^3 \approx 0.5593$ . The scaling function  $f(\alpha)$  is plotted in Figure 1. The proof of the theorem relies on Theorem 1 and upper bounds on Rényi entropies under various constraints, and is given in Appendix B.



**Figure 1.** The scaling function  $f(\alpha)$  for the Heisenberg limit in Theorem 2. Particular values of interest are  $f(1/2) = 1/2$ ,  $f(1) = \sqrt{2\pi}/e^3 \approx 0.5593$ , and the maximum value  $f_{\max} \approx f(0.7471) \approx 0.5823$ .

The lower bounds in Theorem 2 are relatively strong, and indeed the first inequality in Equation (17) is tight, being saturated for number states. In particular, the error distribution is always uniform for this case, as noted below Theorem 1, yielding  $\text{RMSE} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi_{\text{err}} (\phi_{\text{err}})^2 = \pi/\sqrt{3}$ , as per the first lower bound.

Moreover, the third bound in Equation (17) of Theorem 2 is stronger than the Heisenberg limit in Equation (3), both in the numerator and the denominator. In particular, the scaling factor  $f(1) \approx 0.5593$  in Equation (3) is outperformed by  $\approx 4\%$  when compared to  $f_{\max} \approx 0.5823$  in Equation (17). Note that while the derivation of this bound relies on properties of Rényi entropies (see Appendix B), no Rényi entropies appear in the bound itself. The bound thus demonstrates an unambiguous advantage of using Rényi entropies in quantum metrology that is completely independent of their interpretation.

Theorem 2 can also be directly compared to the Fisher information bound

$$\text{RMSE}_{\theta} := \langle (\Theta_{\text{est}} - \theta)^2 \rangle_{\rho_{\theta}}^{1/2} = \left[ \int_{-\infty}^{\infty} d\theta_{\text{est}} (\theta_{\text{est}} - \theta)^2 p(\theta_{\text{est}}|\rho_{\theta}) \right]^{1/2} \geq \frac{1}{2\Delta N}, \quad (19)$$

for the root mean square error of any locally unbiased estimate  $\Theta_{\text{est}}$  of a given phase-shift  $\theta$  of probe state  $\rho$  [25]. Here phase shifts are ‘unwrapped’ from the unit circle to the real line,  $\Delta N$  is the root mean square deviation of the number operator for the probe state, and local unbiasedness is the requirement that  $\langle \Theta_{\text{est}} \rangle_{\rho_{\chi}} = \int_{-\infty}^{\infty} d\theta_{\text{est}} \theta_{\text{est}} p(\theta_{\text{est}}|\rho_{\chi}) = \chi$  for all phase shifts  $\chi$  in some neighbourhood of  $\theta$ . Note the bound implies that  $\text{RMSE}_{\theta}$  becomes infinite for number states. Under the constraint of a maximum photon number  $n_{\max}$ , the probe state minimising the Fisher bound is the simple superposition  $2^{-1/2}(|0\rangle + |n_{\max}\rangle)$ , considered in Section 2, yielding  $\text{RMSE}_{\theta} \geq 1/n_{\max}$ , which approaches zero as  $n_{\max}$  increases [25]. In contrast, for any estimate of a uniformly random phase shift, the first two bounds in Equation (17) of Theorem 2 give the much stronger lower bounds  $\text{RMSE} > \pi/(2\sqrt{3}) \approx 0.9069$  and  $\text{RMSE} \geq \frac{1}{2}$  for this probe state, irrespective of the value of  $n_{\max}$ . Thus, the optimal single-mode probe state for the Fisher information bound is not optimal for estimating uniformly random phase shifts. The underlying reason for this difference is related to the degree of prior information available about the phase shift [4,26], as discussed further in Section 3.1.

Finally, it is of interest to note that each of the bounds in Theorem 2 can be used to obtain a corresponding preparation uncertainty relation for the canonical phase and photon number observables. In particular, defining the standard deviation  $\Delta_{\chi}\Phi$  of the canonical phase observable  $\Phi$  with respect to reference angle  $\chi$  via [24]

$$(\Delta_\chi \Phi)^2 := \int_{\chi-\pi}^{\chi+\pi} d\phi (\phi - \chi)^2 p(\phi|\rho), \tag{20}$$

one has the following corollary of Theorem 2, also proved in Appendix B.

**Corollary 1.** *The canonical phase and photon number of an optical mode satisfy the family of uncertainty relations*

$$L_\beta(N|\rho)\Delta_\chi\Phi \geq \alpha^{\frac{\alpha}{\alpha-1}} f(\alpha), \quad \frac{1}{\alpha} + \frac{1}{\beta} = 2, \tag{21}$$

for Rényi length and standard deviation. In particular, the choices  $\alpha \rightarrow \infty$ ,  $\alpha = 0.5$ , and the value of  $\alpha$  maximising  $f_\alpha$  yield the corresponding number-phase uncertainty relations

$$L_{1/2}(N|\rho)\Delta_\chi\Phi \geq \frac{\pi}{\sqrt{3}}, \quad \Delta_\chi\Phi \geq \max_n p(n|\rho), \quad (\langle N \rangle + \frac{1}{2})\Delta_\chi\Phi \geq f_{\max} \approx 0.5823. \tag{22}$$

The above uncertainty relations are easy to evaluate for many states, and are relatively strong. In particular, the first inequality in Equation (22) is saturated for number states, and hence has the best possible lower bound. Further, the lower bound  $f_{\max}$  in the third inequality is near-optimal, as it cannot be improved to more than  $\pi/(2\sqrt{3})$ , corresponding to the value of the left hand side for the number state  $|0\rangle$ . This suggests the conjecture

$$(\langle N \rangle + \frac{1}{2})\Delta_\chi\Phi \geq \frac{\pi}{2\sqrt{3}} \approx 0.9069. \tag{23}$$

Note this conjecture is consistent with the fact that the right hand side can be no larger than  $2(-z_A/3)^{3/2} \approx 1.3761$  in the asymptotic limit  $\langle N \rangle \rightarrow \infty$ , where  $z_A$  denotes the first (negative) zero of the Airy function [27]. Evidence for a related conjecture,  $(\langle N \rangle + 1)\Delta_\chi\Phi \geq 2(-z_A/3)^{3/2}$ , is given in References [5,28].

### 3. Stronger Metrology Bounds and Uncertainty Relations via Sandwiched Rényi Relative Entropies and Asymmetry

The results of the previous section relied on the known uncertainty relation (4) for Rényi entropies. To obtain stronger results which can, for example, take prior information and nonpurity into account, two approaches are possible. The first is to follow essentially the same strategy as in Section 2, but starting with stronger uncertainty relations, similar to the approach in Reference [29] for standard entropies. The second approach, followed in this paper, is more fundamental, being based on information properties of quantum communication channels that not only yield stronger metrology tradeoff relations, but also lead to stronger uncertainty relations for Rényi entropies than Equation (4). Further, the results are applicable not only to number and phase but to all unitary displacements generated by discrete-valued operators, including rotations generated by angular momentum and time evolution generated by a Hamiltonian with a discrete spectrum.

#### 3.1. Setting the Scene: The Case of Standard Entropies

It is helpful to first briefly review how uncertainty relations and metrology bounds for standard Shannon and von Neumann entropies, such as Equations (1)–(3), follow from upper and lower bounds for quantum information [4,6]. This both motivates and provides a base of comparison for the general case.

For a quantum communication channel in which signal state  $\rho_x$  is transmitted with prior probability density  $p(x)$ , corresponding to the ensemble  $\mathcal{E} \equiv \{\rho_x; p(x)\}$ , the information that can be gained per signal via measurements of observable  $A$  at the receiver is given by the Shannon mutual information

$$I(A : X) = H(A) - H(A|X) = H(A|\rho_{\mathcal{E}}) - \int dx p(x)H(A|\rho_x), \quad \rho_{\mathcal{E}} = \int dx p(x)\rho_x, \tag{24}$$

where  $H(A|X)$  is the conditional entropy  $H(AX) - H(X)$ , and integration is replaced here and below by summation for discrete signal ensembles [30]. A useful upper bound for this information gain is the Holevo quantity  $\chi(\mathcal{E})$ , with [30,31]

$$I(A : X) \leq \chi(\mathcal{E}) := H(\rho_{\mathcal{E}}) - \int dx p(x)H(\rho_x). \tag{25}$$

Consider now the case of signal states generated by a group of unitary transformations  $U_x = e^{-ixG}$ , for some Hermitian generator  $G$  with a discrete spectral decomposition  $G = \sum_k g_k P_k$ , where  $P_k$  is the projector onto the eigenspace of eigenvalue  $g_k$ , so that  $\rho_x = U_x \rho U_x^\dagger$  for some ‘probe’ state  $\rho$ . The Holevo quantity then has the upper bound [4,6]

$$\chi(\mathcal{E}) = H(\rho_{\mathcal{E}}) - H(\rho) \leq A^G(\rho) := H(\rho_G) - H(\rho), \quad \rho_G := \sum_n P_k \rho P_k = \sum_k P_k \rho_{\mathcal{E}} P_k, \tag{26}$$

where the inequality follows because the decoherence map  $\rho \rightarrow \rho_G$  is unital and hence entropy-increasing. The upper bound,  $A^G(\rho)$ , is called the asymmetry of  $\rho$  with respect to  $G$  [32], and more generally extends to a resource measure for groups of unitary displacements of a given state [32–34]. Note that inequality (26) unifies and generalises energy-time uncertainty relations (8), (12) and (E11) of Reference [17], which correspond to discrete prior distributions and uniform continuous prior distributions (see also Section 5).

For any estimate  $A = X_{\text{est}}$  of the shift parameter  $X$ , one also has a simple lower bound for the Shannon mutual information from rate-distortion theory [4,6]:

$$I(X_{\text{est}} : X) = H(X) - H(X|X_{\text{est}}) = H(X) - H(X - X_{\text{est}}|X_{\text{est}}) \geq H(X) - H(X - X_{\text{est}}). \tag{27}$$

Combining this with the Holevo bound (25) and the asymmetry bound (26), and noting that  $H(Z) = H(-Z)$  in general, then gives the strong metrological tradeoff relation [4]

$$H(X_{\text{est}} - X|\rho) + H(\rho_G) \geq H(X) + H(\rho). \tag{28}$$

This will be generalised to Renyi entropies further below.

For example, if  $\rho$  is the state of an optical mode then phase shifts are generated by  $G = N$ , which is nondegenerate. Hence  $H(\rho_N) = H(N|\rho)$ , and the above relation becomes

$$H(\Theta_{\text{est}} - \Theta|\rho) + H(N|\rho) \geq H(\Theta) + H(\rho). \tag{29}$$

Note that this implies and hence is stronger than both Equation (2) and Theorem 1 (for  $\alpha = \beta = 1$ ), which correspond to the special case of a uniformly random phase shift, with  $p(\theta) = \frac{1}{2\pi}$  and  $H(\Theta) = \log 2\pi$ . Further, entropic number-phase uncertainty relation (1) is recovered by choosing  $\Theta_{\text{est}} = \Phi$  and  $p(\theta) = \frac{1}{2\pi}$  in Equation (29), and hence may be viewed as a consequence of the Holevo bound for quantum communication channels [35].

Equation (29) also strengthens the Heisenberg limit in Equation (3) to take into account the degree of purity of the probe state and any prior information about the phase shift, via

$$\text{RMSE} \geq (2\pi e)^{-1/2} e^{H(\rho) - H(N|\rho)} e^{H(\Theta)} \geq \frac{(2\pi e^3)^{-1/2} e^{H(\rho)} e^{H(\Theta)}}{\langle N \rangle + \frac{1}{2}} \tag{30}$$

(using units of nats for the entropies). The first lower bound follows via the well-known bound  $H \leq \log \text{RMSE} + \frac{1}{2} \log 2\pi e$  for Shannon entropy, and the second from inequality (A17) in Appendix B (for  $\alpha = \beta = 1$ ). In particular, the lower bounds increase for less pure probe states (via increasing  $H(\rho)$ ), and decrease for more prior information (via decreasing  $H(\Theta)$ ).

Finally, note that it is the latter property that underlies the different scaling behaviour of the Fisher bound (15), discussed in Section 2. In particular, the first inequality in Equation (30) implies that the only way to obtain a scaling of  $\text{RMSE} \sim 1/n_{\text{max}}$ , for the probe state  $2^{-1/2}(|0\rangle + |n_{\text{max}}\rangle)$ , is if the entropy of the prior probability density  $p(\theta)$  scales

as  $H(\Theta) \sim -\log n_{\max}$ , i.e., if the phase shift is already known to within an accuracy of  $L(\Theta) \sim 1/n_{\max}$  before any estimate is made. This is consistent with the Fisher bound, since the latter only applies to the error of an (unbiased) estimate of a *known* phase shift  $\theta$ . If the phase shift is not known, then  $\text{RMSE}_\theta$  must be generalised to take the prior distribution into account, e.g., via the RMSE, for which the typically stronger bounds in Equation (30) apply (to both biased and unbiased estimates).

To emphasise this point of distinction, note that applying a phase shift  $\theta$  to the probe state  $2^{-1/2}(|0\rangle + |n_{\max}\rangle)$  gives the phase-shifted state  $2^{-1/2}(|0\rangle + e^{-in_{\max}\theta}|n_{\max}\rangle)$ , implying that no measurement on this state (nor on multiple copies thereof) can discriminate between the phase shifts  $\theta, \theta + 2\pi/n_{\max}, \dots, \theta + 2\pi(n_{\max} - 1)/n_{\max}$ . Thus,  $\theta$  cannot be accurately estimated to within an error of  $1/n_{\max}$  via this probe state, as per the Fisher bound, unless it is already known to lie within an interval of length  $2\pi/n_{\max}$ . Finally, it is also worth noting that Fisher information bounds cannot be used to obtain Heisenberg limits in terms of average photon number, as per Equation (30), unless further assumptions are made (in addition to local unbiasedness). For example, the probe state  $\frac{\sqrt{3}}{2} \sum_m 2^{-m} |2^m\rangle$  has  $\Delta N = \infty$  and  $\langle N \rangle = 3/2$  [7], giving a trivial Fisher bound of 0 in Equation (15) but a nontrivial bound in Equation (30). Further discussion, including the case of multiple probe states, may be found in References [4,26,35].

### 3.2. Sandwiched Rényi Relative Entropy and Mutual Information

The use of information bounds for quantum communication channels to obtain strong metrological bounds (28) and (29) points the way to strengthening the results in Section 2. To proceed, however, suitable generalisations of mutual information and the Holevo quantity, i.e., of  $I(A : X)$  and  $\chi(\mathcal{E})$ , are required. These are provided via sandwiched Rényi relative entropies, as discussed below.

The starting point is rewrite  $I(A : X)$  and  $\chi(\mathcal{E})$  in terms of relative entropies:

$$I(A : X) = D(p_{AX} \| p_A p_X) \leq D(\rho_{\mathcal{E}X} \| \rho_{\mathcal{E}} \otimes \rho_X) = \chi(\mathcal{E}). \tag{31}$$

Here  $p_{AX}(a, x) = p(x)p(a|\rho_x)$  denotes the joint distribution for outcome  $A = a$  and signal state  $\rho_x$ , with marginals  $p_A(a) = p(a|\rho_{\mathcal{E}})$  and  $p_X(x) = p(x)$ ;  $\rho_{\mathcal{E}X}$  denotes the joint density operator  $\int dx \rho_x \otimes |x\rangle\langle x|$ , with reduced density operators  $\rho_{\mathcal{E}}$  and  $\rho_X = \int dx p(x)|x\rangle\langle x|$  for some orthogonal basis set  $\{|x\rangle\}$  on an auxiliary Hilbert space; and

$$D(p\|q) := \int dz p(z)[\log p(z)(\log p(z) - \log q(z))] \geq 0, \quad D(\rho\|\sigma) := \text{tr}[\rho(\log \rho - \log \sigma)] \geq 0, \tag{32}$$

are the respective Shannon and von Neumann relative entropies for two probability distributions  $p, q$  and two density operators  $\rho, \sigma$  (defined to be infinite if the support of  $p$  does not lie in the support of  $q$  and similarly for  $\rho$  and  $\sigma$ ). The inequality in Equation (31), corresponding to the Holevo bound (25), is a direct consequence of the data processing inequality [30,31].

Now, the relative entropies appearing in Equation (31) have the alternate form

$$D(p_{AX} \| p_A p_X) = \inf_{q_A} D(p_{AX} \| q_A p_X), \quad D(\rho_{\mathcal{E}X} \| \rho_{\mathcal{E}} \otimes \rho_X) = \inf_{\sigma_{\mathcal{E}}} D(\rho_{\mathcal{E}X} \| \sigma_{\mathcal{E}} \otimes \rho_X) \tag{33}$$

(using, e.g.,  $D(\rho_{\mathcal{E}X} \| \sigma_{\mathcal{E}} \otimes \rho_X) = D(\rho_{\mathcal{E}X} \| \rho_{\mathcal{E}} \otimes \rho_X) + D(\rho_{\mathcal{E}} \| \sigma_{\mathcal{E}})$ ). Hence, if suitable definitions of classical and quantum Rényi relative entropies  $D_\alpha(p\|q)$  and  $D_\alpha(\rho\|\sigma)$  are available and satisfy a data processing inequality, then one can define a classical Rényi mutual information  $I_\alpha(A : X)$  [12,13] and associated Holevo quantity  $\chi_\alpha(\mathcal{E})$  [14,15,36] (also called the quantum Rényi mutual information), satisfying a Rényi Holevo bound, via

$$I_\alpha(A : X) := \inf_{q_A} D_\alpha(p_{AX} \| q_A p_X) \leq \inf_{\sigma_{\mathcal{E}}} D_\alpha(\rho_{\mathcal{E}X} \| \sigma_{\mathcal{E}} \otimes \rho_X) =: \chi_\alpha(\mathcal{E}). \tag{34}$$

Further, suitable definitions of  $D_\alpha(p||q)$  and  $D_\alpha(\rho||\sigma)$  do indeed exist, given by the classical Rényi relative entropy [37,38]

$$D_\alpha(p||q) := \frac{1}{\alpha - 1} \log \int dz p(z)^\alpha q(z)^{1-\alpha} \geq 0 \tag{35}$$

and the quantum sandwiched Rényi relative entropy [14,39]

$$D_\alpha(\rho||\sigma) := \frac{1}{\alpha - 1} \log \text{tr}[(\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}})^\alpha] \geq 0. \tag{36}$$

These reduce to standard relative entropies in the limit  $\alpha \rightarrow 1$ , vanish for  $p = q$  and for  $\rho = \sigma$ , and satisfy the data processing inequality

$$D_\alpha(\nu(\rho)||\nu(\sigma)) \leq D_\alpha(\rho||\sigma), \quad \alpha \geq \frac{1}{2} \tag{37}$$

for all completely positive trace preserving maps  $\nu$  [40]. Further properties and operational interpretations of sandwiched relative entropies are given in References [14,15,39–42].

Finally, note that while the rewriting of  $\chi(\mathcal{E})$  in Equation (31) and the definition of  $\chi_\alpha(\mathcal{E})$  in Equation (34) are perfectly valid for the case of a discrete set of signal states (with integration replaced by summation), there is an important point of rigour to be considered for the case of a continuous set of signal states. In particular, kets  $\{|x\rangle\}$  forming an orthogonal basis set for this case are not normalisable, with  $\langle x|x'\rangle = \delta(x - x')$ , so that  $\rho_X$  and  $\rho_{\mathcal{E}X}$  are not well-defined density operators. This point may be addressed by working with discrete signal ensembles, with  $p(x)$  and  $\rho_x$  replaced by  $p_j$  and  $\rho_j$ , via  $p_j \rho_j = \int_{X_j} dx p(x) \rho_x$  for some countable (possibly finite) partition  $\{X_j\}$  of the range of  $x$ . The existence of a suitable orthonormal basis  $\{|x_j\rangle\}$  is then assured; the integrals defining  $\rho_X$  and  $\rho_{\mathcal{E}X}$  can be replaced by summations over the index  $j$ ; and  $\chi_\alpha(\mathcal{E})$  in Equation (34) rigorously defined as the limit (or supremum) of a suitable sequence of such partitions. (In particular, any such discrete partition  $\mathcal{E}_p \equiv \{p_j; \rho_j\}$  of  $\mathcal{E}$ , with associated orthonormal basis  $\{|x_j\rangle\}$ , can be subpartitioned as  $\mathcal{E}_{p'} \equiv \{p_{jk}; \rho_{jk}\}$  and basis  $\{|x_j\rangle \otimes |k\rangle\}$ , with  $p_j \rho_j = \sum_k p_{jk} \rho_{jk}$ . Data processing inequality (37) then gives  $\chi_\alpha(\mathcal{E}_p) \leq \chi_\alpha(\mathcal{E}_{p'})$ , and so  $\chi_\alpha(\mathcal{E})$  may be defined as the supremum over all such partitions, analogously to Shannon’s general definition of mutual information in Appendix 7 of Reference [43]. A possible alternative approach for continuous ensembles would be to work more generally with  $C^*$ -algebras, where commuting algebras of this type have states formally corresponding to the kets  $|x\rangle$ , and relative entropies are well-defined and satisfy a data-processing inequality [44].) An approach of this type is used, for example, in Reference [17]. Hence, in what follows, the notation in the definition of  $\chi_\alpha(\mathcal{E})$  in Equation (34) will be informally used for both discrete and continuous ensembles, under the implicit assumption that the limiting approach is applied for the continuous case.

### 3.3. Rényi Asymmetry and Upper Bounds for Mutual Information

To generalise the asymmetry bound in Equation (26), one may follow the general approach of Gao et al. [16] to define the Rényi asymmetry of state  $\rho$ , with respect to any Hermitian operator  $G$  with a discrete spectrum, by [17],

$$A_\alpha^G(\rho) := \inf_{\sigma: [\sigma, G]=0} D_\alpha(\rho||\sigma), \quad \alpha \geq \frac{1}{2}. \tag{38}$$

This reduces to  $A^G(\rho)$  in Equation (26) for  $\alpha = 1$ , via the identity  $D(\rho||\sigma) = H(\rho_G) - H(\rho) + D(\rho_G||\sigma)$  for  $[\sigma, G] = 0$ . For later purposes, it is worth noting that, since any state commuting with  $G$  must also commute with  $h(G)$  for any function  $h$ ,

$$A_\alpha^{h(G)}(\rho) = \inf_{\sigma: [\sigma, h(G)]=0} D_\alpha(\rho||\sigma) \leq \inf_{\sigma: [\sigma, G]=0} D_\alpha(\rho||\sigma) = A_\alpha^G(\rho). \tag{39}$$

If relative entropies are regarded as quasi-distances, then the Rényi asymmetry may be interpreted as the distance from  $\rho$  to the closest state that is invariant under the unitary transformation  $U_x$  for all  $x$ , i.e., that commutes with  $G$ . In particular, it vanishes if  $\rho$  itself is invariant i.e., if  $[\rho, G] = 0$ , implying that the asymmetry may also be interpreted as the inherent ‘quantum’ uncertainty of  $G$  [45] (see also Section 5). Further, when  $G$  is nondegenerate, the asymmetry is a measure of coherence, relative to the eigenbasis of  $G$  [18]. This measure corresponds to the relative entropy of coherence for  $\alpha = 1$  [19], and it is of interest to note that

$$A_{1/2}^G(\rho) = -\log[1 - C_g(\rho)], \quad A_\infty^G(\rho) = \log[1 + C_R(\rho)], \quad (40)$$

for nondegenerate  $G$ , where  $C_g(\rho)$  and  $C_R(\rho)$  are the geometric coherence [19] and robustness of coherence [18,19], respectively. However, the utility of asymmetry for the purposes of this paper arises from the following upper bounds on the Rényi Holevo quantity.

**Theorem 3.** For any Hermitian operator  $G$  having a discrete spectral decomposition  $G = \sum_k g_k P_k$ , the Rényi asymmetry  $A_\alpha^G(\rho)$  of an ensemble of signal states  $\mathcal{E} \equiv \{\rho_x = e^{-ixG} \rho e^{ixG}; p(x)\}$  has the upper and lower bounds

$$\chi_\alpha(\mathcal{E}) \leq A_\alpha^G(\rho) \leq H_\beta(G|\rho), \quad \frac{1}{\alpha} + \frac{1}{\beta} = 2, \quad (41)$$

where  $\chi_\alpha(\mathcal{E})$  is the Rényi Holevo quantity in Equation (34) and  $H_\beta(G|\rho)$  is the classical Rényi entropy of the probability distribution  $p(g_k|\rho) = \text{tr}[\rho P_k]$ . The upper bound is saturated when  $\rho$  is pure.

Theorem 3 clearly generalises the information bounds in Equation (26) for standard entropies, which correspond to the special case  $\alpha = 1$ . The lower bound  $\chi_\alpha(\mathcal{E}) \leq A_\alpha^G(\rho)$  also generalises energy-time uncertainty relation (10) of Reference [17], from the special case of a uniform prior distribution to arbitrary prior distributions (see also Section 5). Similarly to Reference [17], the lower bound is a simple consequence of the fact that the transformation mapping  $\rho_x \otimes |x\rangle\langle x|$  to  $\rho \otimes |x\rangle\langle x|$  is a reversible isometry. In particular, data processing inequality (37) implies that  $D_\alpha(\rho_{\mathcal{E}X} \|\sigma_{\mathcal{E}} \otimes \rho_X) = D_\alpha(U\rho_{\mathcal{E}X}U^\dagger \|\ U\sigma_{\mathcal{E}} \otimes \rho_XU^\dagger)$  for the unitary transformation  $U = \int dx U_x^\dagger \otimes |x\rangle\langle x|$ . This transformation maps  $\rho_{\mathcal{E}X} \rightarrow \rho \otimes \rho_X$ , and  $\sigma_{\mathcal{E}} \otimes \rho_X \rightarrow \sigma_{\mathcal{E}} \otimes \rho_X$  when  $[\sigma_{\mathcal{E}}, G] = 0$ , and hence

$$\chi_\alpha(\mathcal{E}) \leq \inf_{\sigma_{\mathcal{E}}: [\sigma_{\mathcal{E}}, G]=0} D_\alpha(\rho_{\mathcal{E}X} \|\sigma_{\mathcal{E}} \otimes \rho_X) = \inf_{\sigma_{\mathcal{E}}: [\sigma_{\mathcal{E}}, G]=0} D_\alpha(\rho \otimes \rho_X \|\sigma_{\mathcal{E}} \otimes \rho_X) = A_\alpha^G(\rho), \quad (42)$$

as desired, where the inequality follows immediately from the definition of  $\chi_\alpha(\mathcal{E})$  in Equation (34). The remainder of Theorem 3 is proved in Appendix C.

The upper bound for asymmetry in Equation (41) has the benefit of being simpler to calculate than the asymmetry itself, and will be seen to underlie the constraint on  $\alpha$  and  $\beta$  in Rényi uncertainty relation (4). It is a consequence of the duality property

$$A_\alpha^G(|\psi\rangle\langle\psi|) = H_\beta(|\psi\rangle\langle\psi|_G), \quad \frac{1}{\alpha} + \frac{1}{\beta} = 2, \quad (43)$$

for the asymmetry of pure states, also proved in Appendix C, which in turn is formally connected to a deeper (and much harder to prove) duality relation for conditional Rényi entropies [39,41].

Finally, it is of interest to note there is an alternative representation of asymmetry that relates it more directly to the Holevo quantity  $\chi_\alpha$ . In particular, defining the continuous ensemble  $\mathcal{E}_r \equiv \{\rho_x; p_r(x)\}$  by  $p_r(x) := 1/(2r)$  for  $|x| < r$  and vanishing otherwise, it follows that  $\lim_{r \rightarrow \infty} \rho_{\mathcal{E}_r} = \rho_G$  (see Appendix C). Hence, using Equation (25),  $\lim_{r \rightarrow \infty} \chi(\mathcal{E}_r) = S(\rho_G) - S(\rho) = A^G(\rho)$ . Thus, the standard asymmetry is equal to the

Holevo quantity in the limiting case of a maximally uniform ensemble of signal states [4]. This property generalises to all Rényi asymmetries, i.e.,

$$\chi_\alpha^\infty := \lim_{r \rightarrow \infty} \chi_\alpha(\mathcal{E}_r) = A_\alpha^G(\rho), \tag{44}$$

as shown in Appendix C. In the case that  $U_x$  is periodic with period  $x_p$ , the limiting ensemble can be replaced by the uniform ensemble over  $[0, x_p)$ , as noted in References [32–34] for  $\alpha = 1$ .

### 3.4. A Convolution Lower Bound for Mutual Information

Unlike the Holevo quantity  $\chi_\alpha(\mathcal{E})$ , the classical Rényi mutual information  $I_\alpha(A : X)$  in Equation (34) can be calculated explicitly [12,13]. In particular, Equations (34) and (35) give

$$I_\alpha(A : X) = \inf_{q_A} \frac{1}{\alpha - 1} \log \int da q_A(a)^{1-\alpha} r_\alpha(a), \quad r_\alpha(a) := \int dx p(x) p(a|x)^\alpha, \tag{45}$$

with integration replaced by summation over discrete ranges of  $A$  and  $X$ , and  $p(a|x) = \text{tr}[\rho_x M_a]$  for a measurement of observable  $A$  described by the POVM  $\{M_a\}$ . A straightforward variation with respect to  $q_A$  under the constraint  $\int da q(a) = 1$  then yields [12,13]

$$I_\alpha(A : X) = H_{\frac{1}{\alpha}}(\tilde{p}_A) + \frac{1}{1-\alpha} \log \int da r_\alpha(a), \quad \tilde{p}_A(a) := r_\alpha(a) / \int da r_\alpha(a). \tag{46}$$

It may be checked that this expression reduces to  $I(A : X)$  in Equation (24) in the limit  $\alpha \rightarrow 1$ , as expected.

However, while this expression allows the mutual information to be explicitly calculated for arbitrary observables  $A$ , the statistical characterisation of the error  $X_{\text{est}} - X$  of some estimate  $X_{\text{est}}$  of  $X$  requires an expression involving the error probability density

$$p_{\text{err}}(y) = \int dx p_{X_{\text{est}}X}(x + y, x), \tag{47}$$

where  $p_{X_{\text{est}}X}(x_{\text{est}}, x) = p(x) \text{tr}[\rho_x M_{x_{\text{est}}}]$  is the joint probability density of  $X_{\text{est}}$  and  $X$  (akin to Equation (A2) of Appendix A). For Shannon mutual information this requirement is achieved via the lower bound in Equation (27), which is partially generalised here via the following theorem and corollary.

**Theorem 4.** *The classical Rényi mutual information, for an estimate  $X_{\text{est}}$  of  $X$  made on the ensemble  $\mathcal{E} \equiv \{\rho_x; p(x)\}$ , has the lower bound*

$$I_\alpha(X_{\text{est}} : X) \geq \inf_q D_\alpha(p_{\text{err}} \| q * p_-), \quad \alpha \geq \frac{1}{2}, \tag{48}$$

where  $p_{\text{err}}(y)$  is the probability density of the error  $X_{\text{est}} - X$ ;  $(f * g)(y) = \int dx f(x)g(y - x)$  denotes the convolution of functions  $f$  and  $g$ ; and  $p_-(x) := p(-x)$ .

Theorem 4 is proved in Appendix D, and the lower bound in Equation (48) will be referred to as the convolution lower bound. This bound has the desirable property of depending only on the prior probability density  $p(x)$  and the error probability density  $p_{\text{err}}$ , similarly to Equation (27) for Shannon mutual information.

**Corollary 2.** *If the prior probability density  $p(x)$  is uniform on an interval  $I$  of length  $\ell_I$ , and vanishes outside this interval, then*

$$I_\alpha(X_{\text{est}} : X) \geq \log \ell_I - H_\alpha(X_{\text{est}} - X | \rho), \quad \alpha \geq \frac{1}{2}. \tag{49}$$

Note for  $\alpha = 1$  that this result is equivalent to Equation (27) in the case that  $p(x)$  is uniform on some interval  $I$ . Corollary 2 is proved in Appendix D, and relies on the following Lemma, which is of some interest in its own right.

**Lemma 1.** *If  $Z$  is a random variable on the real line, and  $\tilde{Z} = Z \bmod I$  for some interval  $I$ , then*

$$H_\alpha(\tilde{Z}) \leq H_\alpha(Z), \quad \alpha \geq 0. \tag{50}$$

This lemma corresponds to the intuition that concentrating a probability density onto an interval will reduce the spread of distribution, and is proved in the last part of Appendix D. Note that if  $Z$  is periodic, with period equal to  $\ell_I$ , then Lemma 1 holds trivially with equality (noting that entropies are translation invariant).

### 3.5. Putting It All Together: Strengthened Metrology Bounds and Uncertainty Relations

Combining the Rényi Holevo bound,  $I_\alpha(X_{\text{est}} : X) \leq \chi_\alpha(\mathcal{E})$  in Equation (34), with the upper bounds for  $\chi_\alpha(\mathcal{E})$  in Theorem 3 and the lower bound for  $I_\alpha(X_{\text{est}} : X)$  in Corollary 2 yields the inequality chain

$$\log \ell_I - H_\alpha(X_{\text{est}} - X|\rho) \leq I_\alpha(X_{\text{est}} : X) \leq \chi_\alpha(\mathcal{E}) \leq A_\alpha^G(\rho) \leq H_\beta(G|\rho), \quad \frac{1}{\alpha} + \frac{1}{\beta} = 2, \tag{51}$$

connecting the entropy of the estimation error to the asymmetry and entropy of the generator  $G$ , for the case of a prior distribution of the shift parameter that is uniform over some interval  $I$ . This chain immediately implies the following general result.

**Theorem 5.** *For any estimate  $X_{\text{est}}$  of a unitary displacement  $X$  applied to a probe state  $\rho$ , where the displacement is generated by a Hermitian operator  $G$  with a discrete spectrum and has a uniform prior distribution over an interval  $I$  of length  $\ell_I$ , the entropy of the estimation error  $X_{\text{est}} - X$  satisfies the tradeoff relation*

$$H_\alpha(X_{\text{est}} - X|\rho) + A_\alpha^G(\rho) \geq \log \ell_I, \quad \alpha \geq \frac{1}{2}. \tag{52}$$

Further, for any nonlinear displacement generated by  $h(G)$ , the tradeoff relation

$$H_\alpha(X_{\text{est}} - X|\rho) + H_\beta(G|\rho) \geq \log \ell_I, \quad \frac{1}{\alpha} + \frac{1}{\beta} = 2, \tag{53}$$

holds for any function  $h$ .

Equation (52) of the theorem generalises the metrology bounds and uncertainty relations in Section 2 to estimates of general unitary displacements, and strengthens them to take into account prior information and any nonpurity of the state. Equation (53), following via asymmetry property (39), shows that a nonlinear generator gives no advantage, in the sense that the Rényi entropy of the estimation error is lower-bounded by  $\log \ell_I - H_\beta(G|\rho)$  for all generators  $h(G)$ , similarly to the case of Shannon entropy [4].

The content of Theorem 5 has the advantage of being independent of, and hence not requiring, any interpretation of the Rényi mutual information  $I_\alpha$  and the Holevo quantity  $\chi_\alpha$ . The following four corollaries provide simple applications of the theorem to RMSE and to phase shifts, with further applications discussed in Section 4.

**Corollary 3.** *For any estimate  $X_{\text{est}}$  of a unitary displacement  $X$  applied to a probe state  $\rho$ , where the displacement is generated by a Hermitian operator  $G$  with discrete spectral decomposition  $\sum_k g_k P_k$ , and  $X$  has a uniform prior distribution over an interval  $I$  of length  $\ell_I$ , the root-mean-square error of the estimate,  $\text{RMSE} = \langle (X_{\text{est}} - X)^2 \rangle^{1/2}$ , has the lower bounds*

$$\text{RMSE} \geq \frac{\alpha^{\frac{\alpha}{\alpha-1}} f(\alpha) \ell_I}{2\pi} e^{-A_\alpha^G(\rho)}, \quad \alpha \geq \frac{1}{2}, \tag{54}$$

(using units of nats for  $A_\alpha^G(\rho)$ ), where  $f(\alpha)$  is the function defined in Equation (18). For the particular choices  $\alpha = \infty$ ,  $\alpha = \frac{1}{2}$  and  $\alpha \rightarrow 1$ , one further has

$$\text{RMSE} \geq \frac{\ell_I}{2\sqrt{3} L_{1/2}(G|\rho)}, \quad \text{RMSE} \geq \frac{\ell_I}{2\pi} \max_k p(g_k|\rho) \quad \text{RMSE} \geq \frac{\ell_I}{\sqrt{2\pi e}} e^{H(\rho) - H(\rho_G)}. \tag{55}$$

where  $L_\alpha(G|\rho)$  is a Rényi length as per Equation (10).

The lower bound in Equation (54) of Corollary 3 is sensitive to the purity of the probe state via  $A_\alpha^G(\rho)$ , and to the prior information via  $\ell_I$ , and follows by noting that the derivation of Equation (A11) in Appendix B goes through for the choice  $\sigma^2 = \text{RMSE}$ , and applying Theorem 5. The first two inequalities in Equation (55) then follow via the upper bound for asymmetry in Equation (51), and generalise the corresponding bounds in Theorem 2 to include prior information about the estimate. The third inequality follows via the expression for asymmetry in Equation (26) for  $\alpha = 1$ , and generalises Equation (30) for phase and photon number to arbitrary discrete generators. Note that all lower bounds in Corollary 3 scale in proportion to the length of the interval,  $\ell_I$ , to which the displacement  $X$  is restricted. Thus, better estimates are possible when more prior information is available.

**Corollary 4.** *The root-mean-square error for any estimate of a phase shift  $\Theta$  with a uniform prior distribution on an interval  $I$  with length  $\ell_I$ , via a measurement on probe state  $\rho$ , satisfies the strong Heisenberg limit*

$$\text{RMSE} \geq \frac{\ell_I}{2\pi} \frac{f_{\max}}{\langle N \rangle + \frac{1}{2}}, \tag{56}$$

where  $f_{\max} \approx 0.5823$  is the maximum value of the function  $f(\alpha)$  in Equation (18).

Corollary 4 is a direct consequence of Equation (54) in Corollary 3 for  $G = N$ , using  $A_\alpha^G(\rho) \leq H_\beta(G|\rho)$  from Equation (51) and applying upper bound (A17) from Appendix B. It generalises the strong Heisenberg limit in Equations (3) and Theorem 2 to include prior information about the phase shift.

**Corollary 5.** *The number and canonical phase observables  $N$  and  $\Phi$  satisfy the family of uncertainty relations previewed in Equation (7), i.e.,*

$$A_\alpha^N(\rho) + H_\alpha(\Phi|\rho) \geq \log 2\pi, \quad \alpha \geq \frac{1}{2}. \tag{57}$$

Corollary 5 follows from Theorem 5 for  $G = N$ ,  $\ell_I = 2\pi$ , and the particular choice of estimate  $\Theta_{\text{est}} = \Phi$ , via  $H(\Theta_{\text{est}} - \Theta|\rho) = H(\Phi|\rho)$  on the unit circle. For  $\alpha = 1$  it is equivalent to uncertainty relation (1) for Shannon entropies when  $\rho$  is a single mode state (noting that  $A_1^G(\rho) = H(G|\rho) - H(\rho)$  for nondegenerate  $G$  as per Equation (26)), and more generally strengthens uncertainty relation (4) for Rényi entropies to take the degree of purity of the state into account.

**Corollary 6.** *The number and canonical phase observables  $N$  and  $\Phi$  satisfy the family of uncertainty relations*

$$A_\alpha^N(\rho) \Delta_\chi \Phi \geq \alpha^{\frac{\alpha}{\alpha-1}} f(\alpha), \quad \alpha \geq \frac{1}{2}, \tag{58}$$

where the function  $f(\alpha)$  and the standard deviation  $\Delta_\chi \Phi$  are defined in Equations (18) and (20).

Corollary 6 follows via an analogous argument to the proof of Corollary 1 given in Appendix B, but using the uncertainty relation of Corollary 5 in place of Equation (4). It strengthens Equation (21) of Corollary 1 when the state is non-pure.

#### 4. Applications to Coherence Measures, Rotations, and Energy-Time Tradeoffs

The results of the previous section have utility well beyond the number-phase examples given therein, as indicated here via several further applications.

##### 4.1. Coherence Bounds

Recall, as per the discussion in Section 3.3, that for nondegenerate  $G$  the Rényi asymmetry  $A_\alpha^G(\rho)$  is a measure of coherence, relative to the eigenbasis of  $G$  [18]. Coherence measures are typically difficult to calculate explicitly, other than for pure states and general qubit states [18,19]. However, the results of Section 2 lead to simple and strong upper and lower bounds for Rényi-related measures of coherence, as per the following theorem and corollary.

**Theorem 6.** *The Rényi entropy of coherence for state  $\rho$ ,  $C_\alpha(\rho) := A_\alpha^M(\rho)$ , relative to a given orthonormal basis  $\{|m\rangle\}$  indexed by a countable set of integers, has the upper and lower bounds*

$$\log 2\pi - H_\alpha(\Phi_\zeta|\rho) \leq C_\alpha(\rho) \leq H_\beta(M|\rho), \quad \frac{1}{\alpha} + \frac{1}{\beta} = 2, \quad (59)$$

where  $M = \sum_m m|m\rangle\langle m|$ ,  $\zeta \equiv \{\zeta_m\}$  is an arbitrary set of reference phases, and  $\Phi_\zeta$  denotes the continuous phase observable corresponding to the POVM  $\{|\phi\rangle\langle\phi|\}$  defined by

$$|\phi\rangle := \frac{1}{\sqrt{2\pi}} \sum_m e^{i\zeta_m} e^{-im\phi} |m\rangle. \quad (60)$$

The upper bound is saturated for pure states.

**Corollary 7.** *The relative entropy of coherence, geometric coherence and robustness of coherence, with respect to basis  $\{|m\rangle\}$ , satisfy the respective bounds*

$$\log 2\pi - H(\Phi_\zeta|\rho) \leq C_{\text{rel ent}}(\rho) = H(M|\rho) - H(\rho), \quad (61)$$

$$1 - \frac{L_{1/2}(\Phi_\zeta|\rho)}{2\pi} \leq C_g(\rho) \leq 1 - \max_m \langle m|\rho|m\rangle, \quad (62)$$

$$2\pi \sup_{\phi \in [0, 2\pi)} p(\phi|\rho) - 1 \leq C_R(\rho) \leq \left( \sum_n \langle m|\rho|m\rangle^{1/2} \right)^2 - 1 \quad (63)$$

for state  $\rho$ , where  $L_\alpha$  denotes the Rényi length in Equation (10). The upper bounds are saturated for pure states.

The lower bounds in Theorem 6 and Corollary 7 hold for both finite and infinite Hilbert spaces, and follow in direct analogy to Corollary 5, noting that  $M$  is invariant under  $|m\rangle \rightarrow e^{i\zeta_m} |m\rangle$  and using Equations (8), (26) and (40). Note also that  $H_\alpha(\Phi_\zeta|\rho)$  may be replaced by  $\inf_\zeta H_\alpha(\Phi_\zeta|\rho)$ , and that a weaker lower bound,  $C_\alpha(\rho) \geq \log \alpha^{\frac{\alpha}{\alpha-1}} f(\alpha) - \log \Delta_\chi \Phi_\zeta$ , follows via Corollary 6. The upper bounds follow directly from Theorem 3, using Equations (26) and (40) and recalling that  $H_\beta(\rho_G) = H_\beta(G|\rho)$  for nondegenerate  $G$ . It follows that a low phase uncertainty implies high coherence, which in turn requires a large uncertainty in  $G$ .

The bounds are relatively strong. The lower bounds are tight for all mixtures of number states, i.e., with zero coherence (noting that  $p(\phi|\rho) = (2\pi)^{-1}$  for such states). Further, the upper bounds are saturated for all pure states, and are the strongest possible upper bounds that depend only on the distribution  $p_m = \langle m|\rho|m\rangle$ , being saturated for the pure state  $\sum_m \sqrt{p_m} |m\rangle$  in particular. Thus, for example, for an optical mode and the choice  $M = N$ , the coherent phase states  $|v\rangle = (1 - |v|^2)^{1/2} \sum_n v^n |n\rangle$  (with  $|v| < 1$ ) not only have excellent phase resolution properties in general [24], but also have the highest possible coherence for given average photon number. Moreover, for  $\alpha = 1$  the bounds are equivalent to uncertainty relation (1) for Shannon entropies, and more generally imply, and hence are stronger than, the uncertainty relation

$$H_\alpha(M|\rho) + H_\beta(\Phi_\zeta|\rho) \geq \log 2\pi, \quad \frac{1}{\alpha} + \frac{1}{\beta} = 2 \tag{64}$$

for Rényi entropies, generalising Equation (4) to all dimensions.

The upper bound for geometric coherence in Equation (62) recovers the upper bound in Theorem 1 of Reference [46]. Further, the lower bound is typically stronger than the corresponding bound in Reference [46], as it depends on both the diagonal and off-diagonal elements of  $\rho$ . For example, for the maximally coherent qubit state  $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$  one obtains  $C_g(\rho) \geq 1 - 8/\pi^2 \approx 0.189$  from Equation (62) (with  $\zeta_n \equiv 0$ ), whereas Theorem 1 of Reference [46] gives a trivial lower bound of zero. Note that  $C_g(\rho) = \frac{1}{2}$  for this state, recalling that upper bound (62) is saturated for pure states.

For the coherence of robustness, the upper bound  $C_R(\rho) \leq \sum_{m,m'} |\langle m|\rho|m'\rangle| - 1$  in Reference [47] is stronger than the upper bound in Equation (63) for nonpure states (since the latter follows from the former via the Schwarz inequality). However, as noted above, Equation (63) gives the strongest possible upper bound that depends only on the distribution of  $M$ . Further, the lower bound in Equation (63) is typically stronger than the corresponding bound in Reference [47]. For example, for the maximally coherent qubit state  $|\psi\rangle$  in the above paragraph, both the lower bound in Equation (63) and the lower bound in Theorem 5 of Reference [47] are saturated, with a value of unity. However, for the coherent phase state  $|v\rangle$  mentioned above, the lower bound in Equation (63) is again saturated, with a value  $2|v|/(1 - |v|)$  following from Equation (37a) of Reference [24], whereas Theorem 5 of Reference [47] gives a trivial lower bound of zero. It would be of interest to compare these two lower bounds further, and to investigate the lower bound in Theorem 6 more generally.

#### 4.2. Rotations

Two important applications of quantum metrology are the estimation of phase shifts, generated by the photon number operator, and the estimation of rotation angles, generated by angular momentum. In the latter case, for example, the strength of a magnetic field may be estimated via the rotation of an ensemble of atomic spins [48].

The formal differences between phase and rotation estimation are small. For example, rotations of GHZ states of  $M$  spin qubits,  $\frac{1}{\sqrt{2}}(\otimes^M |\uparrow\rangle + \otimes^M |\downarrow\rangle)$ , are formally equivalent to phase shifts of single-mode states  $\frac{1}{\sqrt{2}}(|0\rangle + |M\rangle)$  discussed following Theorems 1 and 2, and to phase shifts of two-mode NOON states  $\frac{1}{\sqrt{2}}(|M, 0\rangle + |0, M\rangle)$  [48].

In fact the only significant formal difference between phase estimation and rotation estimation is that the eigenvalues of the photon number operator  $N$  are nonnegative integers, whereas the eigenvalues of an angular momentum component  $J_z$  range over all positive and negative integers. Thus, for example, the optical phase kets in Equation (8) for the canonical phase observable  $\Phi$  are replaced by the rotation kets

$$|\phi_z\rangle := \frac{1}{\sqrt{2\pi}} \sum_{j=-\infty}^{\infty} e^{-ij\phi_z} |j\rangle \tag{65}$$

for the rotation angle observable  $\Phi_z$  conjugate to  $J_z$ , where  $\{|j\rangle\}$  denote the eigenstates of  $J_z$ . Note that  $\Phi_z$  corresponds to a Hermitian operator, with  $\langle \phi_z | \phi'_z \rangle = \delta(\phi_z - \phi'_z)$ .

Given that the general result in Theorem 5 holds for general discrete generators  $G$ , and noting that  $J_z$  and  $N$  only differ in their range of eigenvalues, it follows that all results for phase shifts not directly dependent on properties of the range of  $N$  yield corresponding results for rotations, via the replacement of  $N$  by  $J_z$  and  $\Phi$  by  $\Phi_z$ . Thus, for example, Rényi uncertainty relation (4) holds for angular momentum and angle [8,9], as does Theorem 1 and the metrology tradeoff relation

$$H_\alpha(\Theta_{z,\text{est}} - \Theta_z|\rho) + A_\alpha^{J_z}(\rho) \geq \log \ell_I, \quad \alpha \geq \frac{1}{2} \tag{66}$$

corresponding to Theorem 5. It follows that the estimation error, as characterised by its Rényi entropy, can only be small if the corresponding asymmetry of the state is large. In particular, a GHZ state of  $M$  spin qubits has a relatively low asymmetry, with  $A_\alpha^{J_z}(\rho) = \log 2$  via duality property (43) for pure states, and hence has a relatively poor angular resolution [4].

Further, since the relation between RMSE and entropy in Equation (A11) of Appendix B holds independently of the generator, all results for the RMSE of phase shifts and the standard deviation  $\Delta_\chi\Phi$  of optical phase that do not depend on the eigenvalues of  $N$  yield corresponding results for rotations. Thus, for example, the first two inequalities in Theorem 2 and the lower bounds in Corollaries 3 and 6 hold for angular momentum and angle, as does, e.g., the uncertainty relation

$$\Delta_\chi\Phi_z \geq \max_j p(j|\rho), \tag{67}$$

corresponding to the second equality in Equation (22) of Corollary 1. This implies, for example, that a GHZ state of  $M$  spin qubits has a relatively large standard deviation, with  $\Delta_\chi\Phi_z \geq \frac{1}{2}$ .

Indeed, the only cases in which earlier results for number and phase do not immediately translate into results for angular momentum and angle are those involving the average photon number, such as the third uncertainty relation in Corollary 1 and the strong Heisenberg limits in Equations (3) and (5), Theorem 2 and Corollary 4. This is because such results rely on the upper bound for Rényi length in Equation (A17) in Appendix B, which assumes positive eigenvalues and so must be modified for the case of angular momentum. A suitable modification is given by

$$L_\beta(J_z|\rho) \leq \alpha^{\frac{\alpha}{\alpha-1}} \left( 2\langle |J_z| \rangle + \frac{1}{2}\langle 0|\rho|0 \rangle \right) \leq \alpha^{\frac{\alpha}{\alpha-1}} \left( 2\langle |J_z| \rangle + \frac{1}{2} \right), \quad \frac{1}{\alpha} + \frac{1}{\beta} = 2, \tag{68}$$

as shown in the last part of Appendix B. This leads immediately to the following result for angle estimation, corresponding to Corollary 4.

**Corollary 8.** *The root-mean-square error for any estimate of a rotation  $\Theta$  with a uniform prior distribution on an interval  $I$  with length  $\ell_I$ , via a measurement on probe state  $\rho$ , satisfies the strong Heisenberg limits*

$$\text{RMSE} \geq \frac{\ell_I}{2\pi} \frac{f_{\max}}{2\langle |J_z| \rangle + \frac{1}{2}\langle 0|\rho|0 \rangle} \geq \frac{\ell_I}{2\pi} \frac{f_{\max}}{2\langle |J_z| \rangle + \frac{1}{2}}, \tag{69}$$

where  $f_{\max} \approx 0.5823$  is the maximum value of the function  $f(\alpha)$  in Equation (18).

The lower bounds in Corollary 8 improve on the Heisenberg limit given in endnote [29] of Reference [5], in both the numerators and denominators, in addition to including prior information about the rotation via the factor  $\ell_I/(2\pi)$ .

### 4.3. Energy and Time

The results of Section 3 also apply straightforwardly to the time evolution of quantum systems with discrete energy levels. For example, any estimate  $T_{\text{est}}$  of a time translation  $T$  generated by a Hamiltonian with discrete spectral decomposition  $E = \sum_k E_k P_k$ , for uniform prior probability density  $p(t) = 1/\ell_I$  over an interval of length  $\ell_I$ , satisfies the tradeoff relation

$$H(T_{\text{est}} - T|\rho) + A_\alpha^E(\rho) \geq \log \ell_I, \quad \alpha \geq \frac{1}{2}, \tag{70}$$

as an immediate consequence of Theorem 5. Further, the RMSE of the estimate satisfies the lower bounds in Corollary 3 for  $G = E$ .

If the system is periodic, with period  $\tau = 2\pi/\omega$ , the energy eigenvalues are of the form

$$E_k = \epsilon + \hbar\omega n_k, \quad (71)$$

where  $\epsilon$  denotes the groundstate energy and the  $n_k$  are nonnegative integers. Hence a time translation of the system by an amount  $t$  is formally identical to a phase shift of an optical mode by  $\phi = \omega t$  (for a state of the mode with support restricted to the number states  $\{|n_k\rangle\}$ ), and all previous results for number and phase carry over immediately to analogous results for the energy and time of periodic systems via the replacement of  $\phi$  by  $\omega t$  and  $N$  by  $(E - \epsilon)/(\hbar\omega)$ . For example, Corollary 4 implies the strong Heisenberg limit

$$\text{RMSE} = \langle (T_{\text{est}} - T)^2 \rangle^{1/2} \geq \frac{\hbar\ell_I}{\tau} \frac{f_{\text{max}}}{\langle E - \epsilon \rangle + \frac{1}{2}\hbar\omega} \quad (72)$$

for the RMSE for any estimate of the time shift of a periodic system, if the prior probability density  $p(t)$  is uniform over an interval of length  $\ell_I$ , which strengthens the result in Reference [29] for this case. Similarly, Corollaries 5 and 6 imply the strong energy-time uncertainty relations

$$A_\alpha^E(\rho) + H_\alpha(\mathcal{T}_\tau|\rho) \geq \log \tau, \quad A_\alpha^E(\rho)\Delta_{t_0}\mathcal{T}_\tau \geq \frac{\alpha^{\frac{\alpha}{\alpha-1}}f(\alpha)}{2\pi} \tau, \quad \alpha \geq \frac{1}{2} \quad (73)$$

for the energy and canonical time observables of a periodic system with period  $\tau$ , where  $\mathcal{T}_\tau$  denotes the canonical time observable corresponding to  $\Phi/\omega$  [20,22], and  $\Delta_{t_0}\mathcal{T}_\tau = \langle (\mathcal{T}_\tau - t_0)^2 \rangle^{1/2}$  is the standard deviation about any reference time  $t_0$ . Note that the first of these relations, combined with the asymmetry bound in Theorem 3, implies and hence is stronger than the known Rényi entropic uncertainty relation

$$H_\alpha(E|\rho) + H_\beta(\mathcal{T}_\tau|\rho) \geq \log \tau, \quad \frac{1}{\alpha} + \frac{1}{\beta} = 2 \quad (74)$$

for the energy and time observables of periodic systems [49], analogous to Equation (4) for number and phase.

While some quantum systems, such as harmonic oscillators and qubits, are indeed periodic, most systems with discrete Hamiltonians do not have energy eigenvalues of the form in Equation (71) and so are nonperiodic. This is not an issue for the basic energy-time metrology tradeoff relation (70), which is universal for discrete Hamiltonians and so applies equally well to both periodic and nonperiodic systems, as do the bounds for the RMSE of time estimates in Corollary 4 (choosing  $G = E$ ). However, the question of whether there are time-energy uncertainty relations for nonperiodic systems, that generalise Equations (73) and (74) for periodic systems, is less straightforward.

This question has been addressed for the case of Shannon entropies, via the definition of a canonical time observable  $\mathcal{T}$  that is applicable to both periodic and nonperiodic systems. This observable has almost-periodic probability density  $p_{ap}(t)$  associated with it, and a corresponding almost-periodic Shannon entropy  $H^{ap}(\mathcal{T}|\rho)$ , which satisfies the energy-time-energy entropic uncertainty relation [20,35]

$$H(E|\rho) + H^{ap}(\mathcal{T}|\rho) \geq 0. \quad (75)$$

This relation reduces to Equation (74) for the case of periodic systems, and is strengthened and extended below to general Rényi entropies.

To proceed, it is convenient to first deal with any energy degeneracies, by taking the degree of degeneracy to be the same for each energy eigenvalue (by formally extending the Hilbert space if necessary), so that the energy eigenstates can be formally written as  $|E_k\rangle \otimes |d\rangle$  with the range of the degeneracy index  $d$  independent of  $E_k$ . Defining  $1_D$ :

$= \sum_d |d\rangle\langle d|$ , the canonical time observable conjugate to  $E = \sum_k |E_k\rangle\langle E_k| \otimes 1_D$  is then defined via the almost-periodic POVM  $\mathcal{T} \equiv \{M_t\}$  given by

$$M_t := \sum_{k,k'} e^{-i(E_k - E_{k'})t/\hbar} |E_k\rangle\langle E_{k'}| \otimes 1_D \geq 0, \quad t \in (-\infty, \infty), \tag{76}$$

and associated almost-periodic probability density

$$p_{ap}(t|\rho) := \text{tr}[\rho M_t], \quad t \in (-\infty, \infty). \tag{77}$$

for state  $\rho$  [20]. Note the formal similarity to the canonical phase observable in Section 2.1. For periodic systems, the associated periodic time observable  $\mathcal{T}_\tau$  above has the related POVM  $\{\tau^{-1}\mathcal{T}_t : t \in [0, \tau)\}$ , with associated periodic probability density [20]

$$p_\tau(t|\rho) = \tau^{-1} p_{ap}(t|\rho), \quad t \in [0, \tau). \tag{78}$$

Now, it is easy to check that  $p_{ap}(t|\rho)$  is not normalised with respect to the Lebesgue measure, and indeed that  $\int_{-\infty}^{\infty} dt p_{ap}(t|\rho)$  diverges. Hence an alternative measure is required. This is provided by the Besicovitch measure  $\mu_{ap}[\cdot]$ , defined on the algebra of almost-periodic functions, i.e., functions of the form  $f(t) = \sum_j f_j e^{i\omega_j t}$  with  $\sum_j |f_j|^2 < \infty$ , by [50]

$$\mu_{ap}[f] := \limsup_{s \rightarrow \infty} \frac{1}{s} \int_0^s dt f(t) \tag{79}$$

For  $f(t) = p_{ap}(t|\rho)$  in Equation (77) this yields

$$\mu_{ap}[p_{ap}] = \sum_{k,d} \langle E_k, d | \rho | E_k, d \rangle = 1, \tag{80}$$

and hence the almost-periodic density is normalised, as desired. The average of the almost-periodic function  $f(t)$  with respect to  $p_{ap}(t|\rho)$  can then be defined as [20]

$$\langle f \rangle := \mu_{ap}[p_{ap} f] = \sum_{j,k,k': E_k - E_{k'} = \hbar\omega_j} f_j \langle E_{k'} | \text{tr}_D[\rho] | E_k \rangle, \tag{81}$$

and the almost-periodic Rényi entropy of the canonical time observable by

$$H_\alpha^{ap}(\mathcal{T}|\rho) := \frac{1}{1-\alpha} \log \mu_{ap}[(p_{ap})^\alpha] = \frac{1}{1-\alpha} \log \limsup_{s \rightarrow \infty} \frac{1}{s} \int_0^s dt p_{ap}(t|\rho)^\alpha, \tag{82}$$

generalising the case of almost-periodic Shannon entropy [20,35]. For the case of a periodic system with period  $\tau$ , it follows from Equation (78) that

$$H_\alpha^{ap}(\mathcal{T}|\rho) = H_\alpha(\mathcal{T}_\tau|\rho) - \log \tau, \tag{83}$$

where  $H_\alpha$  denotes the Rényi entropy of the periodic probability density  $p_\tau(t|\rho)$ .

Finally, any almost-periodic function  $f(t)$  can be approximated by a periodic function to any desired accuracy, via a sequence of periodic functions  $f_m(t)$  with respective periods  $\tau_m$ , such that  $\tau_m \rightarrow \infty$  and  $f_m(t)$  converges uniformly to  $f(t)$  in the limit  $m \rightarrow \infty$  [50]. In particular,  $p_{ap}(t|\rho)$  has such a sequence of periodic approximations, each corresponding to the canonical time distribution of a periodic system with energy observable  $E^{(m)} = \sum_k E_k^{(m)} |E_k\rangle\langle E_k| \otimes 1_D$  and period  $\tau_m$ , with  $E_k^{(m)} \rightarrow E_k$  as  $m \rightarrow \infty$  [51]. Further, each such periodic system must satisfy uncertainty relation (73). Hence, using Equation (83) and taking the limit  $m \rightarrow \infty$ , one obtains the following general result.

**Corollary 9.** *The energy and almost-periodic canonical time observables  $E$  and  $\mathcal{T}$  satisfy the family of uncertainty relations*

$$A_\alpha^E(\rho) + H_\alpha^{ap}(\mathcal{T}|\rho) \geq 0, \quad \alpha \geq \frac{1}{2}, \quad (84)$$

for any quantum system with a discrete energy spectrum, where  $H_\alpha^{ap}(\mathcal{T}|\rho)$  is the almost-periodic Rényi entropy in Equation (82).

Corollary 9 generalises Corollary 5 and uncertainty relations (73) and (74), for periodic systems, to any system with a discrete energy spectrum. Moreover, using Equation (26) for the case  $\alpha = 1$  [35], and Theorem 3 more generally, the corollary further leads to the respective energy-time uncertainty relations

$$H(\rho_E) + H^{ap}(\mathcal{T}|\rho) \geq H(\rho), \quad H_\alpha(E|\rho) + H_\beta^{ap}(\mathcal{T}|\rho) \geq 0, \quad \frac{1}{\alpha} + \frac{1}{\beta} = 2, \quad (85)$$

for the Shannon and Rényi entropies of general systems. It may be noted, however, that the function  $f(t) = t^2$  is not almost-periodic, implying that there is no analogue of  $\Delta_{t_0}\mathcal{T}$  for nonperiodic systems and hence no corresponding generalisation of the second uncertainty relation in Equation (73).

Finally, it should be noted that a suggestion in Reference [20], to interpret  $I^{ap}(\rho) := -H^{ap}(\mathcal{T}|\rho)$  as the maximum information that can be gained about a random time shift in the limit of a uniform prior distribution on the real line, via a measurement of  $\mathcal{T}$ , is incorrect. This quantity is in fact a *lower* bound for the information gain in this scenario. In particular, note from Corollary 2 that  $I_\alpha(\mathcal{T}_\tau : T) \geq \log \tau - H_\alpha(\mathcal{T}_\tau - T|\rho) = \log \tau - H_\alpha(\mathcal{T}_\tau|\rho)$  for uniformly random time displacements of a periodic system with period  $\tau$ . Hence, choosing the same sequence of periodic systems as above and using Equation (83), the lower bound

$$I_\alpha(\mathcal{T} : T) \geq -H_\alpha^{ap}(\mathcal{T}|\rho) = I_\alpha^{ap}(\rho) \quad (86)$$

follows for information gain in the limit of a uniform prior distribution on the real line, valid for both periodic and nonperiodic systems. This lower bound can be quite strong for systems with many pairs of resonant energy levels (i.e., with  $E_j - E_{j'} = E_k - E_{k'} \neq 0$ ), but is no greater than  $\log 2$  in the case of no shared resonances and  $\alpha = 1$  [20].

## 5. Discussion

The main results of the paper, embodied in Theorems 1–6 and Corollaries 1–9, are seen to have wide applicability, including lower bounds for the error of any estimate of unitary displacement parameters, such as phase shifts, rotations and time; Heisenberg limits for the scaling of RMSE with average photon number and angular momentum; upper and lower bounds for measures of coherence; and uncertainty relations for canonically conjugate observables. As demonstrated by various examples, the results are typically stronger than existing results in the literature.

Whereas the results in Section 2 are based on known uncertainty relation (4) for Rényi entropies, the results in Sections 3.5 and 4 rely on the upper and lower bounds in Theorems 3 and 4 for the Rényi mutual information of quantum communication channels, which provide a path to far stronger metrology bounds and uncertainty relations. All of these results have the advantage of being independent of, and hence not requiring any interpretation of, the Rényi mutual information itself. Indeed a number of the inequalities in Theorem 2 and Corollaries 1, 4 and 8 do not refer even to Rényi entropies.

There is an interesting subtlety worth noting in regard to entangled states. In particular, if a unitary displacement acts only on one component of an entangled probe state, then there are two distinct scenarios: (i) an estimate is made via a measurement on that component, or (ii) via a measurement on the whole state. The first scenario, by limiting the class of measurements, will in general have an increased estimation error, that is not taken into account in Theorem 5 and its corollaries. Fortunately this is straightforward to remedy, by replacing the state  $\rho$  in those results by its accessible component, i.e., by the partial trace  $\text{tr}_R[\rho]$  over any unmeasured components. Stronger lower bounds for estimation error are thereby obtained in the first scenario, noting that  $A_\alpha^G(\text{tr}_R[\rho]) \leq A_\alpha^G(\rho)$  via the data

processing inequality, which yield correspondingly improved uncertainty relations for observables that act on a component of an entangled state.

The Rényi asymmetry, already known to be a useful resource in various contexts [17,18,32–34] is seen to also be a valuable resource in quantum metrology. In particular, the various lower bounds for the estimation error of unitary displacements, whether measured via its entropy or the RMSE, decrease as the asymmetry increases, making probe states with high asymmetry desirable. Moreover, the strong uncertainty relations derived in the paper, e.g., Corollaries 5, 6 and 9, imply that  $A_\alpha^G(\rho)$  can further be regarded as a measure of the intrinsically ‘quantum’ uncertainty of  $G$  for state  $\rho$ , given that it only vanishes for eigenstates of  $G$  and for any classical mixtures thereof. Together with Theorem 3, this suggests that the ‘total uncertainty’ of  $G$  for state  $\rho$ , as characterised by  $\mathcal{U}_\alpha^{\text{total}}(G|\rho) := H_\alpha(G|\rho)$ , can be decomposed into quantum and classical contributions via

$$\mathcal{U}_\alpha^{\text{total}}(G|\rho) = \mathcal{U}_\alpha^{\text{quantum}}(G|\rho) + \mathcal{U}_\alpha^{\text{classical}}(G|\rho), \quad \alpha \geq \frac{1}{2}, \quad (87)$$

where  $\mathcal{U}_\alpha^{\text{quantum}}(G|\rho) := A_\beta^G(\rho)$ ,  $\mathcal{U}_\alpha^{\text{classical}}(G|\rho) := H_\alpha(G|\rho) - A_\beta^G(\rho)$ , and  $1/\alpha + 1/\beta = 2$ . It follows that the quantum contribution vanishes if and only if the state is classical with respect to  $G$ , i.e.,  $[G, \rho] = 0$ . Conversely, the classical contribution vanishes if and only if the state is pure, i.e., has no classical mixing. For  $\alpha = 1$  this decomposition matches the one introduced in Reference [45] for Shannon entropy. An analogous decomposition of variance into quantum and classical contributions has been given by Luo [52].

It was mentioned in Section 3 that the universal asymmetry bound  $\chi_\alpha(\mathcal{E}) \leq A_\alpha^G(\rho)$  in Equations (26) and (41) unifies and generalises the recent energy-time estimation relations given by Coles et al. [17]. To see this in more detail, note that the main result in Reference [17] for uniform discrete ensembles, in the form given by Equations (3) and (10) thereof, translates in the notation of this paper to

$$-\inf_{\sigma_\mathcal{E}} D_\alpha(\rho_{\mathcal{E}X} \| \sigma_\mathcal{E} \otimes 1_X) + A_\alpha^G(\rho) \geq \log d, \quad (88)$$

where  $\mathcal{E}$  is an ensemble corresponding to  $d$  displaced states  $\rho_{x_k}$  having uniform prior probabilities  $p(x_k) = 1/d$ , and  $1_X$  is the unit operator on a corresponding reference system for these displacements (with orthonormal basis  $\{|x_k\rangle\}$  as in Section 3.2). It is straightforward to check that  $D_\alpha(\rho_{\mathcal{E}X} \| \sigma_\mathcal{E} \otimes 1_X) = D_\alpha(\rho_{\mathcal{E}X} \| \sigma_\mathcal{E} \otimes d^{-1}1_X) - \log d = D_\alpha(\rho_{\mathcal{E}X} \| \sigma_\mathcal{E} \otimes \rho_X) - \log d$  for such ensembles, implying via Equation (34) that the above result is equivalent to  $\chi_\alpha(\mathcal{E}) \leq A_\alpha^G(\rho)$ , as claimed. A similar equivalence holds between the asymmetry bound and Equation (12) of Reference [17] for uniform continuous ensembles. Hence, while the estimation relations in Reference [17] are interpreted via a game, with scores determined by Rényi conditional entropies for estimates of  $G$  and of the displacements it generates, they may also be interpreted as special cases of the asymmetry bound. Alternatively, they may be interpreted via Equation (44) as instances of a general inequality between the quantum Rényi mutual information of a given ensemble and that of the ensemble corresponding to the limit of a maximally uniform prior distribution.

Finally, several topics for future work are suggested by the results. These include using Theorem 4 to obtain explicit lower bounds for the mutual information of ensembles with arbitrary prior probabilities, including discrete prior probability distributions (thus generalising the results in Sections 3.5 and 4); extending the analysis to displacements induced by generators with continuous spectra, such as translations generated by momentum, and to multiparameter displacements (some preliminary results for the case of Shannon entropy are given in Reference [35]); obtaining Heisenberg-type limits for RMSE in terms of the variance of  $N$  rather than of  $\langle N \rangle$  (via corresponding upper bounds on Rényi entropies analogous to Equation (A17) of Appendix B); and further investigating the lower bounds for coherence in Theorem 6 and Corollary 7.

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### Appendix A

**Proof of Theorem 1.** First, applying a uniformly random phase shift  $\theta$  to a probe state  $\rho$  gives the phase-shifted state  $\rho_\theta = e^{-iN\theta}\rho e^{iN\theta}$ , with associated prior probability density  $p(\theta) = 1/(2\pi)$ . Further, any estimate  $\theta_{\text{est}}$  made of  $\theta$  must be described by some POVM  $\Theta_{\text{est}} \equiv \{M_{\theta_{\text{est}}}\}$ , with  $M_{\theta_{\text{est}}} \geq 0$  and  $\oint d\theta_{\text{est}} M_{\theta_{\text{est}}} = \hat{1}$ . Hence the joint probability density of  $\theta$  and  $\theta_{\text{est}}$  is given by

$$p(\theta, \theta_{\text{est}}|\rho) = p(\theta)p(\theta_{\text{est}}|\rho_\theta) = \frac{1}{2\pi} \text{tr}[\rho_\theta M_{\theta_{\text{est}}}], \tag{A1}$$

and the error of the estimate,  $\Theta_{\text{est}} - \Theta$ , has the corresponding marginal probability density

$$p(\theta_{\text{err}}) = \oint d\theta_{\text{est}} p(\theta_{\text{est}} - \theta_{\text{err}}, \theta_{\text{est}}|\rho) = \frac{1}{2\pi} \oint d\theta_{\text{est}} \text{tr}[\rho_{\theta_{\text{est}} - \theta_{\text{err}}} M_{\theta_{\text{est}}}] = \text{tr}[\rho \tilde{M}_{\theta_{\text{err}}}], \tag{A2}$$

where

$$\tilde{M}_{\theta_{\text{err}}} := e^{-iN\theta_{\text{err}}}\tilde{M}_0 e^{iN\theta_{\text{err}}}, \quad \tilde{M}_0 := \frac{1}{2\pi} \oint d\theta_{\text{est}} e^{iN\theta_{\text{est}}} M_{\theta_{\text{est}}} e^{-iN\theta_{\text{est}}}, \tag{A3}$$

and the cyclic property of the trace has been used. Note the useful property  $\langle n|\tilde{M}_0|n\rangle = 1$ .

Now,  $\tilde{M}_0 \geq 0$  follows from  $M_{\theta_{\text{est}}} \geq 0$ , implying it can be written in the form  $\tilde{M}_0 = \sum_{\tilde{m}} |\tilde{m}\rangle\langle\tilde{m}|$  for some set of kets  $\{|\tilde{m}\rangle\}$ . Defining the completely-positive trace-preserving map  $\mu(\rho) = \sum_{\tilde{m}} A_{\tilde{m}}\rho A_{\tilde{m}}^\dagger$  with  $A_{\tilde{m}} = \sum_{n=0}^\infty \langle\tilde{m}|n\rangle |n\rangle\langle n|$  [20,24], it is then straightforward to calculate the canonical phase distribution of  $\Phi$  for state  $\mu(\rho)$  as

$$p(\phi|\mu(\rho)) = p(\theta_{\text{est}} - \theta = \phi|\rho), \tag{A4}$$

i.e., it is identical to the distribution of the error  $\Theta_{\text{est}} - \Theta$ . One further finds

$$p(n|\mu(\rho)) = \langle n|\mu(\rho)|n\rangle = \langle n|\rho|n\rangle = p(n|\rho), \tag{A5}$$

i.e., the number distributions of  $\mu(\rho)$  and  $\rho$  are identical. Finally, it may be checked that  $\mu(\hat{1}) = \hat{1}$ , i.e.,  $\mu$  is a unital map, implying that the von Neumann entropy increases under  $\mu$  [30], i.e.,

$$H(\mu(\rho)) \geq H(\rho). \tag{A6}$$

Hence, uncertainty relation (1) for standard entropies immediately implies that

$$H(\Theta_{\text{est}} - \Theta|\rho) + H(N|\rho) = H(\Phi|\mu(\rho)) + H(N|\mu(\rho)) \geq \log 2\pi + H(\mu(\rho)) \geq \log 2\pi + H(\rho) \tag{A7}$$

as per Equation (2), while the Rényi uncertainty relation (4) similarly yields

$$H_\alpha(\Theta_{\text{est}} - \Theta|\rho) + H_\beta(N|\rho) \geq \log 2\pi, \quad \frac{1}{\alpha} + \frac{1}{\beta} = 2, \tag{A8}$$

as per Equation (12) of Theorem 1.  $\square$

### Appendix B

**Proof of Theorem 2.** The proof proceeds by establishing upper bounds on Rényi entropies under various constraints. First, consider the variation of the quantity  $\int dx p(x)^\alpha$  with respect to probability density  $p(x)$  for  $\alpha \geq 1/2$  and  $\alpha \neq 1$ , under the constraints  $\int dx x^2 p(x) = \sigma^2$

and  $\int dx p(x) = 1$ , where integration is over the real line. Applying the method of Lagrange multipliers then gives an extremal probability distribution of the form

$$p(x) = \frac{1}{\lambda} p_1(x/\lambda), \quad p_1(x) = K(1 \pm x^2)^{\frac{-1}{1-\alpha}}. \tag{A9}$$

where the + sign is chosen for  $1/2 \leq \alpha < 1$  and the – sign for  $\alpha > 1$  [53], and in the latter case one takes  $p_1(x) = 0$  for  $|x| > 1$ . These choices correspond to a maximum and minimum of  $\int dx p(x)^\alpha$ , respectively, and hence to a maximum of the associated Rényi entropy. The values of  $\lambda$  and  $K$  are determined by the constraints as [53]

$$\lambda = \sigma \left( \frac{3\alpha - 1}{|1 - \alpha|} \right)^{1/2}, \quad K = \begin{cases} \frac{\Gamma(\frac{1}{1-\alpha})}{\sqrt{\pi} \Gamma(\frac{1}{1-\alpha} - \frac{1}{2})}, & 1/2 \leq \alpha < 1 \\ \frac{\Gamma(\frac{\alpha}{\alpha-1} + \frac{1}{2})}{\sqrt{\pi} \Gamma(\frac{\alpha}{\alpha-1})}, & \alpha > 1 \end{cases}. \tag{A10}$$

Since phase error is restricted to a subset of the real line, it follows that its Rényi entropy can be no greater than the maximum for  $p(x)$  with  $\sigma = \text{RMSE}$ . Hence,

$$\begin{aligned} H_\alpha(\Theta_{\text{est}} - \Theta|\rho) &\leq H_\alpha[p] = \log \lambda + H_\alpha[p_1] = \log \lambda + \frac{1}{1-\alpha} \log \int dx p_1(x)^\alpha \\ &= \log \text{RMSE} + \log \left[ \left( \frac{3\alpha - 1}{|1 - \alpha|} \right)^{1/2} \left( \frac{2\alpha}{3\alpha - 1} \right)^{\frac{1}{1-\alpha}} \right] - \log K, \end{aligned} \tag{A11}$$

where the last line follows by direct calculation. It may be checked that this inequality also holds in the limit  $\alpha \rightarrow 1$ , for which it becomes equivalent to the standard bound  $H \leq \log \text{RMSE} + \frac{1}{2} \log 2\pi e$  for Shannon entropy. Inverting the above result, and using  $H_\alpha(\Theta_{\text{est}} - \Theta|\rho) \geq \log 2\pi - H_\beta(N|\rho)$  from Equation (12) of Theorem 1, then gives the lower bound

$$\text{RMSE} \geq \frac{\alpha^{\frac{\alpha}{\alpha-1}} f(\alpha)}{L_\beta(N|\rho)}, \quad \frac{1}{\alpha} + \frac{1}{\beta} = 2, \tag{A12}$$

where  $f(\alpha)$  is defined in Equation (18).

To obtain the first lower bound in Theorem 2, consider the limit  $\alpha \rightarrow \infty$  in Equation (A12) above. The constraint on  $\alpha$  and  $\beta$  gives  $\beta = \frac{1}{2}$ , and one finds

$$\lim_{\alpha \rightarrow \infty} \alpha^{\frac{\alpha}{\alpha-1}} f(\alpha) = 2\sqrt{\pi} \frac{\Gamma(\frac{3}{2})}{\Gamma(1)} \lim_{\alpha \rightarrow \infty} \alpha^{\frac{1}{\alpha-1}} \left( \frac{\alpha - 1}{3\alpha - 1} \right)^{1/2} = \frac{\pi}{\sqrt{3}} \tag{A13}$$

(corresponding to the maximum value of  $\alpha^{\frac{\alpha}{\alpha-1}} f(\alpha)$ ), yielding the required first bound. Similarly, substitution of  $\alpha = \frac{1}{2}$  into Equation (A12) gives unity for the numerator and  $L_\infty(N|\rho) = (\max_n p(n|\rho))^{-1}$  for the denominator, yielding the second lower bound in Theorem 2.

The third and final bound requires an upper bound for the Rényi length  $L_\beta(N|\rho)$  as a function of  $\langle N \rangle$ . This is obtained by considering the variation of the quantity  $L_\beta(X) = [\int dx p(x)^\beta]^{1/(1-\beta)}$  for  $\beta \geq \frac{1}{2}$  and  $\beta \neq 1$ , under the constraints  $\int dx xp(x) = \bar{x}$  and  $\int dx p(x) = 1$ , where integration is now over  $x \geq 0$ . Applying the method of Lagrange multipliers as before, the probability distribution maximising  $L_\beta(X)$  is found to have the form

$$\tilde{p}(x) = \frac{1}{\tilde{\lambda}} \tilde{p}_1(x/\tilde{\lambda}), \quad \tilde{p}_1(x) = \tilde{K}(1 \pm x)^{\frac{-1}{1-\beta}}, \tag{A14}$$

analogous to Equation (A9). The + sign is chosen for  $1/2 \leq \beta < 1$  and the – sign for  $\beta > 1$ , where in the latter case one takes  $\tilde{p}_1(x) = 0$  for  $x > 1$ . The values of  $\tilde{\lambda}$  and  $\tilde{K}$  follow from the constraints and  $1/\alpha + 1/\beta = 2$  as

$$\tilde{\lambda} = \bar{x} \frac{2\beta - 1}{|1 - \beta|} = \frac{\bar{x}}{|1 - \alpha|}, \quad \tilde{K} = \frac{\beta}{|1 - \beta|} = \frac{\alpha}{|1 - \alpha|}, \tag{A15}$$

yielding the upper bound

$$L_\beta(X) \leq \tilde{\lambda} L_\beta[\tilde{p}_1] = \tilde{\lambda} \tilde{K}^{-1} (1 \pm \bar{x}/\tilde{\lambda})^{\frac{1}{1-\beta}} = \alpha^{\frac{\alpha}{\alpha-1}} \bar{x} \tag{A16}$$

for the Rényi length of any random variable  $X$  on  $[0, \infty)$ . Choosing the particular random variable having probability density  $p(x) = p(n|\rho)$  for  $0 \leq n \leq x < n + 1$  gives  $\bar{x} = \sum_n p(n|\rho) \int_n^{n+1} dx x = \sum_n (n + \frac{1}{2}) p(n|\rho) = \langle N \rangle + \frac{1}{2}$ , and  $L_\beta(X) = L_\beta(N|\rho)$ , and hence the above upper bound reduces to

$$L_\beta(N|\rho) \leq \alpha^{\frac{\alpha}{\alpha-1}} (\langle N \rangle + \frac{1}{2}). \tag{A17}$$

This also holds in the limit  $\alpha \rightarrow 1$ , for which it gives the strong upper bound  $H(N|\rho) \leq \log(\langle N \rangle + \frac{1}{2})e$ , which is close to the maximum entropy  $\log(\langle N \rangle + 1) + \langle N \rangle \log(1 + 1/\langle N \rangle)$  (to within second order in  $1/\langle N \rangle$ ), corresponding to a thermal state. Finally, substitution of Equation (A17) into Equation (A12) yields the third bound in Theorem 2, as required.  $\square$

**Proof of Corollary 1.** The proof of Equation (A11) above goes through with  $\sigma$  and RMSE replaced by  $\Delta_\chi \Phi$ , the maximising density  $p(x)$  in Equation (A9) by  $p(x - \chi)$ , and  $H_\alpha(\Theta_{\text{est}} - \Theta|\rho)$  by  $H(\Phi|\rho)$ . Using uncertainty relation (4) then leads to Equation (21) of Corollary 1, in analogy to Equation (A12). The first two bounds in Equation (22) of the corollary correspond to the cases  $\alpha \rightarrow \infty$  and  $\alpha = \frac{1}{2}$ , while the third bound follows via the upper bound in Equation (A17) above.  $\square$

**Proof of Equation (68).** As noted in the main text, Equation (68) for angular momentum is obtained via a suitable modification of the derivation of upper bound (A17) above. This achieved by expanding the domain of integration in the derivation to the full real number line, and replacing the constraint  $\int dx xp(x) = \bar{x}$  by  $\int dx |x|p(x) = \bar{x}$ . This leads to a maximising probability density of the same form as in Equation (A14), but extended to the negative numbers and with  $x$  replaced by  $\bar{x}$ . This increases the corresponding Rényi entropy by  $\log 2$  and hence upper bound (A16) increases by a factor of 2, to

$$L_\beta(X) \leq 2\alpha^{\frac{\alpha}{\alpha-1}} \bar{x}. \tag{A18}$$

Finally, choosing the random variable having probability density  $p(x) = p(j|\rho)$  for  $j - \frac{1}{2} \leq x < j + \frac{1}{2}$ , where  $p_j = \langle j|\rho|j \rangle$  is the distribution of  $J_z$ , gives

$$\bar{x} = \sum_j p_j \int_{j-\frac{1}{2}}^{j+\frac{1}{2}} dx |x| = \frac{1}{4} p_0 + \sum_{j \neq 0} p_j |j| = \frac{1}{4} p_0 + \langle |J_z| \rangle, \tag{A19}$$

and substitution into Equation (A18) gives Equation (68) as desired.  $\square$

### Appendix C

**Proof of Theorem 3.** The lower bound in Equation (41) of Theorem 3 was proved in the main text. The upper bound is obtained by considering a purification  $|\psi\rangle\langle\psi|$  of state  $\rho$  on the joint Hilbert space of the probe and a reference ancilla  $R$ . If  $1_R$  denotes the unit operator for the ancilla, then

$$A_\alpha^{1_R \otimes G}(|\psi\rangle\langle\psi|) = \inf_{\sigma_{RE}: [\sigma_{RE}, 1 \otimes G] = 0} D_\alpha(|\psi\rangle\langle\psi| \| \sigma_{RE}) \geq \inf_{\sigma_E: [\sigma_E, G] = 0} D_\alpha(\rho \| \sigma_E) = A_\alpha^G(\rho), \tag{A20}$$

where the inequality follows by applying data processing inequality (37) to operation of tracing over the ancilla. Further, duality property (43), proved further below, implies that the left hand side is given by

$$\begin{aligned}
 A_{\alpha}^{1_R \otimes G}(|\psi\rangle\langle\psi|) &= H_{\beta}(|\psi\rangle\langle\psi|_{1_R \otimes G}) \\
 &= H_{\beta}\left(\sum_k (1_R \otimes P_k)|\psi\rangle\langle\psi|(1_R \otimes P_k)\right) \\
 &= \frac{1}{1-\beta} \operatorname{tr} \left[ \left( \sum_k \frac{|\psi_k\rangle\langle\psi_k|}{\langle\psi_k|\psi_k\rangle} \langle\psi_k|\psi_k\rangle \right)^{\beta} \right] \\
 &= \frac{1}{1-\beta} \sum_k \langle\psi_k|\psi_k\rangle^{\beta},
 \end{aligned} \tag{A21}$$

where  $|\psi_k\rangle := (1_R \otimes P_k)|\psi\rangle$ , and the last line follows noting that the  $|\psi_k\rangle\langle\psi_k|/\langle\psi_k|\psi_k\rangle$  terms are mutually orthogonal rank-1 projectors. Finally, if  $|\psi\rangle = \sum_{\lambda} \sqrt{w_{\lambda}}|u_{\lambda}\rangle_R \otimes |v_{\lambda}\rangle$  is the Schmidt decomposition of  $|\psi\rangle$ , so that  $\rho = \sum_{\lambda} w_{\lambda}|v_{\lambda}\rangle\langle v_{\lambda}|$ , it follows that  $\langle\psi_k|\psi_k\rangle = \sum_{\lambda} w_{\lambda}\langle v_{\lambda}|P_k|v_{\lambda}\rangle = \operatorname{tr}[\rho P_k] = p(G = g_k|\rho)$ , and substituting in Equations (A20), (A21) gives

$$A_{\alpha}^G(\rho) \leq A_{\alpha}^{1_R \otimes G}(|\psi\rangle\langle\psi|) = \frac{1}{1-\beta} \sum_k p(G = g_k|\rho)^{\beta} = H(G|\rho), \tag{A22}$$

thus yielding the upper bound in Theorem 3. Finally, if  $\rho$  is pure, then duality property (43) immediately implies this bound is saturated.  $\square$

**Proof of duality property (43).** The Rényi asymmetry of a pure state  $|\psi\rangle$  follows from definitions (36) and (38) as

$$\begin{aligned}
 A_{\alpha}^G(|\psi\rangle\langle\psi|) &= \inf_{\sigma: [\sigma, G]=0} D_{\alpha}(|\psi\rangle\langle\psi||\sigma) \\
 &= \inf_{\sigma: [\sigma, G]=0} \frac{1}{\alpha-1} \log \operatorname{tr} \left[ \left( \frac{\sigma^{\frac{1-\alpha}{2\alpha}} |\psi\rangle\langle\psi| \sigma^{\frac{1-\alpha}{2\alpha}}}{\langle\psi|\sigma^{\frac{1-\alpha}{\alpha}}|\psi\rangle} \langle\psi|\sigma^{\frac{1-\alpha}{\alpha}}|\psi\rangle \right)^{\alpha} \right] \\
 &= \inf_{\sigma: [\sigma, G]=0} \frac{1}{\alpha-1} \log \langle\psi|\sigma^{\frac{1-\alpha}{\alpha}}|\psi\rangle^{\alpha} \\
 &= \inf_{\sigma: [\sigma, G]=0} \frac{\alpha}{\alpha-1} \log \operatorname{tr} \left[ |\psi\rangle\langle\psi| \sum_k P_k \sigma^{\frac{1-\alpha}{\alpha}} P_k \right] \\
 &= \inf_{\sigma: [\sigma, G]=0} \frac{\alpha}{\alpha-1} \log \operatorname{tr} [|\psi\rangle\langle\psi|_G \sigma^{\frac{1-\alpha}{\alpha}}],
 \end{aligned} \tag{A23}$$

where the third line follows because the fractional expression in the round brackets of the second line is a rank-1 projector, the fourth line from  $[\sigma, G] = 0$ , and the last line from the cyclic property of the trace and the definition of  $\rho_G$  in Equation (26). To determine the infimum over all  $\sigma$ , note that variation of  $\operatorname{tr}[\rho \sigma^{\frac{1-\alpha}{\alpha}}]$  with respect to arbitrary  $\sigma$ , under the constraint  $\operatorname{tr}[\sigma] = 1$ , yields  $\sigma = \tilde{\sigma} := \rho^{\frac{\alpha}{2\alpha-1}} / \operatorname{tr}[\rho^{\frac{\alpha}{2\alpha-1}}]$ , and the corresponding extremal value  $(\operatorname{tr}[\rho^{\frac{\alpha}{2\alpha-1}}])^{\frac{2\alpha-1}{\alpha}}$ . Hence, noting that  $[\tilde{\sigma}, G] = 0$  for  $[\rho, G] = 0$ , and that  $\beta = \alpha/(2\alpha - 1)$  for  $1/\alpha + 1/\beta = 2$ , it follows for the choice  $\rho = |\psi\rangle\langle\psi|_G$  that

$$A_{\alpha}^G(|\psi\rangle\langle\psi|) = \frac{\alpha}{\alpha-1} \log(\operatorname{tr}[\rho^{\frac{\alpha}{2\alpha-1}}])^{\frac{2\alpha-1}{\alpha}} = \frac{2\alpha-1}{\alpha-1} \log(\operatorname{tr}[\rho^{\frac{\alpha}{2\alpha-1}}])^{\frac{2\alpha-1}{\alpha}} = H_{\beta}(|\psi\rangle\langle\psi|_G), \tag{A24}$$

as per Equation (43).  $\square$

**Proof of Equation (44) relating asymmetry to maximally uniform ensembles.** Let  $\rho_{\mathcal{E}_X}^r$  and  $\rho_X^r$  denote the states in Equation (31) for the prior distribution  $p_r(x) := 1/(2r)$  for  $r < |x|$  and vanishing otherwise. Further, let  $\mathcal{E}_r \equiv \{U_x \rho U_x^{\dagger}; p_r(x)\}$  denote the correspond-

ing continuous ensemble, and  $U = \int dx U_x^\dagger \otimes |x\rangle\langle x|$  be the controlled unitary transformation used in Section 3.3. Fixing  $\epsilon > 0$  and letting  $\sigma_r$  be a state which achieves the infimum in Equation (34) to within less than  $\epsilon$ , it follows that

$$\begin{aligned} \chi_\alpha(\mathcal{E}_r) + \epsilon &> D_\alpha(\rho_{\mathcal{E}_X}^r \| \sigma_r \otimes \rho_X^r) \\ &= D_\alpha(U\rho_{\mathcal{E}_X}^r U^\dagger \| U\sigma_r \otimes \rho_X^r U^\dagger) \\ &= D_\alpha(\rho \otimes \rho_X^r \| \int dx p_r(x) U_x^\dagger \sigma_r U_x \otimes |x\rangle\langle x|) \\ &\geq D_\alpha(\rho \| \tilde{\sigma}_r), \end{aligned} \tag{A25}$$

where  $\tilde{\sigma}_r := \int dr p_r(x) U_x^\dagger \sigma_r U_x$  and the last line follows by applying data processing inequality (37) to the partial trace operation. Hence, recalling the definition  $\chi_\alpha^\infty := \lim_{r \rightarrow \infty} \chi_\alpha(\mathcal{E}_r)$  in Equation (44),

$$\chi_\alpha^\infty + \epsilon > D(\rho \| \tilde{\sigma}_\infty). \tag{A26}$$

However, using  $\sum_k P_k = 1$ , one has

$$\tilde{\sigma}_\infty = \lim_{r \rightarrow \infty} \sum_{k,k'} \frac{1}{2r} \int_{-r}^r dx e^{ixG} P_k \sigma_r P_{k'} e^{-ixG} = \sum_{k,k'} \lim_{r \rightarrow \infty} \frac{1}{2r} \int_{-r}^r dx e^{ix(g_k - g_{k'})} P_k \sigma_r P_{k'} = \sum_k P_k \sigma_\infty P_k, \tag{A27}$$

and thus  $[\tilde{\sigma}_\infty, G] = 0$ . It then follows from Equation (A26) that

$$\chi_\alpha^\infty + \epsilon > \inf_{\sigma: [\sigma, G]=0} D(\rho \| \sigma) = A^G(\rho). \tag{A28}$$

Combined with  $\chi_\alpha^\infty = \lim_{r \rightarrow \infty} \chi(\mathcal{E}_r) \leq \lim_{r \rightarrow \infty} A_G(\rho) = A_G(\rho)$  from Theorem 3, this yields

$$A_G(\rho) - \epsilon < \chi_\alpha^\infty \leq A_G(\rho) \tag{A29}$$

for all  $\epsilon > 0$ . Hence  $A_G(\rho) = \chi_\alpha^\infty$  as per Equation (44).  $\square$

### Appendix D

**Proof of Theorem 4.** Defining the translation isometry  $\mathcal{T} : p(a, x) \rightarrow p(x + a, x)$ , analogous to the unitary transformation  $U$  in Equation (A25), and the marginalisation operation  $\mathcal{M} : p(a, x) \rightarrow \int dx p(a, x)$ , analogous to the partial trace operation, it follows from Equation (34) and data processing inequality (37) for  $A = X_{\text{est}}$  that

$$\begin{aligned} I_\alpha(X_{\text{est}} : X) &= \inf_{q_A} D_\alpha(p_{AX} \| q_A p_X) \\ &= \inf_{q_A} D_\alpha(\mathcal{T} p_{AX} \| \mathcal{T} q_A p_X) \\ &\geq \inf_{q_A} D_\alpha(\mathcal{M} \mathcal{T} p_{AX} \| \mathcal{M} \mathcal{T} q_A p_X) \\ &= \inf_q D_\alpha(p_{\text{err}} \| q * p_-) \end{aligned} \tag{A30}$$

as per Theorem 4, where the last line follows using  $(\mathcal{M} \mathcal{T} p_{AX})(y) = p_{\text{err}}(y)$  via Equation (47) and

$$(\mathcal{M} \mathcal{T} q p_X)(y) = \int dx q(y + x) p(x) = \int dx q(y - x) p_-(x) = (q * p_-)(y). \tag{A31}$$

$\square$

**Proof of Corollary 2.** Define the intervals  $I_j := [(j - \frac{1}{2})|I|, (j + \frac{1}{2})|I|)$  for integer  $j$ , and the concentration operation  $\mathcal{C} : p(y) \rightarrow p(y \bmod I_0)$ , that shifts the distribution of  $y$  onto interval  $I_0$ . The probability density  $p_{\text{err}}$  of  $X_{\text{err}} = X_{\text{est}} - X$   $\tilde{p}_{\text{err}}$  is then related to the

probability density  $\tilde{p}_{\text{err}}$  of  $\tilde{X}_{\text{err}} := X_{\text{err}} \bmod I_0$  via  $\tilde{p}_{\text{err}} = \mathcal{C}p_{\text{err}}$ . More generally, for a general density  $r$  one has by construction that

$$(\mathcal{C}r)(y) = \sum_j r(y + n\ell_I), \quad y \in I_0. \tag{A32}$$

Now, data processing inequality (37) and Equation (A30) above give

$$I_\alpha(X_{\text{est}} : X) \geq \inf_q D_\alpha(\mathcal{C}p_{\text{err}} \| \mathcal{C}q * p_-) = \inf_q D_\alpha(\tilde{p}_{\text{err}} \| \mathcal{C}q * p_-), \tag{A33}$$

where choosing  $r = q * p_-$  in Equation (A32) and  $p(x) = 1/\ell_I$  on some interval  $I$  gives

$$\begin{aligned} (\mathcal{C}q * p_-)(y) &= \sum_j (q * p_-)(y + n\ell_I) \\ &= \frac{1}{\ell_I} \sum_j \int_I dx q(y + x + n\ell_I) \\ &= \frac{1}{\ell_I} \sum_j \int_{I+n\ell_I} dx q(y + x) \\ &= \frac{1}{\ell_I} \int_{-\infty}^{\infty} dx q(y + x) \\ &= \frac{1}{\ell_I} \end{aligned} \tag{A34}$$

(with the second line following via Equation (A31)). Hence Equation (A33) simplifies to

$$I_\alpha(X_{\text{est}} : X) \geq D_\alpha(\tilde{p}_{\text{err}} \| \ell_I^{-1}) = \log \ell_I - H_\alpha(\tilde{p}_{\text{err}}) = \log \ell_I - H_\alpha(\tilde{X}_{\text{err}}) \geq \log \ell_I - H_\alpha(X_{\text{err}}) \tag{A35}$$

as per Corollary 2, with the final inequality following from Lemma 1 (proved below).  $\square$

**Proof of Lemma 1.** Entropy is invariant under translations and hence, for the purposes of proving Lemma 1, it is sufficient to consider  $\tilde{Z} = Z \bmod I_0 = [-\frac{1}{2}\ell_I, \frac{1}{2}\ell_I)$  for the interval  $I_0$  in the proof of Corollary 2 above. If  $q$  denotes the probability density of  $Z$ , then the corresponding probability density of  $\tilde{Z}$  is then  $\tilde{q} = \mathcal{C}q$ , as per Equation (A32).

It is convenient to represent  $q$  by a mixture of non-overlapping probability densities, supported on the corresponding non-overlapping intervals  $I_j = I_0 + n\ell_I$ . In particular, one has the identity

$$q(z) = \sum_j w_j q_j(z), \quad w_j := \int_{I_j} dz q(z), \quad q_j(z) := \begin{cases} q(z)/w_j, & z \in I_j, \\ 0, & \text{otherwise,} \end{cases} \tag{A36}$$

and it follows that  $\tilde{q} = \sum_j w_j \tilde{q}_j$ , where  $\tilde{q}_j = \mathcal{C}q_j$  is supported on  $I_0$  and  $q_j(z) = \tilde{q}_j(z + n\ell_I)$ .

For the case of Shannon entropies, i.e.,  $\alpha = 1$ , it follows that

$$H(Z) = - \sum_j \int_{I_j} dz w_j q_j(z) \log w_j q_j(z) = - \sum_j \int_{I_0} dz w_j \tilde{q}_j(z) \log w_j \tilde{q}_j(z) = H(J) + H(\tilde{Z}|J), \tag{A37}$$

where  $J$  is the discrete random variable with distribution  $\{w_j\}$ . Hence

$$H(Z) - H(\tilde{Z}) = H(J) + H(\tilde{Z}|J) - H(\tilde{Z}) = H(J|\tilde{Z}) \geq 0, \tag{A38}$$

yielding Lemma 1 for the case  $\alpha = 1$ .

Further, for  $\alpha \neq 1$  the Rényi entropies of  $Z$  and  $\tilde{Z}$  have the forms

$$H_\alpha(Z) = \frac{1}{1-\alpha} \log \sum_j \int_{I_j} dz [w_j q_j(z)]^\alpha = \frac{1}{1-\alpha} \log \int_{I_0} dz \sum_j [w_j \tilde{q}_j(z)]^\alpha, \quad (\text{A39})$$

$$H_\alpha(\tilde{Z}) = \frac{1}{1-\alpha} \log \int_{I_0} dz \left[ \sum_j w_j \tilde{q}_j(z) \right]^\alpha. \quad (\text{A40})$$

Defining  $p_j(z) := w_j \tilde{q}_j(z) / \sum_j w_j \tilde{q}_j(z)$ , it follows that  $\sum_j p_j(z) = 1$ ,  $p_j(z) \leq 1$  and  $\frac{1}{p_j(z)} \geq 1$ . Thus, letting  $\sum'_j$  denote summation over nonzero values of  $p_j(z)$ ,

$$\frac{\sum_j [w_j \tilde{q}_j(z)]^\alpha}{\left[ \sum_j w_j \tilde{q}_j(z) \right]^\alpha} = \sum'_j p_j(z)^\alpha = \begin{cases} \sum'_j p_j(z) \frac{1}{p_j(z)^{1-\alpha}} \geq \sum'_j p_j(z) = 1, & \alpha \leq 1, \\ \sum'_j p_j(z) p_j(z)^{\alpha-1} \leq \sum'_j p_j(z) = 1, & \alpha \geq 1. \end{cases} \quad (\text{A41})$$

Hence  $\sum_j [w_j \tilde{q}_j(z)]^\alpha$  is greater than or equal to (less than or equal to)  $\left[ \sum_j w_j \tilde{q}_j(z) \right]^\alpha$  if  $\alpha < 1$  ( $\alpha > 1$ ), and it immediately follows from the above forms for  $H_\alpha(Z)$  and  $H_\alpha(\tilde{Z})$  that

$$H_\alpha(Z) \geq H_\alpha(\tilde{Z}) \quad (\text{A42})$$

for the case  $\alpha \neq 1$ , yielding Lemma 1 as desired.  $\square$

## References

1. Coles, P.J.; Berta, M.; Tomamichel, M.; Wehner, S. Entropic uncertainty relations and their applications. *Rev. Mod. Phys.* **2017**, *89*, 015002. [\[CrossRef\]](#)
2. Bialynicki-Birula, I.; Mycielski, J. Uncertainty relations for information entropy in wave mechanics. *Commun. Math. Phys.* **1975**, *44*, 129. [\[CrossRef\]](#)
3. Frank, R.L.; Lieb, E.H. Entropy and the uncertainty principle. *Ann. Henri Poincaré* **2012**, *13*, 1711–1717. [\[CrossRef\]](#)
4. Hall, M.J.W.; Wiseman, H.M. Does nonlinear metrology offer improved resolution? Answers from quantum information theory. *Phys. Rev. X* **2012**, *2*, 041006. [\[CrossRef\]](#)
5. Hall, M.J.W.; Berry, D.W.; Zwierz, M.; Wiseman, H.M. Universality of the Heisenberg limit for estimates of random phase shifts. *Phys. Rev. A* **2012**, *85*, 041802(R). [\[CrossRef\]](#)
6. Nair, R. Fundamental quantum limits in optical metrology from rate-distortion theory. *J. Phys. A* **2018**, *51*, 434001. [\[CrossRef\]](#)
7. Berry, D.W.; Tsang, M.; Hall, M.J.W.; Wiseman, H.M. Quantum Bell-Ziv-Zakai bounds and Heisenberg limits for waveform estimation. *Phys. Rev. X* **2015**, *5*, 031018. [\[CrossRef\]](#)
8. Maassen, H.; Uffink, J.B.M. Generalized entropic uncertainty relations. *Phys. Rev. Lett.* **2008**, *60*, 1103–1106. [\[CrossRef\]](#)
9. Bialynicki-Birula, I. Formulation of the uncertainty relations in terms of the Rényi entropies. *Phys. Rev. A* **2006**, *74*, 052101. [\[CrossRef\]](#)
10. Damgård, I.B.; Fehr, S.; Renner, R.; Salvail, L.; Schaffner, C. A Tight High-Order Entropic Quantum Uncertainty Relation with Applications. In *Advances in Cryptology—CRYPTO 2007*; Menezes, A., Ed.; Springer: Berlin, Germany, 2007; pp. 360–378. [\[CrossRef\]](#)
11. Kriváchy, T.; Fröwis, F.; Brunner, N. Tight steering inequalities from generalized entropic uncertainty relations. *Phys. Rev. A* **2018**, *98*, 062111. [\[CrossRef\]](#)
12. Sibson, R. Information radius. *Z. Wahrscheinlichkeitstheorie Verwandte Geb.* **1969**, *14*, 149–160. [\[CrossRef\]](#)
13. Ho, S.W.; Verdú, S. Convexity/concavity of the Rényi entropy and  $\alpha$ -mutual information. In Proceedings of the 2015 IEEE International Symposium on Information Theory, Hong Kong, China, 14–19 June 2015; pp. 745–749. [\[CrossRef\]](#)
14. Wilde, M.M.; Winter, A.; Yang, D. Strong converse for the classical capacity of entanglement-breaking and Hadamard channels via a sandwiched Rényi relative entropy. *Commun. Math. Phys.* **2014**, *331*, 593–622. [\[CrossRef\]](#)
15. Hayashi, M.; Tomamichel, M. Correlation Detection and an Operational Interpretation of the Rényi Mutual Information. *J. Math. Phys.* **2016**, *57*, 102201. [\[CrossRef\]](#)
16. Gao, L.; Junge, M.; LaRacuente, N. Unifying entanglement with uncertainty via symmetries of observable algebras. *arXiv* **2017**, arXiv:1710.10038.
17. Coles, P.J.; Katariya, V.; Lloyd, S.; Marvian, I.; Wilde, M.M. Entropic energy-time uncertainty relation. *Phys. Rev. Lett.* **2019**, *122*, 100401. [\[CrossRef\]](#) [\[PubMed\]](#)
18. Chitambar, E.; Gour, G. Comparison of incoherent operations and measures of coherence. *Phys. Rev. A* **2016**, *94*, 052336. [\[CrossRef\]](#)
19. Streltsov, A.; Adesso, G.; Plenio, M.B. Colloquium: Quantum coherence as a resource. *Rev. Mod. Phys.* **2017**, *89*, 041003. [\[CrossRef\]](#)
20. Hall, M.J.W. Almost-periodic time observables for bound quantum systems. *J. Phys. A* **2008**, *41*, 255301. [\[CrossRef\]](#)

21. Helstrom, C.W. Estimation of a displacement parameter of a quantum system. *Int. J. Theor. Phys.* **1974**, *11*, 357–378. [[CrossRef](#)]
22. Holevo, A.S. *Probabilistic and Statistical Aspects of Quantum Theory*; North-Holland: Amsterdam, The Netherlands, 1982.
23. Hall, M.J.W. Universal geometric approach to uncertainty, entropy, and information. *Phys. Rev. A* **1999**, *59*, 2602–2615. [[CrossRef](#)]
24. Hall, M.J.W. Phase resolution and coherent phase states. *J. Mod. Opt.* **1993**, *40*, 809–824. [[CrossRef](#)]
25. Boixo, S.; Flammia, S.T.; Caves, C.M.; Geremia, J.M. Generalized limits for single-parameter quantum estimation. *Phys. Rev. Lett.* **2007**, *98*, 090401. [[CrossRef](#)] [[PubMed](#)]
26. Hall, M.J.W. Metrology with entangled coherent states—A quantum scaling paradox. In Proceedings of the First International Workshop on Entangled Coherent States and Its Application to Quantum Information Science, 2013; Usuda, T.S., Kato, K., Eds.; Tamagawa University Quantum ICT Research Institute: Tokyo, Japan, 2013; pp. 19–26. Available online: <https://www.tamagawa.jp/research/quantum/discourse/> (accessed on 26 October 2022).
27. Bandilla, A.; Paul, H.; Ritze, H.-H. Realistic quantum states of light with minimum phase uncertainty. *Quantum Opt.* **1991**, *3*, 267–282. [[CrossRef](#)]
28. Berry, D.W.; Hall, M.J.W.; Zwierz, M.; Wiseman, H.M. Optimal Heisenberg-style bounds for the average performance of arbitrary phase estimates. *Phys. Rev. A* **2012**, *86*, 053813. [[CrossRef](#)]
29. Hall, M.J.W.; Wiseman, H.M. Heisenberg-style bounds for arbitrary estimates of shift parameters including prior information. *New J. Phys.* **2012**, *14*, 033040. [[CrossRef](#)]
30. Nielsen, M.A.; Chuang, I.L. *Quantum Computation and Quantum Information*, 10th Anniv. ed.; Cambridge University Press: Cambridge, UK, 2010.
31. Yuen, H.P.; Ozawa, M. Ultimate information carrying limit of quantum systems. *Phys. Rev. Lett.* **1993**, *70*, 363–366. [[CrossRef](#)]
32. Vaccaro, J.A. Group theoretic formulation of complementarity. In Proceedings of the 8th International Conference on Quantum Communication, Measurement and Computing, Tsukuba, Japan, 28 November–3 December 2006; Hirota, O., Shapiro, J.H., Sasaki, M., Eds.; NICT Press: Tokyo, Japan, 2006; pp. 421–424.
33. Vaccaro, J.A.; Anselmi, F.; Wiseman, H.M.; Jacobs, K. Tradeoff between extractable mechanical work, accessible entanglement, and ability to act as a reference system, under arbitrary superselection rules. *Phys. Rev. A* **2008**, *77*, 032114. [[CrossRef](#)]
34. Gour, G.; Marvian, I.; Spekkens, R.W. Measuring the quality of a quantum reference frame: The relative entropy of frameness. *Phys. Rev. A* **2009**, *80*, 012307. [[CrossRef](#)]
35. Hall, M.J.W. Entropic Heisenberg limits and uncertainty relations from the Holevo information bound. *J. Phys. A* **2018**, *51*, 364001. [[CrossRef](#)]
36. Sharma, N.; Warsi, N.A. On the strong converses for the quantum channel capacity theorems. *arXiv* **2012**, arXiv:1205.1712.
37. Rényi, A. On measures of entropy and information. In Proceedings of the 4th Berkeley Symposium on Mathematics, Statistics and Probability, Berkeley, CA, USA, 20–30 June 1961; Neyman, J., Ed.; University of California Press: Berkeley, CA, USA, 1961; pp. 547–561. Available online: <https://projecteuclid.org/ebook/Download?urlid=bsmsp/1200512181&isFullBook=false> (accessed on 26 October 2022).
38. Van Erven, T.; Harremoës, P. Rényi divergence and Kullback–Leibler divergence. *IEEE Trans. Inf. Theory* **2014**, *60*, 3797–3820. [[CrossRef](#)]
39. Müller-Lennert, M.; Dupuis, F.; Szehr, O.; Fehr, S.; Tomamichel, M. On quantum Rényi entropies: A new generalization and some properties. *J. Math. Phys.* **2013**, *54*, 122203. [[CrossRef](#)]
40. Frank, R.L.; Lieb, E.H. Monotonicity of a relative Rényi entropy. *J. Math. Phys.* **2013**, *54*, 122201. [[CrossRef](#)]
41. Beigi, S. Sandwiched Rényi divergence satisfies data processing inequality. *J. Math. Phys.* **2013**, *54*, 122202. [[CrossRef](#)]
42. Mosonyi, M.; Ogawa, T. Quantum hypothesis testing and the operational interpretation of the quantum Rényi relative entropies. *Commun. Math. Phys.* **2015**, *334*, 1617–1648. [[CrossRef](#)]
43. Shannon, C.E. A mathematical theory of communication. *Bell Syst. Tech. J.* **1948**, *27*, 623–656. [[CrossRef](#)]
44. Uhlmann, A. Relative entropy and the Wigner-Yanase-Dyson-Lieb concavity in an interpolation theory. *Commun. Math. Phys.* **1977**, *54*, 21–32. [[CrossRef](#)]
45. Korzekwa, K.; Lostaglio, M.; Jennings, D.; Rudolph, T. Quantum and classical entropic uncertainty relations. *Phys. Rev. A* **2014**, *89*, 042122. [[CrossRef](#)]
46. Zhang, H.J.; Chen, B.; Li, M.; Fei, S.M.; Long, G.L. Estimation on geometric measure of quantum coherence. *Commun. Theor. Phys.* **2017**, *67*, 166–170. [[CrossRef](#)]
47. Piani, M.; Cianciaruso, M.; Bromley, T.R.; Napoli, C.; Johnston, N.; Adesso, G. Robustness of asymmetry and coherence of quantum states. *Phys. Rev. A* **2016**, *93*, 042107. [[CrossRef](#)]
48. Tóth, G.; Apellaniz, I. Quantum metrology from a quantum information science perspective. *J. Phys. A* **2014**, *47*, 424006. [[CrossRef](#)]
49. Rastegin, A.E. On entropic uncertainty relations for measurements of energy and its “complement”. *Ann. Phys.* **2019**, *531*, 1800466. [[CrossRef](#)]
50. Besicovitch, A.S. *Almost Periodic Functions*; Dover: New York, NY, USA, 1954; pp. 1–109.
51. Hall, M.J.W.; Pegg, D.T. Comment on “Quantum phase for an arbitrary system with finite-dimensional Hilbert space”. *Phys. Rev. A* **2012**, *86*, 056101. [[CrossRef](#)]

- 
52. Luo, S.L. Quantum versus classical uncertainty. *Theoret. Math. Phys.* **2005**, *143*, 681–688. [[CrossRef](#)]
  53. Costa, J.; Hero, A.; Vignat, C. On solutions to multivariate maximum  $\alpha$ -entropy problems. In Proceedings of the International Workshop on Energy Minimization Methods in Computer Vision and Pattern Recognition, Lisbon, Portugal, 7–9 July 2003; Rangarajan, A., Figueiredo, M., Zerubia, J., Eds.; Springer: Berlin, Germany, 2003; pp. 211–226. [[CrossRef](#)]