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Markov property and operads

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Abstract: We define heat kernel measure on punctured spheres. The random field which is got by this procedure is not Gaussian. We define a stochastic line bundle on the loop space, such that the punctured sphere corresponds to a generalized parallel transport on this line bundle. Markov property along the sewing loops corresponds to an operadic structure of the stochastic W.Z.N.W. model.

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1 Introduction

In conformal field theory, people look at a Riemann surface Σ with boundary $\partial\Sigma$, and the set of maps from Σ into a Riemannian manifold M. The case which will be of interest for us in this present work is when the genus of the Riemann surface is 0. This corresponds to a punctured sphere. We suppose that there are one input loop and n output loop. The map from Σ into M are chosen at random, with the formal probability law:

$$d\mu(\psi) = 1/Z \exp[-I(\psi)]dD(\psi) \tag{1}$$

where dD is the formal Lebesgue measure, $I(\psi)$ the energy of the map and Z a normalizing constant called the partition function destinated to get a probability law. Segal [46] has given a series of axioms which should be satisfied by this theory. In particular, conformal field theory predicts the existence of an Hilbert space Ξ associated to each loop space such that the surface Σ realizes a map from $\Xi^{\otimes n}$ into Ξ, if we consider the case of the $(n+1)$-punctured sphere. $Hom(\Xi^{\otimes n}, \Xi)$ is the archetype of an operad. Namely, if we consider n elements of $Hom(\Xi^{\otimes n_i}, \Xi)$ and an element of $Hom(\Xi^{\otimes n}, \Xi)$, we deduce by composition an element of $Hom(\Xi^{\otimes \sum n_i}, \Xi)$. This composition operation will correspond to the operation of glueing n $1 + n_i$ punctured spheres in a sphere with $(1 + \sum n_i)$ punctured points. For the literature about this statement, we refer to [22], [24], [21], [49]. For material about operads, we refer to the proceedings of Loday, Stasheff and Voronov ([39]).

The problem of the measure $d\mu$ is that it is purely hypothetical: in the case when the manifold M is replaced by R, it is a Gaussian measure, which gives random distributions (See [42], [48], [19]). But it is difficult to say what are distributions that live on a manifolds.

Our statement is the following:

-)Define a measure over the space of spheres with $1 + n$ punctured points.

-)Define an Hilbert space Ξ associated to each loop space given the punctured points on the sphere.

-)Define associated to the sphere with $1+n$ punctured points an element of $Hom(\Xi^{\otimes n}, \Xi)$, such that the application is compatible with the action of sewing spheres along their boundary.

For that, we use the theory of infinite dimensional process, especially the theory of Brownian motion over a loop group of Airault-Malliavin [1] and Brzezniak-Elworthy [7]. Let us recall that the theory of infinite dimensional processes over infinite dimensional manifolds has a lot of aspects. The first who have studied Brownian motion over infinite dimensional manifolds is Kuo [27]. The Russian school has its own version [4], [11], [5]. The theory of Dirichlet forms allows to study Ornstein-Uhlenbeck processes over some loop spaces [12], [2]. Our study is related to the theory of Airault-Malliavin, but in order to produce random cylinders, Airault-Malliavin consider a 1+1 dimensional theory: the first 1 is related to the dimension of the propagation time of the dynamics and the second 1 is involved with the internal time of the theory (The loop space). Our theory is 1+2 dimensional, because the internal time of the theory is 2 dimensional.

1+2 dimensional theories were already studied by Léandre in [31] in order to study the Wess-Zumino-Novikov-Witten model on the torus, in [32] in order to study Brownian cylinders attached to branes and in [35] in order to study one of the concretisation of Segal's axiom by using C^k random fields. In [30] and in [31], stochastic line bundles are used. In [29], we give a general construction of $1+n$ dimensional theory, and we perform a theory of large deviation, in order to compute the action of the theory. In [33], we study stochastic cohomology of the space of random spheres, related to operads (For the aspect of operads related to n-fold loop space, we refer to the proceeding of Loday-Stasheff-Voronov [39]). The problem in [35] is that there is no Markov property of the random field, such that we cannot realize an operad by sewing punctured spheres.

Our goal is to construct a $1+2$ dimensional Wess-Zumino-Novikov-Witten model on the punctured sphere, which is Markovian on the boundary on the sphere. This Markov property allow us to realize an operad, by sewing random spheres along their boundary. For the material of sewing surfaces, by using the formal measure of physicist, we refer to the surveys of Gawedzki ([19], [16], [17]).

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2 Punctured random spheres and markov property

In order to construct a sphere with $1+n$ punctured points, we define first a sphere with $1+2$ punctured points (a pant), and we sew the pants along their boundary.

We consider a compact connected Lie group G of dimension d, equipped with its bi-invariant metric. We can imbedd it isometrically in a special orthogonal group.

We consider the Hilbert space H of maps from $S^1\times[0,1]$ into the real line R endowed with the following Hilbert structure:

$$\|h\|^2_{S^1\times[0,1]} = \int_{S^1\times[0,1]} |h(S)|^2 dS + \int_{S^1\times[0,1]} |\partial/\partial s h(S)|^2 dS + \int_{S^1\times[0,1]} |\partial/\partial t h(S)|^2 dS + \int_{S^1\times[0,1]} |\partial^2/\partial s\partial t h(S)|^2 dS \tag{2}$$

where $S=(s,t)$ belongs to $S^1\times[0,1]$ We can consider the free loop space of maps from S^1 into R with the Hilbert structure:

$$\|h\|^2_{S^1} = \int_0^1 |h(s)|^2 ds + \int_0^1 |h'(s)|^2 ds \tag{3}$$

We can find an element $e(s)$ of this Hilbert space such that

$$h(0) = <h, e> \tag{4}$$

where $e(s)=\lambda\exp[-s]+\mu\exp[s]$ for $0\le s\le 1$ such that $e(0)=e(1)$ but $e'(0)\neq e'(1)$.

We add in (2) the Neumann boundary condition:

$$\partial/\partial t h(s,0) = \partial/\partial t h(s,1) = 0 \tag{5}$$

Let us recall that the Green kernel over $[0,1]$ associated to the Hilbert space of functions from $[0,1]$ into R with Neumann boundary condition, associated to the Hilbert structure:

$$\int_0^1 |h(t)|^2 dt + \int_0^1 |h'(t)|^2 dt \tag{6}$$

satisfies to

$$e_t(t') = (\mu_t^- \exp[-t'] + \lambda_t^- \exp[t'])1_{t' \leq t} + (\mu_t^+ \exp[-t'] + \lambda_t^+ \exp[t'])1_{t' > t} \tag{7}$$

where $\mu_t^-, \lambda_t^-, \mu_t^+, \lambda_t^+$ depend smoothly on t. The Green kernel associated to the Hilbert structure (2) are the product of the one dimensional Green kernel $e_s(s')e_t(t') = E_{s,t}(s',t')$.

We would like to consider the same Hilbert space with the constraint $h(s^1,1) = h(s^2,1) = 0$ for two given times $s^1 < s^2$ (We can choose another condition, but we choose the simplest condition for the sake of simplicity). When we add this condition, we get another Hilbert space H^1 which is a finite codimensional subspace of the initial Hilbert space H.

We can find an orthonormal basis of the orthogonal complement of H^1 constituted from two maps $h^1(s,t)$ and $h^2(s,t)$ which are smooth in (s,t). Let us consider the Brownian motion with values in H. It is a 3 dimensional Gaussian process $B_u(s,t)$ where u denotes the propagation time and (s,t) the internal time. The covariance between $B_.(s,t)$ and $B_.(s',t')$ is $E_{s,t}(s',t')$. The Brownian motion with values in H^1 can be seen as

$$B_{1,u}(s,t) = \alpha^1 B_u(s,t) + \beta^1 B_u^1 h^1(s,t) + \gamma^1 B_u^2 h^2(s,t) \tag{8}$$

where $(\alpha^1, \beta^1, \gamma^1)$ are deterministic constants and B_u^1 and B_u^2 are two R-valued independent Brownian motion. In the sequel, we will choose this procedure in order to construct the Brownian motion $B_{1,u}(S)$ with values in H^1.

Let us consider the time $t = 1$ where the loop splits in two loops given by s_1 and s_2. We get after this splitting two circles. We consider the Hilbert space H^2 of maps from $S^1 \times [0,1]$ into R submitted to the boundary conditions $h(s,0) = h(s,1) = 0$ with the Hilbert structure:

$$\int_{S^1\times[0,1]} |\partial^2/\partial s \partial t h(S)|^2 dS + \int_{S^1\times[0,1]} |\partial/\partial t h(S)|^2 dS \tag{9}$$

In fact we should introduce some normalizing constant due to the fact that we do not consider the normalized Lebesgue measure over each circles given by splitting the circle into 2 circles. The Green kernel associated to this problem is the product of the Green kernel associated to (3) and the Green kernel associated to the Hilbert space of functions from $[0,1]$ into R equal to 0 in $t = 0$ and $t = 1$ associated to the Hilbert structure $\int_0^1 |h'(t)|^2 dt$. The Green kernel associated to this Hilbert space are of the type

$$e_t^2(t') = a_t t' 1_{t \geq t'} + b_t(t'-1)1_{t < t'} \tag{10}$$

where a_t and b_t are smooth. Therefore the Green kernel, $E_{s,t}^2(s',t')$, associated to the Hilbert space H^2 satisfy to

$$E_{s,t}^2(s',t') = e_s(s')e_t^2(t') \tag{11}$$

We consider an analogous Hilbert space H^3 with the Hilbert structure (9) and the boundary condition $h(s,0) = 0$ (without the boundary condition $h(s,1) = 0$). The Green kernel in t are of the type

$$e_t^3(t') = a_t t \wedge t' \tag{12}$$

and the global Green kernel satisfy to

$$E_{s,t}^3(s',t') = e_s(s')e_t^3(t') \tag{13}$$

Over each Hilbert space, we consider the Brownian motion $B_{i,.}(.,.)$. Let Σ be a pant (The elementary surface). Its boundary is constituted of circles, and we get tubes near the output boundary $S^1 \times [0,1/2]$ and tube near the input boundary $S^1 \times [1/2,1]$. Near the boundary, we consider the Brownian motion with values in H^3, by taking care that the starting condition $h(s,0) = 0$ is inside Σ for an output boundary and this condition is outside Σ for an input boundary. We choose 3 independent Brownian motion $B^3_.(.)$ over H^3. We multiply these Brownian motions by a deterministic function $g(t)$ equal to 0 only at 0 and 1 such that $g(1/2)B^3_.(.,1/2)$ corresponds to a normalized circle of length 1. Outside these boundary tubes, we consider over the cylinder with constraint $h(s_1,1) = h(s_2,1) = 0$, a Brownian motion with values in H^1, chosen independently of the others Brownian motions, but which intersect the input boundary tube on the cylinder $S^1 \times [1-\epsilon,1]$: we multiply by a smooth function $g(t) > 0$ which is 0 only in $1-\epsilon$. When the loop $s \to h(s,t)$ splits in two loops, we get two loops: we **add** the Brownian motion with values in H^2 over each (Two independent one modulo some normalizing constants), and we get two cylinders which intersect the exit tube $S^1 \times [0,1/2]$ over the tube $S^1 \times [0,\epsilon]$. We mutiply these Brownian motion by a smooth function $g(t) > 0$, and which is 0 on ϵ.

After performing all these glueing operations, we get an infinite dimensional Gaussian process parametrized by $[0,1] \times \Sigma$ $u \to B_{tot,u}(.)$, Which is an infinite dimensional Brownian motion with values in a suitable Hilbert space of functions on Σ which satisfies to the following properties:

-)For all $S \in \Sigma$, $u \to B_{tot,u}(S)$ is a Gaussian martingale.

-)$(u,S) \to B_{tot,u}(S)$ is almost surely Hölder, and if $<,>$ denotes the right bracket of the martingale theory, we get for $u \leq 1$

$$< B_{tot,.}(S), B_{tot,.}(S') > \leq Cd(S,S')^{1/2} \tag{14}$$

over each elementary parts of the pant Σ where the construction is done. Moreover, over the pant Σ, $(u,S) \to B_{tot,u}(S)$ is almost surely continuous.

c)Over each boundary of the pant, $u \to B_{tot,u}(S)$ are independent.

In order to curve these Gaussian processes, we use the theory of Brownian motion over a loop group of Airault-Malliavin [1] and Brzezniak-Elworthy [7].

Let e_i be a basis of the Lie algebra of G. Lert $B^i_{tot,.}(.)$ be d independent copies of $B_{tot,..}(S)$. We write $d_u B_{tot,u}(S) = \sum e_i d_u B^i_{tot,u}(S)$. We consider the equation in Stratonovitch sense:

$$d_u g_u(S) = g_u(S) d_u B_{tot,u}(S) \tag{15}$$

starting from e, the unit element in the group G..

We get (See [29], [31]) for proof in a closed context.

Theorem 2.1: Over each elementary part of the pant where the leading Brownian motion is constructed, the random field $S \to g_1(S)$ is almost surely $1/2 - \epsilon$ Hölder. Moreover, the random field on Σ: $S \to g_1(S)$ is almost surely continous, and its restriction on each circle on the boundary are independent.

In order to get a general $(1+n)$ punctured sphere, we sew successively pants, which are independent, except on the boundary, with a glueing condition. This glueing condition is, when we sew an exit loop of a pant to an input loop of another pant, we choose the same Brownian motion on H^3. We can do that, because the restriction to $S^1 \times \{1/2\}$ are the same. We get by that a tree or a punctured sphere $\Sigma(1,n)$. We get:

Theorem 2.2: Over each punctured sphere $\Sigma(1,n)$, the random field $S \to g_1(S)$ got after this sewing procedure is almost surely continuous.

By using this procedure, if we consider a $(1+n)$ punctured spheres $\Sigma(1,n)$ and n punctured spheres $\Sigma(1,n_i)$, we can glue the input loop to each $\Sigma(1,n_i)$ to the output loop of $\Sigma(1,n)$ and we get a sphere $\Sigma(1,\Sigma n_i)$. We suppose that all the data in this sewing procedure are independents, except for the Brownian motion in H^3 when we sew an output boundary in $\Sigma(1,n)$ to an input boundary in $\Sigma(1,n_i)$. Let us suppose that the random fields are sewed on the loops $(\partial\Sigma)_i$.

We get some thing like a Markov property along the sewing boundary:

Theorem 2.3: The random field $S \to g_1(S)$ over $\Sigma(1,\sum n_i)$ are conditionally independent over each $\Sigma(1,n_i)$ and over $\Sigma(1,n)$ conditionally to each $(\partial\Sigma)_i$.

Proof: We remark that for H^3

$$< B^3_.(s,t+1/2) - B^3_.(s,1/2), B^3_.(s',1/2) >= 0 \tag{16}$$

and that

$$< B^3_.(s,t+1/2) - B^3_.(s,1/2), B^3_.(s',1/2-t') - B^3_.(s',1/2) >= 0 \tag{17}$$

because in the t direction in H^3 , we have the covariance of a Brownian motion. This shows that the process $B^3_.(.,t+1/2) - B^3_.(.,t)$ and $B^3_.(.,1/2-t') - B^3_.(.,1/2)$ are independent. The only problem in establishing the Markov property lies near the boundary. But if we we write

$$g_1(S) = Id + \sum \int_{0<u_1..<u_n<1} dB_{tot,u_1}(S)..dB_{tot,u_n}(S) \tag{18}$$

we get that, after imbedding the group G in a matrix algebra

$$g_1(S) - g_1(S') = \sum \int_{0<u_1<..<u_n<1} (dB_{tot,u_1}(S)..dB_{tot,u_n}(S) - dB_{tot,u_1}(S')....dB_{tot,u_n}(S')) \tag{19}$$

and we write $dB_{tot,u}(S') = dB_{tot,u}(S') - dB_{tot,u}(S) + dB_{tot,u}(s)$ and we distribute in (18). Let us choose two points on the same component of the boundary S_1, S_2 in the boundary, and two points S' and S" not on the side of the boundary. We get that $g_1(S') - g_1(S_1)$ and $g_1(S") - g_1(S_2)$ are conditionnally independent when we suppose given the random field $g_1(S)$ on the boundary. Therefore the result.

$\diamondsuit$

3 Line integrals

When we consider the random punctured sphere $\Sigma(1,n)$, we get vertical loops given by $s \to g_1(s,t)$. Since $\Sigma(1,n)$ is built from elementary pants $\Sigma(1,2)$, it is enough to look each vertical loop $s \to g_1(s,t)$ over each elementary pants.

They are of 4 types:

-)The loop near the input boundary (Hilbert space $H^1 \oplus H^2$).

-)The loops in the body of the pants (Hilbert space H^1).

-)The two loops which are created from a big loop (Hilbert space $H^1 \oplus H^2$).

-)The loops near the exit boundary (Hilbert space $H^2 \oplus H^3$).

Let us consider a one form ω over G. We would like to define for each type of this loop the stochastic Stratonovitch integral:

$$\int_0^1 < \omega(g_1(s,t)), d_s g_1(s,t) > \tag{20}$$

We extend conveniently the one form ω in a smooth form bounded as well as all its derivatives over the matrix algebra where the matrix group is imbeddded. The technics are very similar to the technics of [31], part III.

Let dB_u be a Brownian motion with values in the Lie algebra of G. We consider the solution of the stochastic differential equation which gives the Brownian motion from e in the Lie group G:

$$d_u g_u = g_u d_u B_u \tag{21}$$

The equation of the differential of the differential of the stochastic flow associated to (21) is given (See [23], [26], [6]) by

$$d_u \phi_u = \phi_u d_u B_u \tag{22}$$

and the inverse of the differential of the the flow is given by an analoguous equation. It can be identified to g_u.

Let us consider a finite dimensional family $B_u(\alpha)$ of Brownian motion in the Lie algebra of G depending smoothly of a finite dimensional parameter α where α lives in a finite dimensional family of Brownian motion. We consider the stochastic differential equation depending on a parameter:

$$dg_u(\alpha) = g_u(\alpha) d_u B_u(\alpha) \tag{23}$$

The solution of the equation (15) has a smooth version in the finite dimensional parameter α.

$\partial/\partial\alpha g_u(\alpha)$ is for instance the solution of the linear stochastic differential equation with second member:

$$d_u \partial/\partial_u(\alpha) = \partial/\partial\alpha g_u(\alpha) d_u B_u(\alpha) + g_u(\alpha) d_u \partial/\partial\alpha B_u(\alpha) \tag{24}$$

This equation can be solved by the method of variation of the constant. We get:

$$\frac{\partial}{\partial\alpha} g_u(\alpha) = \phi_u(\alpha) \int_0^u \phi_v^{-1}(\alpha) d_v \frac{\partial}{\partial\alpha} B_v(\alpha) \tag{25}$$

We will write $s \to B_{tot,.}(s,t) = B_.(s)$, and in order to define stochastic line integrals, we will follow the method of [30] and [31], but in this case, it is much more simpler, because there is no conditioning. By using the properties of the Hilbert structure given H^1, H^2 and H^3, the covariance between $B_.(s)$ and $B_.(s')$ is given by $e(s-s')$. Let us suppose that $0 \leq s \leq s+\Delta s \leq t \leq t+\Delta t \leq 1$, and let us compute the covariance of $B_.(s+\Delta s) - B_.(s)$ and of $B_.(t+\Delta t) - B_.(t)$. It is given by

$$\begin{aligned} e(s+\Delta s - t - \Delta t) - e(s - t - \Delta t) - e(s - t + \Delta s) + e(s-t) = \\ Ce"(s-t)\Delta t \Delta s + O(\Delta t + \Delta s)^3 \end{aligned} \tag{26}$$

because e is smooth over $[-1,0] \sim [0,1]$(We use the periodicity assumption over $e(.)$. The only singularity in $e(.)$ comes from 0 identified to 1 in the circle).

This shows us that we can diagonalize the four non independent Brownian motions $B_.(s), B_.(s+\Delta s), B_.(t), B_.(t+\Delta t)$. We find 2 couples of independent Brownian motions $(w_.(1), w_.(2))$ and $(w_.(3), w_.(4))$ such that:

$$\begin{aligned} B_.(s) =& w_.(1) \\ B_.(s+\Delta s) =& \alpha(s,\Delta s) w_.(1) + \beta(s,\Delta s) w_.(2) \\ B_.(t) =& w_.(3) \\ B_.(t+\Delta t) =& \alpha(t,\Delta t) w_.(3) + \beta(t,\Delta t) w_.(4) \end{aligned} \tag{27}$$

Moreover t does not belong to $[s, s+\Delta s]$, such that the covariance of $B_.(s+\Delta s) - B_.(s)$ and $B_.(t)$ behaves as Δs because $e(s+\Delta s - t) - e(s-t). = e'(s-t)\Delta s + O(\Delta s)^2$.

Moreover,

$$\alpha(s,\Delta s) = C + C\Delta s + O(\Delta s)^{3/2} \tag{28}$$

$$\beta(s,\Delta s) = C\sqrt{\Delta s} + C\Delta s + O(\Delta s)^{3/2} \tag{29}$$

because $e(s+\Delta s - s) - e(0) = e'_+(0)\Delta s = +O(\Delta s)^2$ because e has semi-derivatives in 0 and $\Delta s > 0$ and $B_.(s+\Delta s)$ has a constant variance. From (26), we deduce that $< w_.(1), w_.(4) >= O(\sqrt{\Delta t})$, $< w_.(3), w_.(2) >= O(\sqrt{\Delta s})$ and that the correlator $< w_.(2), w_.(4) >= O(\sqrt{\Delta s \Delta t})$. We remark that $\frac{\partial}{\partial\sqrt{\Delta s}}\alpha(s,\Delta s)_{\Delta s=0} = 0$.

We imbed G isometrically in a space of linear matrices. It follows from the previous considerations that in law

$$g_.(s+\Delta s) = g_.(s) + \sqrt{\Delta s} g^1_.(s) + \Delta s g^2_.(s) + o(\Delta s)^{3/2} \tag{30}$$

where $g^1_.(s) = \phi_.(w_.(1)) \int_0^. \phi_u(w_.(1))^{-1} \frac{\partial}{\partial\sqrt{\Delta s}}\beta(s,0) dw_u(2)$. We don't write the analoguous expression for $g^2_.(s)$. There is a double integral in $dw_.(2)$ where the simple derivative of $\beta(s,\Delta)$ in $\sqrt{\Delta s}$ appear and a simple integral where the second derivative in $\sqrt{\Delta s}$ of $\alpha(s,\Delta s)$ and $\beta(s,\Delta s)$ appear. (.) is the time of the differential equation (15). Moreover, in law:

$$g_.(t+\Delta t) = g_.(t) + \sqrt{\Delta t} g^1_.(t) + \Delta t g^2_.(t) + O(\Delta t)^{3/2} \tag{31}$$

Let f and h be 2 smooth functions over the matrix space. We suppose they are bounded as well as their derivatives of all orders. We have the estimate which follows from the properties listed after (27), (28) (29):

$$E[f(g_u(s))g_u^1(s)h(g_v(t))g_v^1(t)] = C(s,t)\sqrt{\Delta s\Delta t} + O(\sqrt{\Delta s} + \sqrt{\Delta t})^{3/2} \tag{32}$$

where $C(s,t)$ is continuous. Namely, we conditionate by $w_.(2)$ and $w_.(4)$. There are terms which are $w_.(1)$ and $w_.(3)$ measurables in the expression we want to estimate. When we conditionate by $w_.(2)$ and $w_.(4)$, the expressions which are got belong to all the Sobolev spaces of Malliavin Calculus in $w_.(2)$ and $w_.(4)$. We can apply Clark-Ocone fortmula ([43]) to these expressions. We deduce since $< w_.(3), w_.(2) >= O(\sqrt{\Delta s})$and $< w_.(1), w_.(4) >= O(\sqrt{\Delta t})$ that the Itô integral which appears in the Clark-Ocone formula are in $O(\sqrt{\Delta s})dw_.(2)$ and in $O(\sqrt{\Delta t})dw_.(4)$. These leads to expressions of the type,

$$O(\sqrt{\Delta s})\int_{[0,1]^3} \alpha(s_1, s_2, s_3)dw_{s_1}(2)dw_{s_2}(2)dw_{s_3}(4) \tag{33}$$

where we used either Itô integral or Stratonovitch integral. We convert it in Skorokhod integral (whose expectation is 0) and we find a counterterm in $O(\Delta s)$ (We can suppose that $\Delta s = \Delta t$ as we will do in the sequel). For that we used the following result: let f a smooth functional with bounded derivatives of all orders in a finite number of $g_u(s)$ or in $g_u(t)$. Let F the associated Wiener cylindrical functional. Let $\tilde{F} = E[F|w_.(2), w.(4)]$. It is a smooth functional in the sense of Malliavin Calculus in $w_.(2)$, $w_.(4)$ and its derivatives $D^k\tilde{F}(t_1, .., t_k)$ have an estimate in $O(\sqrt{\Delta s})^k$

We consider a smooth 1-form ω_v in the spaces of matrices with bounded derivatives of all orders which depends smoothly from a finite dimensional parameter v. We suppose that the derivatives in the parameter v are bounded.

We consider 2^N, N being a big integer, and the dyadic subdivision of [0,1] associated to 2^N. We call it s_i with $s_i < s_{i+1}$ such that $s_{i+1} - s_i = 2^{-N}$. If $s \in [s_i, s_{i+1}]$, we call

$$g_u^N(s) = g_u(s_i) + \frac{s - s_i}{s_{i+1} - s_i}(g_u(s_{i+1}) - g_u(s_i)) \tag{34}$$

$s \to g_1^N(s)$ is piecewise differentiable. We consider the random variable:

$$A_v^N = \int_0^1 < \omega(g_1^N(s), d_s g_1^N(s) > \tag{35}$$

Let us give the following decomposition of A_v^N:

$$\begin{aligned} A_v^N = \sum \int_{s_i}^{s_{i+1}} &< \omega(g_1^N(s)) - \omega(g_1^N(s_i)), d_s g_s^N(s) > \\ + \sum \int_{s_i}^{s_{i+1}} &< \omega(g_1^N(s_i), d_s g_s^N(s) >= A_v^N(<,>) + A_v^N(\delta) \end{aligned} \tag{36}$$

The Itô term is $A_v^N(\delta)$ and the Stratonovitch counterterm is $A_v^N(<,>)$. The Itô term can be divided into two pieces: the first one is when in (30) we take the term in $g_.^1(s)$ and the second one is when we take in (31) the term in $g_.^2(s)$. We get the decomposition, of the Itô term in

$A_v^N(\delta_1) + A_V^N(\delta_2)$. The term which diverges "a priori" is $A_v^N(\delta_1)$. But we can use (32), and show that when $N \to \infty$,

$$E[A_v^N(\delta_1)^2] \to \int_{S_1 \times S_1} C(s,t)dsdt + \int_{S_1} C(s)ds \tag{37}$$

where $C(s,t)$ is continuous.

Moreover, the second part in the Itô term checks clearly:

$$E[A_v^N(\delta_2)^2] \to \int_{S_1 \times S_1} C_1(s,t)dsdt + \int_{S_1} C_1(s)ds \tag{38}$$

Since the counterterm which is due to the Stratonovitch correction is a "a priori" less diverging, we can see in an analoguous way that:

$$E[A_v^N(<,>)^2] \to \int_{S_1 \times S_1} C_2(s,t)dsdt + \int_{S_1} C_2(s)ds \tag{39}$$

These remarks justify but not prove the following proposition:

Proposition 3.1: When $N \to \infty$, the sequence of random variables A_v^N tends in L^2 to a limit random variable called $\int_{S^1} < \omega_v(g_1(s)), d_s g_1(s) >= A_v$. Moreover, there exists a smooth version of the line integral A_v in v.

Proof: Let us forget for the moment the parameter v. Let us write:

$$A^N = \sum_i \int_{[s_i, s_{i+1}]} < \omega(g_1^N(s)), d_s g_1^N(s) >= \sum (B_i^N + C_i^N) \tag{40}$$

where B_i^N is the Bracket term

$$B_i^N = \int_{[s_i, s_{i+1}]} < \omega(g_1^N(s)) - \omega(g_1^N(s_i)), d_s g_1^N(s) > \tag{41}$$

and C_i^N is the Itô term:

$$C_i^N =< \omega(g_1(s_i)), \Delta_s g_1(s_i) > \tag{42}$$

We write

$$C_i^N = D_i^N + E_i^N + O(2^{-3N/2}) \tag{43}$$

where

$$D_i^N = \sqrt{s_{i+1} - s_i} < \omega(g_1(s_i)), g_1^1(s_i) > \tag{44}$$

and

$$E_i^N = (s_{i+1} - s_i) < \omega(g_1(s_i)), g_1^2(s_i) > \tag{45}$$

First step: convergence of $\sum E_i^N$.

In $g_1^2(s_i)$ whose writing is derived from (24) by taking another derivative, there is a linear integral which comes from the second derivative of $\alpha(s_i + \Delta s_i)$, from a second derivative in $\beta(s, \Delta s)$ in $\sqrt{\Delta s}$ and a double integral which comes from taking only one derivative in $\beta(s, \Delta s)$. The term in the linear integral can be treated in the following way: we get $\sum E_{i,1}^N$. If $M > N$

$$(\sum E_{i,1}^N - \sum E_{j,1}^M)^2 = (\sum_i (\sum_{[s_j, s_{j+1}] \subseteq [s_i, s_{i+1}]} E_{i,1}^N - E_{j,1}^M))^2 \tag{46}$$

In order to compute $\sum_{[s_j,s_{j+1}]\subseteq[s_i,s_{i+1}]} E^N_{i,1} - E^M_{j,1}$, we write $s_{i+1} - s_i = \sum s_{j+1} - s_j$ such that we can write the sum to estimate

$$\sum_{[s_j,s_{j+1}]\subseteq[s_i,s_{i+1}]} (s_{j+1} - s_j)(< \omega(g_1(s_i), \tilde{g}_1(s_i) > - < \omega(g_1(s_j)), \tilde{g}_1(s_j) >) \tag{47}$$

$\tilde{g}_1(s_i)$ is the term in the simple integral where we take the second derivatives in $\sqrt{\Delta s}$ of $\alpha(s, \Delta s)$ and $\beta(s, \Delta s)$. The terms which are integrated depend continuously from s. Therefore the contribution where we take two derivatives of $\alpha(s, \Delta s)$ vanish. It remains to consider the contribution where we take two derivatives of $\beta(s, \Delta s)$. We can replace the terms considered by

$$\sum_i \sum_{[s_j,s_{j+1}]\subseteq[s_i,s_{i+1}]} < \omega(g_1(s_i)), \overline{g}_1(s_i) > - < \omega(g_1(s_j)), \overline{g}_1(s_j) > \tag{48}$$

where we have replaced the term in two derivatives by $\sqrt{\Delta s_j} B_{.}(s_j + \Delta s_j) - B_{.}(s_j)$. We write $B_{.}(s + \Delta s_i) - B_{.}(s_i) = \sum B_{.}(s_j + \Delta s_j) - B_{.}(s_j)$ and we see that $< B_{.}(s_j + \Delta s_j) - B_{.}(s_j), B_{.}(s_{j'} + \Delta s_{j'}) - B_{.}(s_{j'}) >= O(\Delta s_j \Delta s_{j'})$ if $j \neq j'$ and equal to $O(\Delta s_j))$ if $j = j'$. This shows that the L^2 norm of

$$\sum_{[s_j,s_{j+1}]\subseteq[s_i,s_{i+1}]} (< \omega(g_1(s_i)), \overline{g}_1(s_i) > - < \omega(g_1(s_j)), \overline{g}_1(s_j) >) \tag{49}$$

behaves as $O(1/N)\Delta s_j$ because $\omega(g_1(s))$ depends continuously of s and after using the desintegration argument used after (32).

The problem arises when we take the double integral. In order to study the behaviour of its sum, we can replace $w_{.}(2)$ in (27) by $B_{.}(s_i + \Delta s_i) - B_{.}(s_i)$ and take the double stochastic integral which is associated by taking the derivative of the flow $\phi_u(s_i)$ associted to the equation $dg_u(s_i) = g_u(s_i) dB_u(s_i)$. Namely, we consider a double integral of the type

$$\int_{0<u<v<1} \sqrt{\Delta s_i} \phi_u^{-1} dw_u(2) \sqrt{\Delta s_i} \phi_v^{-1} dw_v(2) \tag{50}$$

which behaves modulo an error term in $O(\Delta s_i)^{3/2}$ as

$$\int_{0<u<v<1} \phi_u^{-1} \Delta_{s_i} B_u(s_i) \phi_v^{-1} \Delta_{s_i} B_v(s_i) \tag{51}$$

For the convergence of E^N_i, we can assimilate $(s_{i+1} - s_i) g^2_u(s_i)$ with the double integral $\alpha_u(s_i)$ after performing these replacements. Let $N' > N$ and s_j be the dyadic subdivision which is associated. We sum over $[s_j, s_{j+1}] \subseteq [s_i, s_{i+1}]$. We get:

$$\begin{gathered} < \omega(g_t(s_i)), \alpha_t(s_i)) > - \sum_j < \omega(g_t(s_j)), \alpha_t(s_j) >= \\ \sum_j (< \omega(g_t(s_i)) - \omega(g_t(s_j)), \alpha_t(s_j) > + < \omega(g_t(s_i)), \alpha_t(s_i) - \sum_j \alpha_t(s_j)) > \\ = \delta^N_i + \epsilon^N_i \end{gathered} \tag{52}$$

The sum of the first term tends to 0 in L^2. The difficult term is to estimate the term in ϵ_i^N. In the double integral which compose $\alpha_t(s_i)$, we write

$$B_.(s_i+\Delta s_i)-B_.(s_i)=\sum_{[s_j,s_{j+1}]\subseteq[s_i,s_{i+1}]}B_.(s_j+\Delta s_j)-B_.(s_j) \tag{53}$$

We distribute the integrands. Over each $dB_.(s_i+\Delta s_i)-dB_.(s_i)$, there is in the double integral a term which $B_.(s_i)$ measurable, which is adapted and which depends on a continuous way of s_i. Since it depends on a continuous way from s_i, we can replace it when we distibute by the corresponding term in s_j in $\alpha_t(s_i)$. After distributing in $\alpha_t(s_i)-\sum\alpha_t(s_j)$, the diagonal term are substracting, and it remains to study the process

$$\begin{aligned}\delta_t^N=\sum_i &< \omega(g_t(s_i)),\\ \sum_{[s_j,s_{j+1}]\subseteq[s_i,s_{i+1}],[s_{j'},s_{j'+1}]\subseteq[s_i,s_{i+1}]j\neq j'}&\int_{0<u<v<t}r_u(s_j)d_u\Delta_{s_j}B_u(s_j)r_v(s_{j'})d_v\Delta_{s_{j'}}B_v(s_{j'})>\end{aligned} \tag{54}$$

We decompose the semi martingale δ_t^N into a finite variational part which converges by using (26) to 0 and a martingale part M_t^N. Namely, we can convert the double Stratonovitch integral which appears in (54) in an Itô integral. The boring term arises when we replace the double Stratonovitch integral by an Itô integral in (54). We would like to show that this martingale tends to 0. For that, we compute its quadratic variation. We get a sum over all quadruple $[s_{j_1},s_{j_1+1}]$, $[s_{j_2},s_{j_2+1}]$, $[s_{j_3},s_{j_3+1}]$ and $[s_{j_4},s_{j_4+1}]$.

-**First case**: let us suppose that all the elements of the quadruple are different. The contribution of each quadruple is in $2^{-4N'}$ by the properties listed after (27), (28), (29) which express that the covariance of $B_.(s_j+\Delta s_j)-B_.(s_j)$ and of $B_.(s_{j'+1})-B_.(s_{j'})$ in term of $\Delta s_j\Delta s_{j'}$ and the covariance of $(B_.(s_j+\Delta s_j)-B_.(s_j)$ and of $B_.(t)$ in Δs_j if t does not belong to $[s_j,s_{j+1}]$. Namely, if the intervals $[s_{j_1},s_{j_1+1}]$, $[s_{j_2},s_{j_2+1}]$ do not intersect and if s_{j_3} and s_{j_4} do not belong to these intervals, we have only to show by using the Itô formula that

$$\begin{aligned}E[\int_{o<u<v}r_u(s_{j_1})d_u\Delta_{s_{j_1}}B_u(s_{j_1})\int_{o<u<v}r_u(s_{s_{j_2}})d_u\Delta_{s_{j_2}}B_u(s_{j_2})r_v(s_{J_3})r_v(s_{j_4})]\\ =O(\Delta s_{j_1}\Delta s_{j_2})\end{aligned} \tag{55}$$

because the right Bracket between $\Delta_{s_{j_3}}B(s_{j_3}$ and $\Delta_{s_{j_4}}B(s_{j_4})$ is in $O(\Delta s_{j_3}\Delta s_{j_4})$ We take the conditional expectation of $r_v(s_{j_3})$ and $r_v(s_{j_4})$ along the Gaussian space spanned by $B_.(s_{j_1})$, $B_.(s_{j_2})$, $\Delta_{s_{j_1}}B(s_{j_1})$ and $\Delta_{s_{j_2}}B_.(s_{j_2})$. We can suppose that $r_v(s_{j_3})$ and $r_v(s_{j_3})$ are measurable over this Gaussian space. But r_v is solution of the stochastic differential equation giving the flow of the Brownian motion over the Lie group, and is therefore a stochastic integral. We use the following rules for calculating different conditional expectation for the solution of this flow. We consider the solution of the stochastic differential equation starting from the identity:

$$dA_t=A_t(dB_t+d\tilde{B}_t) \tag{56}$$

where B_t and $\tilde{B}_t$ are two independent Brownian motions. We can write $A_t = W_t V_t$ where $dV_t = V_t dB_t$ and $dW_t = W_t V_t d\tilde{B}_t V_t^{-1}$. after using this remark in order to calculate the conditional expectation, we desintegrate along $\Delta_{s_{j_1}} B_.(s_{j_1})$ and $\Delta_{s_{j_2}} B_.(s_{j_2})$ as in (32), and we conclude by using the consideration following (27), (28), (29).

They are at most $2^{2N}2^{4(N'-N)}$ such possibilities. The total contribution is 2^{-2N} which tends to 0 when $N \to \infty$.

-)**Second case**: there are 3 different intervals $[s_j, s_{j+1}]$. This can come from a concatenation of two times d_v for $u < v$ in the stochastic integral (54) after converting it in a double Itô integral or a concatenation of the same term d_u in the stochastic integral (54). The contribution of each term is $2^{-3N'}$ by doing as in the first case.. They are at most $2^N 2^{N'-N)} 2^{2(N'-N)} = 2^{3N'}2^{-2N}$ such possibilities. The total contribution behaves in 2^{-2N} which tends to 0 when $N \to \infty$.

-)**Third case**: there are 2 different intervals $[s_j, s_{j+1}]$. The contribution of each element which appears is in $2^{-2N'}$ by doing as in the first case. There are at most $2^N 2^{2(N'-N)}$ such terms. The total contribution is in 2^{-N} which converges to 0 when $N \to \infty$.

This shows us that $\sum E_i^N$ is a Cauchy sequence in L^2.

Second Step: convergence of the Itô term $\sum D_i^N$.

We write

$$\alpha_i^N = D_i^N - \sum_{[s_j,s_{j+1}]\subseteq[s_i,s_{i+1}]} D_j^{N'} \tag{57}$$

and we would like to show that $\sum \alpha_i^N \to 0$ in L^2.

They are two terms to study:

-)The contribution of $E[\alpha_i^N \alpha_{i'}^N]$ for $i \neq i'$. By (32),

$$\sum_{i\neq i'} E[\alpha_i^N \alpha_{i'}^N] \to 2\int_{S^1\times S^1} C_2(s,t)dsdt - 2\int_{S^1\times S^1} C_2(s,t)dsdt = 0 \tag{58}$$

-) The contribution of $\sum_i E[(\alpha_i^N)^2]$. By using the consideration of the first step, we can write modulo a term which vanish that

$$\begin{aligned} \alpha_i^N =&< \omega(g_1(s_i)), \Delta_{s_i} g_i(s_i) > - \sum_{[s_j,s_{j+1}]} < \omega(g_1(s_j)), \Delta_{s_j} g_1(s_j) > \\ =& \sum_{[s_j,s_{j+1}]\subseteq[s_i,s_{i+1}]} < \omega(g_1(s_i)) - \omega(g_1(s_j)), \Delta_{s_j} g_1(s_j) >= \sum_j \beta_j^N \end{aligned} \tag{59}$$

To study its convergence, we write:

$$\begin{gathered} B_.(s_i) = w_.(1) \\ B_.(s_j) = \alpha(s_i, s_j) w_.(1) + \beta(s_i, s_j) w_.(2) \\ B_.(s_j = \Delta s_j) = \alpha(s_i, s_j, \Delta s_j) w_.(1) + \beta(s_i, s_j,\Delta s_j) w_.(2) + \gamma(s_i, s_j, \Delta s_j) w_.(3) \\ B_.(s_{j'}) = \alpha(s_i, s_{j'}) w_.(1) + \beta(s_i, s_{j'}) w_.(4) \\ B_.(s_{j'} + \Delta s_{j'}) = \alpha(s_i, s_{j'}, \Delta s_{j'}) w_.(1) + \beta(s_i, s_{j'},\Delta s_{j'}) w_.(4) + \gamma(s_i, s_{j'}, \Delta s_{j'}) w_.(5) \end{gathered} \tag{60}$$

We have $\gamma(s,t,\Delta t) = C(s,t)\sqrt{\Delta t} + O(\Delta t)$, $\beta(s,t,\Delta t) - \beta(s,t) = C(s,t)\Delta t + O(\Delta t)^{3/2}$ and $\alpha(s,t,\Delta t) - \alpha(s,t) = C'(s,t)\Delta t + O(\Delta t)^{3/2}$. We deduce that $< w_.(5), w_.(3) >= o(\Delta s_j)$, $<$

$w_{.}(5), w_{.}(2) >= O(\sqrt{\Delta s_j})$ and $< w_{.}(5), w_{.}(1) >= O(\sqrt{\Delta s_j})$. In a similar way, we have $< w_{.}(3), w_{.}(1) >= O(\sqrt{\Delta s_j})$, $< w_{.}(3), w_{.}(4) >= O(\Delta s_j)$ (We used the fact that $\Delta s_j = \Delta s_{j'}$). With this decomposition, we write the analoguous of (30) and (3 1) for $g_{.}(s_j + \Delta s_j)$ by doing the conditional expectation along the Gaussian processes $w_{.}(5), w_{.}(4), w_{.}(2), w_{.}(3)$ and for $g_{.}(s_{j'} + \Delta s_{j'})$. We find if $j \neq j'$ $E[\beta_j^N \beta_{j'}^N] = o(I/N)2^{-2N'}$ and in the other cases $E[|\beta_j^N|^2] = o(1/N)2^{-N'}$. Therefore, $E[|C_i^N|^2] = o(1/N)2^{-N}$ and $\sum_i E[|C_i^N|^2] \to 0$.

Third step: study of the convergence of $\sum B_i^N$.

We write

$$\omega(g_1^N(s)) - \omega(g_1(s_i)) = \frac{s - s_i}{\sqrt{s_{i+1} - s_i}} g_1^1(s_i)\alpha(g_1(s_i)) + O(s - s_i) \tag{61}$$

and

$$d_s g_1^N(s) = \frac{ds}{\sqrt{s_{i+1} - s_i}} g_1^1(s_i) + ds g_1^2(s_i) + ds O(s_{i+1} - s_i) \tag{62}$$

The more singular singular tem in B_i^N is

$$\begin{aligned} \alpha_i^N = \int_{s_i}^{s_{i+1}} \frac{s - s_i}{s_{i+1} - si} < g_1^1(s_i), \alpha(g_1(s_i), g_1^1(s_i) > ds = \\ (s_{i+1} - s_i) < g_1^1(s_i)\alpha(g_1(s_i)), g_1^1(s_i) > \end{aligned} \tag{63}$$

There is in the previous contribution a quadratic expression in $g_1^1(s_i)$. These expressions can be treated exactly as in the first step of the convergence of $\sum E_i^N$, by writing $< g_1^1(s_i), g_1^1(s_i) >$ as a double integral and relpacing $(s_{i+1} - s_i) < g_1^1(s_i), g_1^1(s_i) >$ by a double stochastic integral where we have removed $\sqrt{\Delta s_i} w_{.}(1)$ by $\Delta_{s_i} B_{.}(s_i)$. The sum of the others terms tends clearly to 0.

In order to show that $\int_{S^1} < \omega_v(g_1(s)), d_s g_1(s) >$ has a smooth version, we show that the system of derivatives of A_v^N in v converges in L^2. We conclude by using the embedding Sobolev theorem as in [23].

$\diamondsuit$

We consider a more intrinsic approximation of the line integral. We use if $g_1(s_i, t), g_1(s_{i+1}, t)$ are close,

$$F_N(s, g_1(s_i, t), g_1(s_{i+1}, t)) = \exp[\frac{s - s_i}{s_{i+1} - s_i} \log(g_1(s_{i+1}, t) g_1(s_i, t)^{-1})] g_(s_i, t) \tag{64}$$

conveniently extended over the set of all matrices. We put:

$$\tilde{g}_1^N(s, t) = F_N(s, g_1(s_i, t), g_1(s_{i+1}, t)) \tag{65}$$

We consider $\tilde{A}_v^N$ as in (35) with this new approximation. If we look the asymptotic expansion of F_N, we see that the more singular term in $d_s \tilde{g}_1^N(s, t)$ and $d_s g_1^N(s, t)$ coincides. This justify the following theorem:

Theorem 3.2: $\tilde{A}_v^N$ tends in L^2 for the C^k topology over each compact of the parameter set to the Stratonovitch integral $\int_{S^1} < \omega_v(g(s, t)), d_s g(s, t) >$ which has a smooth version in v.

Remark: We don't know if the Stratonovitch integrals of Theorem III.2 and of Proposition III.1 coincide. In the sequel, we will use the version of Theorem III.1, because it is a geometrical version.

Remark: Instead of integrating over a circle, we can integrate over a segment.

4 Integral of a two form

We decompose the pant $\Sigma(1,2)$ in elementary cylinders $S^1 \times [0,1] = D$. Let $B_.(s,t) = B_{tot,.}(s,t)$ be the Brownian motion parametrized by these elementary cylinders. Each correlators check all the properties listed in the part IV of [31] such that each correlator is smooth outside the diagonals and its derivative has half limits on the diagonals, such that we can apply the technics of the part IV of [31]. The requested properties which come from the properties of the correlator are for elementary cylinders which constitute the pant:

Property H1

$$< B_.(s+\Delta s,t) - B_.(s,t), B_.(u,v) >= O(\Delta s) \tag{66}$$

if u does not belong to $]s, s+\Delta s[$ and the symmetric property.

Property H2

$$< B_.(s+\Delta s),t) - B_.(s,t), B_.(u,v+\Delta v) - B_.(u,v) >= O(\Delta s \Delta v) \tag{67}$$

if u does not belong to $]s, s+\Delta s[$ and t does not belong to $]v, v+\Delta v[$.

Property H3

$$< B_.(s+\Delta s,t) - B_.(s,t), B_.(s'+\Delta s',u) - B_.(s',u) = O(\Delta s \Delta s') \tag{68}$$

if $]s', s'+\Delta s'[\cap]s, s+\Delta s[= \emptyset$ and the symmetric property.

Property H4: If $t' \geq t$,

$$< B_.(s+\Delta s,t') - B_.(s,t'), B_.(s+\Delta s,t) - B_.(s,t) >= C(t,t')\Delta s \tag{69}$$

where $C(t,t')$ is continuous, the same being true for the symmetric case.

We imbedd G into a matrix algebra isometrically. Let $g(s,t)$ be the random field parametrized by the torus with values in G. Let 2^N be an integer, and s_i be the associated dyadic subdivision of S^1 and t_j be the associated dyadic subdivision of a copy of $[0,1]$. We consider the polygonal approximation of $g(s,t)$, if $(s,t) \in [s_i, s_{i+1}] \times [t_j, t_{j+1}] = T_{i,j}$.

$$\begin{aligned} g^N(s,t) = g(s_i,t_j) &+ \frac{s-s_i}{s_{i+1}-s_i}(g(s_{i+1},t_j) - g(s_i,t_j)) + \frac{t-t_j}{t_{j+1}-t_j}(g(s_i,t_{j+1}) - g(s_i,t_j)) \\ &+ \frac{t-t_j}{t_{j+1}-t_j}\frac{s-s_i}{s_{i+1}-s_i}(g(s_{i+1},t_{j+1}) - g(s_i,t_{j+1}) - g(s_{i+1},t_j) + g(s_i,t_j)) \\ &= g(s_i,t_j) + \alpha_1^N(s) + \alpha_2^N(t) + \alpha_3^N(s,t) \end{aligned} \tag{70}$$

Let us consider a two form ω over G, conveniently extended in a two form ω over the matrix algebra bounded with bounded derivatives of all orders. We suppose that the two form depends on a finite dimensional parameter v. We consider

$$A_v^N = \int_D (g^N)^* \omega_v = \int_{S^1 \times [0,1]} < \omega_v(g^N(s,t)), d_s g^N(s,t), d_t g^N(s,t) > \tag{71}$$

Let us denote by $\Delta_{t_j} g(s_i, t_j)$ the quantity $g(s_i, t_{j+1}) - g(s_i, t_j)$, by $\Delta_{s_i} g(s_i, t_j)$ the quantity $g(s_{i+1}, t_j) - g(s_i, t_j)$ where we have imbedded the group G in a linear space. If $i \neq i'$, $j \neq j'$, we will see later that

$$E[\Delta_{s_i} g(s_i, t_j)\Delta_{t_j} g(s_i, t_j)\Delta_{s_{i'}} g(s_{i'}, t_{j'})\Delta_{t_{j'}} g(s_{i'}, t_{j'})] = O(\Delta s_i \Delta t_j \Delta s_{i'} \Delta t_{j'}) \tag{72}$$

where we take a quadratic expression homogeneous in each term in each increment. The most diverging term in the quantity A_v^N is

$$\sum_{i,j} < \omega_v(g(s_i, t_j)), \Delta_{s_i} g(s_i, t_j), \Delta_{t_j} g(s_i, t_j) > \tag{73}$$

When the length of the subdivision tends to zero, the L^2-norm of this expression tends to

$$\begin{aligned} &\int_{D\times D} C(s,t,s',t')ds ds' dt dt' + \int_{S^1\times D} C(s,t,t')ds dt dt' \\ &+ \int_{D\times[0,1]} C(s,s',t) ds ds' dt + \int_D C(s,t) ds dt \end{aligned} \tag{74}$$

This justifies without to prove the following proposition:

Proposition 4.1: When $N \to \infty$, the traditional integral A_v^N tends for the C^k topology over each compact of the parameter space in L^2 to the stochastic integral in Stratonovich sense:

$$\int_D g^* \omega_v = \int_{S^1\times[0,1]} < \omega(g(s,t)), d_s g(s,t), d_t g(s,t) > \tag{75}$$

where the stochastic integral $\int_D g^*\omega_v$ has a smooth version in v.

Proof: We suppose first that there is no auxiliary parameter. We can write:

$$\begin{aligned} A^N = &\int_D < \omega(g^N(s,t)), d_s\alpha_1^N(s), d_t\alpha_2^N(t) > \\ &+ \int_D < \omega(g^N(s,t)), d_s\alpha_1^N(s), d_t\alpha_3^N(s,t) > \\ &+ \int_D < \omega(g^N(s,t)), d_s\alpha_3^N(s,t), d_t\alpha_2^N(t) > \\ &+ \int_D < \omega(g^N(s,t)), d_s\alpha_3^N(s,t), d_t\alpha_3^N(s,t) > = A_1^N + A_2^N + A_3^N + A_4^N \end{aligned} \tag{76}$$

STEP I: convergence of A_1^N. We repeat the considerations of the part III for $s \to B_.(s, t_j)$ and $t \to B_.(s_i, t)$. If we fix t_j, we get by (30) an asymptotic expansion in order 3. We get expressions in the asymptotic expansion in $g_.^{1;\cdot}(s_i, t_j)$, $g_.^{2;\cdot}(s_i; t_j)$ and $g^{3;\cdot}(s_i, t_j)$. If we fix s_i, we go in (30) to an asymptotic expansion at order 3. We get derivatives in law $g_.^{\cdot;1}(s_i, t_j)$, $g^{\cdot;2}(s_i, t_j)$ and $g_.^{\cdot;3}(s_i, t_j)$.

We get:

$$\begin{aligned} A_1^N = &\sum_{i,j} < \omega(g(s_i, t_j)), g(s_{i+1}, t_j) - g(s_i, t_j), g(s_i, t_{j+1}) - g(s_i, t_j) > \\ &+ \sum_{i,j} \int_{T_{i,j}} < \omega(g^N(s,t)) - \omega(g(s_i, t_j)), d_s\alpha_1^N(s), d_t\alpha_2^N(t) > = B_1^N + B_2^N \end{aligned} \tag{77}$$

B_1^N is the Itô term, which is apparently the most diverging when $N \to \infty$. B_2^N is the Stratonovitch counterterm.

Step I.1: convergence of the Itô term B_1^N.

We write as in (30)

$$\begin{array}{c} g(s_{i+1},t_j) - g(s_i,t_j) = \sqrt{s_{i+1}-s_i}g^{1;\cdot}(s_i,t_j) + \\ (s_{i+1}-s_i)g^{2;\cdot}(s_i,t_j) + (s_{i+1}-s_i)^{3/2}g^{3;\cdot}(s_i,t_j) + O(s_{i+1}-s_i)^2 \end{array} \tag{78}$$

and we write as in (30)

$$\begin{array}{c} g(s_i,t_{j+1}) - g(s_i,t_j) = \sqrt{t_{j+1}-t_j}g^{\cdot;1}(s_i,t_j) \\ +(t_{j+1}-t_j)g^{\cdot;2}(s_i,t_j) + (t_{j+1}-t_j)^{3/2}g^{\cdot;3}(s_i,t_j) + O(s_{i+1}-s_i)^2 \end{array} \tag{79}$$

This will lead to stochastic integrals in $\sqrt{s_{i+1}-s_i}g^{1;\cdot}(s_i,t_j)$ and in $\sqrt{t_{j+1}-t_j}g^{\cdot;1}(s_i,t_j)$ which apparently do not converge and to integrals in $(s_{i+1}-s_i)g^{2;\cdot}(s_i,t_j)$ as in $(t_{j+1}-t_j)g^{\cdot;2}(s_i,t_j)$ which will lead to classical integrals. We deduce the following decomposition of the Itô term B_1^N:

$$B_1^N = C_1^N + C_2^N + C_3^N + C_4^N + C_5^N + error \tag{80}$$

-)C_1^N is the double stochastic integral in the time direction s and in the time direction t:

$$C_1^N = \sum_{i,j} \sqrt{\Delta s_i}\sqrt{\Delta t_j} < \omega(g(s_i,t_j)), g^{1;\cdot}(s_i,t_j), g^{\cdot;1}(s_i,t_j) > \tag{81}$$

-)C_2^N is a stochastic integral in the direction s and a classical integral in the direction t:

$$C_2^N = \sum_{i,j} \sqrt{\Delta s_i}\Delta t_j < \omega(g(s_i,t_j)), g^{1.\cdot}(s_i;t_j), g^{\cdot;2}(s_i;t_j) > \tag{82}$$

-)C_3^N is a vanishing term:

$$\begin{aligned} C_3^N = & \sum_{i,j} \sqrt{\Delta s_i}\Delta t_j^{3/2} < \omega(g(s_i,t_j)), g^{1;\cdot}(s_i,t_j), g^{\cdot;3}(s_i,t_j) > \\ & + \sum_{i,j} (\Delta s_i)^{3/2}\sqrt{\Delta t_j} < \omega(g(s_i,t_j)), g^{3;\cdot}(s_i,t_j), g^{\cdot;1}(s_i,t_j) > \end{aligned} \tag{83}$$

-)C_4^N is a classical integral in the time direction s and a stochastic integral in the time direction t:

$$C_4^N = \sum_{i,j} \Delta s_i\sqrt{\Delta t_j} < \omega(g(s_i,t_j)), g^{2;\cdot}(s_i,t_j), g^{\cdot;1}(s_i,t_j) > \tag{84}$$

-)C_5^N is a classical integral in the time direction s and in the time direction t.

$$C_5^N = \sum_{i,j} \Delta s_i\Delta t_j < \omega(g(s_i,t_j)), g^{2;\cdot}(s_i,t_j), g^{\cdot;2}(s_i,t_j) > \tag{85}$$

C_1^N is the more "a priori" divergent term when N tends to ∞ and C_5^N will lead to a double classical integral on the torus.

Step I.1.1: For integers N, N' such that $N' > N$, we consider $C_1^N = \sum_{i,j} C_{i,j,1}^N$.
We consider a bigger integer N' than N and we consider

$$D_{i,j,1}^{N'} = C_{i,j,1}^N - \sum_{T_{i',j'} \subseteq T_{i,j}} C_{i',j',1}^{N'} \tag{86}$$

Let us consider first the case where $0 \leq s+\Delta s \leq s' \leq s'+\Delta s' \leq 1$ and $0 \leq t+\Delta t \leq t' \leq t'+\Delta t' \leq 1$. We get if f and g are smooth functions with bounded derivatives of all orders:

$$\begin{gathered} E[f(g(s,t))h(g(s',t'))g^{1;\cdot}(s,t)g^{\cdot;1}(s,t)g^{1;\cdot}(s',t')g^{\cdot;1}(s',t')] = \\ C(s,t,s',t')\sqrt{\Delta s}\sqrt{\Delta t}\sqrt{\Delta s}\sqrt{\Delta t'} + error \end{gathered} \tag{87}$$

In order to see that, we begin by diagonalizing $B_{.}(s,t)$ and $B_{.}(s',t')$.

$$B_{.}(s,t) = w_{.}(1) \tag{88}$$

We write:

$$\begin{gathered} B_{.}(s+\Delta s,t) = \alpha(s,t,\Delta s)w_{.}(1)+\beta(s,t,\Delta s)w_{.}(3) \\ B_{.}(s,t+\Delta t) = \alpha(s,t,\Delta t)w_{.}(1) + \beta(s,t,\Delta t)w_{.}(4) \end{gathered} \tag{89}$$

and the analoguous formulas for $B_{.}(s'+\Delta s',t')$ and $B_{.}(s',t'+\Delta t')$ with some other new auxiliary Brownian motions $w_{.}(5)$ and $w_{.}(6)$. Moreover

$$\alpha(s,t,\Delta s) = C + C\sqrt{\Delta s} + C\Delta s^{3/2} + O(\Delta s)^2 \tag{90}$$

and

$$\beta(s,t,\Delta s) = C\sqrt{\Delta s} + C\Delta s + C(\Delta s)^{3/2} + O(\Delta s)^2 \tag{91}$$

the same asymptotic results being true when we reverse the role of s, t.
The main result is the following:

$$< B_{.}(s+\Delta s,t) - B_{.}(s,t), B_{.}(u,v) >= O(\Delta s) \tag{92}$$

if u does not belong to $]s, s+\Delta s[$, the same equality being true if we reverse the role of s and t. We use the fact that the Green kernel associated to the two dimensional problem is the product of the Green kernels associated to the one dimensional problem by the remark following (6).
Moreover

$$< B_{.}(s+\Delta s,t) - B_{.}(s,t), B_{.}(u,v+\Delta v) - B_{.}(u,v) >= O(\Delta s\Delta v) \tag{93}$$

It is equal namely to

$$\begin{gathered} e(s+\Delta s-u)e(t-v-\Delta v) - e(s-u)(t-v-\Delta v) + e(s+\Delta s-u)e(t-v)- \\ e(s_u)e(t-v) = (e(s+\Delta s-u) - e(s-u))(e(t-v-\Delta v) - e(t_v)) \end{gathered} \tag{94}$$

if u does not belong to $]s, s+\Delta s[$ and t does not belong to $]v, v+\Delta v[$. Moreover,

$$< B_{.}(s+\Delta s,t) - B_{.}(s,t), B_{.}(s'+\Delta s',u) - B_{.}(s',u) >= O(\Delta s\Delta s') \tag{95}$$

if $]s', s' + \Delta s'[\cap]s, s + \Delta s[= \emptyset$ by analoguous reasons, and using the fact that the Green kernel associated to $B_.(s,t)$ is the products of the one dimensional Green kernels.

In order to simplify the exposure, we writte $\Delta t = \Delta t' = \Delta s = \Delta s'$. We conditionate $B_.(s,t)$ and $B_.(s',t')$ by $w_.(3)$,, $w_.(4)$, $w_.(5)$, $w_.(6)$. We use the formula (56) in order to compute this conditionating for $g(s,t)$ and $g(s',t')$, and after the Clark-Ocone formula (See [43]) in order to compute the conditional of $h(g(s,t))$ as an Itô integral in $w_.(3)$, $w_.(4)$, $w_.(5)$ and $w_.(5)$ with term bounded by $\sqrt{\Delta s}$ by (92). We get to take the expectation of the product of four Itô integral or 5 or 6. We can estimate its expectation by using the Itô formula and (93), (94) by applying iteratively the Itô formula and the Clark-Ocone formula. We reduce iteratively the length of the iterated integral we have to compute. The same result holds by the same arguments for:

$$\begin{aligned} &E[f(g(s,t'))h(g(s',t))g^{1;\cdot}(s,t')g^{\cdot;1}(s,t')g^{1;\cdot}(s',t)g^{\cdot;1}(s',t)] \\ &\qquad = C(s,t,s',t')\sqrt{\Delta s}\sqrt{\Delta t}\sqrt{\Delta s'}\sqrt{\Delta t'} + error \end{aligned} \tag{96}$$

if we suppose that $\Delta s = \Delta s' = \Delta t = \Delta t'$.

We deduce from the previous considerations that:

$$E[\sum_{i\neq:i';j\neq j'} D^{N'}_{i,j,1}D^{N'}_{i',j',1}] \to 2\int_{D^2} C(s,t,s',t')dsdtds'dt' - 2\int_{D^2} C(s,t,s',t')dsdtds'dt' = 0 \tag{97}$$

Let us now study the behaviour of

$$E[\sum_{i,j\neq j'} D^{N'}_{i,j,1}D^{N'}_{i,j',1}] \tag{98}$$

when $N' \to \infty$.

By the previous considerations, the contributions of the $T_{k,l}$ strictly interior to $T_{i,j}$ and of the $T_{k',l'}$ strictly interior to $T_{i,j'}$ vanish. Therefore, it is enough to study the contribution of

$$\begin{aligned} C^{1,N'}_{i,j,1} &= \sqrt{\Delta s_i}\sqrt{\Delta t_j}, < \omega(g(s_i,t_j), g^{1;\cdot}(s_i,t_j), g^{\cdot;1}(s_i;t_j) > \\ &- \sum_{i'} \sqrt{\Delta s_{i'}}\sqrt{\Delta t_j} < \omega(g(s_{i'},t_j)), g^{1;\cdot}(s_{i'},t_j), g^{\cdot;1}(s_{i'},t_j) > \end{aligned} \tag{99}$$

for $[s_{i'}, s_{i'+1}] \subseteq [s_i, s_{i+1}]$. We would like to show that $E[\sum_{i,j\neq j'} C^{1,N'}_{i,j,1}C^{1,N'}_{i,j',1}]$ tends to 0 when $N' \to \infty$. We will see later (See Step I.1.2 , Step I.1.3 and Step I.1.4) that we can replace $\sqrt{\Delta s_i}g^{1;\cdot}(s_i,t_j)$ by $\Delta_{s_i}g(s_i,t_j)$ and $\sqrt{\Delta t_j}g(s_i,t_j)$ by $\Delta_{t_j}g(s_i,t_j)$. it is enough therefore to consider the behaviour of

$$\begin{aligned} C^{2,N'}_{i,j,1} &=< \omega(g(s_i,t_j)), \Delta_{s_i}g(s_i,t_j), \Delta_{t_j}g(s_i,t_j) > \\ &- \sum_{i'} < \omega(g(s_{i'},t_j)), \Delta_{s_{i'}}g(s_{i'},t_j), \Delta_{t_j}g(s_{i'},t_j) > \end{aligned} \tag{100}$$

and to show that $E[\sum_{i,j\neq j'} C^{2,N'}_{i,j,1}C^{2,N'}_{i,j',1}]$ tends to 0.

But

$$\sum \Delta_{s_{i'}}g(s_{i'},t_j) = \Delta_{s_i}g(s_i,t_j) \tag{101}$$

Therefore

$$
\begin{aligned}
C^{2,N'}_{i,j,1} &= \sum < \omega(g(s_i,t_j)) - \omega(g(s_{i'},t_j)), \Delta_{s_{i'}} g(s_{i'},t_j), \Delta_{t_j} g(s_i,t_j) > \\
+ \sum &< \omega(g(s_{i'},t_j), \Delta_{s_{i'}} g(s_{i'},t_j), \Delta_{t_j} g(s_i,t_j) - \Delta_{t_j} g(s_{i'},t_j) >= C^{3,N'}_{i,j,1} + C^{4,N'}_{i,j,1}
\end{aligned}
\tag{102}
$$

By using the technics of the next steps, we can replace $\Delta_{s_{i'}} g(s_{i'},t_j)$ by $\sqrt{\Delta s_{i'}} g^{1;\cdot}(s_{i'},t_j)$ and $\Delta_{t_j} g(s_{i'},t_j)$ by $\sqrt{\Delta t_j} g^{\cdot;1}(s_{i'},t_j)$ and $\Delta_{t_j} g(s_{i'},t_j)$ by $\sqrt{\Delta t_j} g^{\cdot;1}(s_{i'},t_j)$ and $\Delta_{t_j} g(s_i,t_j)$ by $\sqrt{\Delta t_j} g^{\cdot;1}(s_i,t_j)$. We get two quantities $C^{5,N'}_{i,j,1}$ and $C^{6,N'}_{i,j,1}$

We compute $\sum_{i,j\neq j'} E[(C^{5,N'}_{i,j,1} C^{5,N'}_{i,j',1})]$. There are two contributions. The first one is when we consider twice the same $s_{i'}$. There are 4 types of increments which appear $(s_i,t_j),(s_{i'},t_j),(s_i,t_{j'}$ and $(s'_i,t_j j')$. We take the conditional expectation along $\Delta_{s_{i'}} B_{\cdot}(s_{i'}),t_j)$, $\Delta_{t_j} B_{\cdot}(s_i,t_j)$, $\Delta_{s_{i'}} B(s_{i'},t_{j'})$ and $\Delta_{t_{j'}} B_{\cdot}(s_i,t_{j'})$ or more precisely along the Brownian motion which arise from the diagonalisation (89) of the Brownian motions $B_{\cdot}(s_i,t_j)$, $B_{\cdot}(s_{i'},t_j)$, $B_{\cdot}(s_i,t_{j'})$ and $B_{\cdot}(s_{i'},t_{j'})$. The Stratonovitch integrals $g^{1;\cdot}(s,t)$ and $g^{\cdot;1}(s,t)$ are in fact Itô integrals. Moreover we can compute the conditional law of $g(s_i,t_j), g(s_{i'},t_j), g(s_i,t_{j'})$ $g(s_{i'},t_{j'})$ by using (56) and the Clark -Ocone formula to express the quantities which appear in this way as stochastic integral which are martingales and whose bracket with the others tems can be estimated by (89). There is a product of Martingale Itô integrals, whose expectation can be estimated by using succesivly the Itô formula and the Clark Ocone formula. We conclude by using (4.27), (4.28) and (4.30). We get that the contribution when there is one coincidence leads to a term in $O(1/N)\Delta s_{i'} \Delta t_j \Delta t_{j'}$. When there is no coincidence, we condition by $\Delta_{s_{i'}} B_{\cdot}(s_{i'},t_j)$, $\Delta_{t_j} B_{\cdot}(s_i,t_j)$, $\Delta_{s_{i"}} B_{\cdot}(s_{i"},t_j)$ and $\Delta_{t_j} B_{\cdot}(s_i,t_{j'})$, or more precisely by the Brownian motions arising from the diagonalisation (89). We proceed as before, and we get a contribution in $o(1/N)\Delta_{s_{i'}}\Delta_{s_{i"}}\Delta_{t_j}\Delta_{t_{j'}}$ Therefore $E[\sum_{i,j\neq j'} C^{5,N'}_{i,j,1} C^{5,N'}_{i,j',1}] \to 0$.

By the same type of trick and performing the conditional expectation along the increment $\Delta_s B_{\cdot}(s,t)$ and $\Delta_t B_{\cdot}(s,t)$ or more precisley by conditioning along the Brownian motions which appears in the diagonalisation (89) in $C^{6,N'}_{i,j,1} C^{6+,N'}_{i,j',1}$ and after using the Clark-Ocone formula, we see that the quantity $\sum_{i,j\neq j'} E[C^{6,N'}_{i,j,1} C^{6,N'}_{i,j',1}] \to 0$. The same holds for $E[\sum_{i,j\neq j'} C^{5,N'}_{i,j,1} C^{6,N'}_{i,j',1}]$.

Let us study the behaviour of $E[\sum_{i,j} (D^{N'}_{i,j,1})^2]$. By the considerations which will follow in the next step, it is enough to study the behaviour of

$$
\begin{aligned}
&< \omega(g(s_i,t_j)), \sum \Delta_{s_{i'}} g(s_{i'},t_j), \sum \Delta_{t_{j'}} g(s_i,t_{j'}) > \\
&- \sum < \omega(g(s_{i'},t_{j'}), \Delta_{s_{i'}} g(s_{i'},t_{j'}), \Delta_{t_{j'}} g(s_{i'},t_{j'}) >= \\
&\{ \sum_{i',j'} < \omega(g(s_i,t_j)), \Delta_{s_{i'}} g(s_{i'},t_{j'}), \Delta_{t_{j'}} g(s_i,t_{j'}) > \\
&- \sum_{i',j'} < \omega(g(s_i,t_j)), \Delta_{s_{i'}} g(s_{i'},t_{j'}), \Delta_{t_{j'}} g(s_{i'},t_{j'}) > \} \\
&+ \sum < \omega(g(s_{i'},t_{j'}) - \omega(g(s_i,t_j)), \Delta_{s_{i'}} g(s_{i'},t_{j'}), \Delta_{t_{j'}} g(s_{i'},t_{j'}) = \tilde{G}^{N'}_{i,j,1} + G^{3,N'}_{i,j,1}
\end{aligned}
\tag{103}
$$

where we do the summation over $[s_{i'}, s_{i'+1}] \subseteq [s_i, s_{i+1}]$ and $[t_{j'}, t_{j'+1}] \subseteq [t_j, t_{j+1}]$. In $\tilde{G}^{N'}_{i,j,1}$, we write:

$$\begin{aligned}
&\Delta_{s_{i'}} g(s_{i'}, t_j)\Delta_{t_{j'}}(g(s_i, t_{j'}) - \Delta_{s_{i'}} g(s_{i'}, t_{j'})\Delta_{t_{j'}} g(s_{i'}, t_{j'}) \\
&= (\Delta_{s_i} g(s_i, t_j) - \Delta_{s_i} g(s_i,t_{j'}))\Delta_{t_{j'}} g(s_i, t_{j'}) \\
&+\Delta_{s_{i'}} g(s_{i'}, t_{j'})(\Delta_{t_{j'}} g(s_i, t_{j'}) - \Delta_{t_{j'}} g(s_{i'}, t_{j'}))
\end{aligned} \tag{104}$$

and we deduce a decomposition of $\tilde{G}^{N'}_{i,j,1}$ into $G^{1,N'}_{i,j,1} + G^{2,N'}_{i,j,1}$ In $G^{1,N'}_{i,j,1}$, $G^{2,N'}_{i,j,1}$ and $G^{2,N'}_{i,j,1}$, we can replace $\Delta_{s_{i'}} g(_{i'}, t_j)$, $\Delta_{t_{j'}} g(s_i, t_{j'})$ by $\sqrt{\Delta s_{i'}} g^{1;\cdot}(s_{i'}, t_j)$. We can replace $\sqrt{\Delta t_{j'}} g^{\cdot;1} g(s_i, t_{j'})$ and $\Delta_{s_{i'}} g(s_{i'}, t_{j'})$ by $\sqrt{\Delta s_{i'}} g^{1;\cdot}(s_{i'}, t_{j'})$ and $\Delta_{t_{j'}} g(s_{i'}, t_{j'})$ by $\sqrt{\Delta t_{j'}} g^{\cdot;1}(s_{i'}, t_{j'})$ by $\sqrt{\Delta t_{j'}} g^{\cdot;1}(s_{i'}, t_{j'})$. We get $G^{3,N'}_{i,j,1}$ and $G^{4,N'}_{i,j,1}$.

We have 6 terms to estimate: $E[\sum_{i,j}(G^{1,N'}_{i,j,1})^2]$, $E[\sum_{i,j}(G^{2,N'}_{i,j,1})^2]$, $E[\sum_{i,j}(G^{3,N'}_{i,j,1})^2]$, $E[\sum_{i,j} G^{1,N'}_{i,j} G^{2,N'}_{i,j}]$, $E[\sum_{i,j} G^{1,N'}_{i,j,1} G^{3,N'}_{i,j,1}]$, $E[\sum G^{2,N'}_{i,j,1} G^{3,N'}_{i,j,1}]$. We can do the multiplication term by term in each product which appear. In each term, we distribute another time. There are 4 terms where two expressions in $g^{1;\cdot}$ and $g^{\cdot;1}$ appear. We condition by the set of increments in the leading Brownian motion which appears in these expressions, or more precisely of the terms which appear after the diagonalisation (89) in $\Delta_s B(s,t)$ and $\Delta_t B(s,t)$. We use (57) and the Clark-Ocone formula (See [43]). We use (89), and (93). When we develop, there is the possibility that we get exactly 4 times $s_{i'}$, $s_{i"}$, $t_{j'}$ and $t_{j"}$, which lead to a contribution in $O(1/N)\sum_{i'\neq i", j'\neq j"} \Delta s_{i'}\Delta s_{i"}\Delta t_{j'}\Delta t_{j"}$. There is a contribution when there are 3 different $s_i, t_{j'}, t_{j"}$ or $s_{i'}, s_{i"}, t_j$ which lead to a contribution in $\sum_{i,j'\neq j"} O(1/N)\Delta s_i \Delta t_{j'}\Delta t_{j"}$ or $\sum_{i'\neq i", j} O(1/N)\Delta s_{i'}\Delta s_{i"}\Delta t_j$ and a contribution where we get only two times s_i and t_j which leads to a contribution in $\sum_{i,j} O(1/N)\Delta s_i \Delta t_j$. Therefore, $\sum_{i,j} G^{3,N'}_{i,j,1}$ tends to 0 in L^2.

By the same argument, $\sum_{i,j} G^{1,N'}_{i,j,1}$ and $\sum^{2,N'}_{i,j}$ tend to 0 in L^2. By using this type of argument, we can get the requested limits.

Step I.1.2 Study of the convergence of the terms C^N_2 and C^N_4 where we mix stochastic integral and classical integral.

This term is simpler to treat than the double stochastic integral, which is most diverging, which appears. But it leads to some complications, because in $g^{\cdot;2}(s,t)$, there are some double stochastic integral in the dynamical time u which appears. We write

$$C^N_2 = \sum_{i,j} C^N_{i,j,2} \tag{105}$$

We consider a bigger integer N' and we write:

$$D^{N'}_{i,j,2} = C^N_{i,j,2} - \sum_{T_{i',j'} \subseteq T_{i,j}} C^{N'}_{i',j',2} \tag{106}$$

We have the following behaviour:

$$\begin{aligned}
E[f(g(s,t))h(g(s',t'))g^{1;\cdot}(s,t)g^{\cdot;2}(s,t)g^{1;\cdot}(s',t')g^{\cdot;2}(s',t')] = \\
C(s,t,s',t')\sqrt{\Delta s}\sqrt{\Delta s} + error
\end{aligned} \tag{107}$$

If $\Delta s = \Delta t$ and if $0 \leq s \leq s + \Delta s \leq s' \leq s' + \Delta s' \leq 1$ and $0 \leq t \leq t + \Delta t \leq t' \leq t' + \Delta t' \leq 1$. $C(s,t,s',t')$ is continuous. Namely, $g^{\cdot;2}(s,t)$ and $g^{\cdot;2}(s',t')$ are given by double stochastic integrals in the term $w_{\cdot}(3)$ or $w_{\cdot}(4)$ which appear in (89). It is the far most complicated term, the terms in simple stochastic integrals can be treated as before. We condition after by the increments $\Delta_t B_{\cdot}(s,t)$, $\Delta_{t'} B_{\cdot}(s',t')$, $\Delta_s B_{\cdot}(s,t)$ and $\Delta_{s'} B_{\cdot}(s',t')$ or more precisely by the terms which arise from the diagonalisation in (89). We write the double Stratonovitch integral which appears in $g^{\cdot;2}(s,t)$ or $g^{\cdot;2}(s',t')$ as double Itô integral and a simple integral. After using the Clark-Ocone formula, the expectation of the product of at most 8 term and at least 2 Itô integrals hasto be computed. We use Itô formula successivly and Clark-Ocone formula successivly in order to get our estimate.

We have analoguous formulas we don't write. Therefore:

$$\begin{aligned} E[\sum_{i\neq i';j\neq j'} D^{N'}_{i,j,2} D^{N'}_{i',j',2}] \rightarrow 2\int_{T^4} C(s,t,s',t') ds ds' dt dt' \\ -2\int_{T^4} C(s,t,s',t') ds ds' dt dt' = 0 \end{aligned} \tag{108}$$

Let us study now the behaviour of

$$E[\sum_{i,j\neq j'} D^{N'}_{i,j,2} D^{N'}_{i,j',2}] \tag{109}$$

By the considerations which will follow, it is enough to study

$$\begin{aligned} C^{N'}_{i,j,2} = \Delta t_j < \omega(g(s_i,t_j)), \Delta_{s_i} g(s_i,t_j), g^{\cdot;2}(s_i,t_j) > \\ -\sum_{i',j'} \Delta t_{j'} < \omega(g(s_{i'},t_{j'}), \Delta_{s_{i'}} g(s_{i'},t_{j'}), g^{\cdot,2}(s_{i'},t_{j'}) > \end{aligned} \tag{110}$$

But we can write:

$$\Delta_{s_i} g(s_i,t_j) = \sum \Delta_{s_{i'}} g(s_{i'},t_j) \tag{111}$$

such that:

$$\begin{aligned} C^{N'}_{i,j,2} = \Delta t_j < \omega(g(s_i,t_j)), \sum \Delta_{s_{i'}} g(s_{i'},t_j), g^{\cdot,2}(s_i,t_j) > \\ -\sum \Delta t_{j'} < \omega(g(s_{i'},t_{j'})), \Delta_{s_{i'}} g(s_{i'},t_{j'}), g^{\cdot,2}(s_{i'},t_{j'}) > \\ = \{\sum_{i',j'} \Delta t_{j'} < \omega(g(s_i,t_j)), \Delta_{s_{i'}} g(s_{i'},t_j), g^{\cdot,2}(s_i,t_j) > \\ -\Delta t_{j'} < \omega(g(s_i,t_j), \Delta_{s_{i'}} g(s_{i'},t_{j'}), g^{\cdot,2}(s_{i'},t_{j'}) >\} \\ +\sum_{i',j'} \Delta t_{j'} \{< \omega(g(s_i,t_j)) - \omega(g(s_{i'},t_{j'}), \Delta_{s_i} g(s_{i'},t_{j'}), g^{\cdot,2}(s_{i'},t_{j'})) >\} \\ = C^{1,N'}_{i,j,2} + C_{i,j,2}{}^{2,N'} \end{aligned} \tag{112}$$

In $C^{1,N'}_{i,j,2}$ and $C^{2,N'}_{i,j,2}$, we can replace, by the considerations which will follow, $\Delta_{s_{i'}}(g(s_{i'},t_{j'})$ by the quantity $\sqrt{\Delta s_{i'}} g^{1,\cdot}(s_{i'},t_{j'})$ and $\Delta_{s_{i'}}(g(s_{i'},t_j)$ by $\sqrt{\Delta s_{i'}} g^{1,\cdot}(s_{i'},t_j)$. We get expressions $C^{3,N'}_{i,j,2}$ and $C^{4,N'}_{i,j,2}$. We distribute the term which appear in $\sum(C^{4,N'}_{i,j,2} C^{4,N'}_{i,j',2}$, there are 4 terms with increments

$\sqrt{\Delta s_{i'}}g^{1;\cdot}(s_{i'},t_{j'})$ $\sqrt{\Delta s_{i"}}g^{1;\cdot}(s_{i"},t_{j"})$ and $\Delta t_{j'}g^{\cdot;2}(s_{i'},t_{j'})$ and $\Delta t_{j"}g^{\cdot;2}(s_{i"},t_{j"})$ which appear. We condition by the Brownian motions which are got after diagonalising the increments of the leadings Brownian motions which appear in these formulas and we get as before a norm in L^2 which tends to 0.

We have to study 3 terms: $E[\sum i,j \neq j' C^{3,N'}_{i,j,2}C^{3,N'}_{i,j',2}]$, $E[\sum_{i,j\neq j'} C^{4,N'}_{i,j,2}C^{4,N'}_{i,j',2}]$ and the last one $E[\sum_{i,j\neq j'} C^{3,N'}_{i,j,2}C^{4,N'}_{i,j',2}]$. The behaviour of $E[\sum_{i,j\neq j'} C^{3,N'}_{i,j,2}C^{3,N'}_{i,j',2}]$ is the most complicated to treat.

We write:

$$
\begin{aligned}
C^{3,N'}_{i,j,2} &= \{\sum_{i',j'} \sqrt{\Delta s_{i'}}\Delta t_{j'} < \omega(g(s_i,t_j)), g^{1,\cdot}(s_{i'},t_j), g^{\cdot,2}(s_i,t_j) > - \\
&\sum_{i',j'} \sqrt{\Delta s_{i'}}\Delta t_{j'} < \omega(g(s_i,t_j)), g^{1;\cdot}(s_{i'},t_{j'}), g^{\cdot,2}(s_i,t_j) >\} \\
&+\{\sum_{i',j'} \sqrt{\Delta s_i}\Delta t_{j'} < \omega(g(s_i,t_j)), g^{1;\cdot}(s_{i'},t_{j'}), g^{\cdot;2}(s_i,t_j) - g^{\cdot;2}(s_{i'},t_j) >\} \\
&+\{\sum_{i',j'} \sqrt{\Delta s_{i'}}\Delta t_{j'} < \omega(g(s_i,t_j)), g^{1;\cdot}(s_{i'},t_{j'}), g^{\cdot,2}(s_{i'},t_j) - g^{\cdot;2}(s_{i'},t_{j'}) >\} \\
&= C^{5,N'}_{i,j,2} + {C_{i,j,2}}^{6,N'} + C^{7,N'}_{i,j,2}
\end{aligned}
\tag{113}
$$

By the previous considerations, we have only to estimate $E[\sum_{i,j\neq j'} C^{5,N}_{i,j,2}C^{5,N'}_{i,j',2}]$, $E[\sum_{i,j\neq j'} C^{6,N'}_{i,j,2}C^{6,N'}_{i,j',2}]$ and $E[\sum_{i,j\neq j'} C^{7,N'}_{i,j,2}C^{7,N'}_{i,j',2}]$ as well as the sum where there exist other coincidences of indices i, i', j, j'. We have to estimate the analoguous quantities where we mix $C^{5,N'}_{i,j,2}$ and $C^{6,N'}_{i,j',2}$, the term where we mix $C^{5,N'}_{i,j,2}$ and $C^{6,N'}_{i,j',2}$ and $C^{7,N'}_{i,j',2}$ and the term where we mix $C^{6,N'}_{i,j,2}$ and $C^{7,N'}_{i,j',2}$. We will omit to write the details of the convergence of these mixed term to 0. Clearly,

$$
E[\sum_{i,j\not j'} C^{5,N'}_{i,j,2}C^{5,N'}_{i,j',2}] \to 0 \tag{114}
$$

Namely, if we do the multiplication of each term in the sum, there are 6 increments which appear $\Delta_{s_{i'_1}}B(s_{i'_1},t_{j_1})$, $\Delta_{s_{i'_1}}B(s_{i'_1},t_{j'_1})$, $\Delta_{t_{j_1}}B(s_{i_1},t_{j_1})$, $\Delta_{s_{i'_2}}B(s_{i'_2},t_{j_2})$, $\Delta_{s_{i'_2}}B(s_{i'_2},t_{j'_2})$ and $\Delta_{t_{j_2}}B(s_{i_2},t_{j_2})$. Their mutual covariances satisfy to (92), (93) and (95) because $j_1 \neq j_2$ and because we don't have to consider when we do the multiplication term by term to consider the interaction between $\Delta_{s_{i'_1}}(s_{i'_1},t_{j_1})$ and $\Delta_{s_{i'_1}}B(s_{i'_1},t_{j'_1})$ and the interaction between $\Delta_{s_{i'_2}}B(s_{i'_2},t_{j'_2})$ and $\Delta_{s_{i'_2}}B(s_{i'_2},t_{j_2})$. We conclude after conditioning along these increments, or more precisely the Brownian motions which appear when we use the diagonalization (89). This allows us to show (114).

Moreover,

$$
E[\sum_{i,j\neq j'} C^{6,N'}_{i,j,2}C^{6,N'}_{i,j',2}] \to 0 \tag{115}
$$

Namely, when we do the product term by term in (115), there are 6 increments which appear $\Delta_{s_{i'_1}}B(s_{i'_1},t_{j'_1})$, $\Delta_{t_{j_1}}B(s_{i_1},t_{j_1})$, $\Delta_{t_{j_1}}B(s_{i'_1},t_{j_1})$, $\Delta_{t_{j_1}}B(s_{i_1},t_{j_1})$, $\Delta_{s_{i'_2}}B(s_{i'_2},t_{j'_2})$, and the terms $\Delta_{t_{j_2}}B(s_{i_2},t_{j_2})$ $\Delta_{t_{j_2}}B(s_{i'_2},t_{j_2})$. We can apply (92), (93) and (95) to these increments because we don't have to take the covariance between $\Delta_{t_{j_1}}B(s_{i_1},t_{j_1})$ and $\Delta_{t_{j_1}}B(s_{i'_1},t_{j_1})$ and the covariance between $\Delta_{t_{j_2}}B(s_{i_2},t_{j_2})$ and $\Delta_{t_{j_2}}(s_{i'_2},t_{j_2})$.

Let us consider the most complicated term $C^{7,N'}_{i,j,2}$ because in $g^{\cdot;2}(s_{i'},t_j)$ and in $g^{\cdot;2}(s_{i'},t_{j'})$ in (114), it is not the same subdivision in t_j. But since we consider

$$E[\sum_{i,j\neq j'} C^{7,N'}_{i,j,2}C^{7,N'}_{i,j',2}] \tag{116}$$

there are 6 increments to see. They are $\Delta_{s_{i'_1}}B(s_{i'_1},t_{j'_1})$, $\Delta_{t_{j_1}}B(s_{i'_1},t_{j_1})$, $\Delta_{t_{j'_1}}B(s_{i'_1},t_{j'_1})$, $\Delta_{s_{i'_2}}B(s_{i'_2},t_{j'_2})$, $\Delta_{t_{j_2}}B(s_{i'_2},t_{j_2})$ and $\Delta_{t_{j'_2}}B(s_{i'_2},t_{j'_2})$ and we don't have to consider the correlation between $\Delta_{t_{j_1}}B(s_{i'_1},t_{j_1})$ and $\Delta_{t_{j'_1}}B(s_{i'_1},t_{j'_1})$ and the correlation $\Delta_{t_{j_2}}B(s_{i'_2},t_{j_2})$ and $\Delta_{t_{j'_2}}(s_{i'_2},t_{j'_2})$.We can apply (92), (93), (95) for the correlations we consider, and we can conclude as previously.

By the same reason

$$\sum_{i\neq i',j} E[C^{5,N'}_{i,j,2}C^{5,N'}_{i',j,2}] \to 0 \tag{117}$$

$$\sum_{i\neq i',j} E[C^{6,N'}_{i,j,2}C^{6,N'}_{i',j,2}] \to 0 \tag{118}$$

The same arguments arise when we consider:

$$\sum_{i\neq i',j} E[C^{7,N'}_{i,j,2}C^{7,N'}_{i',j,2}] \tag{119}$$

It remains to treat the case where there are two coincidences, that is to treat the case of $\sum E[(C^{5,N'}_{i,j,2})^2]$, $\sum E[(C^{6,N'}_{i,j,2})^2]$ and $\sum E[(C^{7,N'}_{i,j,2})^2]$, after doing the same restriction about the mixed terms. But as a matter of fact, we can show simply that

$$\sum_{i,j} E[(C^{5,N'}_{i,j,2})^2] \to 0 \tag{120}$$

We have namely the correlators between the following increments to consider:$\Delta_{s_{i'_1}}B(s_{i'_1},t_j)$, $\Delta_{s'i_1}B(s_{i'_1},t_{j'_1})$, $\Delta_{t_j}B(s_i,t_j)$, $\Delta_{s_{i'_2}}(s_{i'_2},t_j)$ and $\Delta_{s'_{i_2}}B(s_{i'_2},t_{j'_1})$. But we have $t_{j'_1}\geq t_j$ and $t_{j'_2}\geq t_j$. Therefore:

$$\begin{gathered}< \Delta_{s_{i'_1}}B(s_{i'_1},t_{j'_1}),\Delta_{s'_1}B(s_{i'_1},t_j >= \\ e(t_{j'_1}-t_j)(e(-\Delta s_{i'_1})+e(\Delta s_{i'_1})-2e(0)) = C\Delta s_{i'_1}e(t_{j'_1}-t_j)\end{gathered} \tag{121}$$

because $t_{j'_1}\geq t_j$ and because e has half derivatives in 0. This remark allows us to repeat the previous considerations as well as to use (92), (93) and (95).

Moreover

$$\sum E[(C^{6,N'}_{i,j,2})^2] \to 0 \tag{122}$$

We have no difficulty to show that because we don't have to consider the covariance of a $g^{1;0}(s_{i'},t_j)$ and a $g^{1;0}(s_{i'},t_{j'})$ and because $< g^{1;\cdot}(s_{i'},t_j), g^{1;\cdot}(s_{i"},t_j) >= CO(\sqrt{\Delta s_{i'}\Delta_{s_{i"}}})$.

The difficult part is to show that $\sum E[(C^{7,N'}_{i,j,2})^2]\to 0$, because two different subdivision $[t_{j'},t_{j'+1}]$ and $[t_j,t_{j+1}]$ appear and because $t_{j'}\in[t_j,t_{j+1}]$. We write the details of this limit, because it is the most complicated, the others limits are simpler. We write:

$$\begin{aligned} C_{i,j,2}^{7,N'} &= \sum \sqrt{\delta s_{i'}}\Delta t_{j'} < \omega(g(s_i,t_j)), g^{1;\cdot}(s_{i'},t_j), g^{\cdot;2}(s_{i'},t_j) - g^{\cdot;2}(s_{i'},t_{j'}) > \\ &+ \sum \sqrt{\Delta s_{i'}}\Delta t_{j'} < \omega(g(s_i,t_j)), g^{1;\cdot}(s_{i'},t_{j'}) - g^{1;\cdot}(s_{i'},t_j), g^{\cdot;2}(s_{i'},t_j) > \\ &- \sum < \sqrt{\Delta s_{i'}}\Delta t_{j'} < \omega(g(s_i,t_j)), g^{1;\cdot}(s_{i'},t_{j'}) - g^{1;\cdot}(s_{i'},t_j), g^{\cdot;2}(s_{i'},t_{j'}) > \\ &= C_{i,j,2}^{8,N'} + C_{i,j,2}^{9,N'} + C_{i,j,2}{}^{10,N'} \end{aligned} \tag{123}$$

By the previous considerations, the terms $E[\sum(C_{i,j,2}^{9,N'})^2]$ and $E[\sum(C_{i,j,2}^{10,N'})^2]$ tend to 0. The main difficulty is to show that

$$E[\sum_{i,j}(C_{i,j,2}^{8,N'})^2] \to 0 \tag{124}$$

If these results are true, the term where we mix $C_{i,j,2}^{8,N'}$, $C_{i,j,2}^{9,N'}$ and $C_{i,j,2}^{10,N'}$ can be treated by Cauchy-Schwartz inequality. We proceed for that as it was done in the previous part. We remark, by the same considerations as in the first part, that it is enough to replace $\Delta t_j g^{\cdot;2}(s_{i'},t_j)$ by a double stochastic iterated integral $\int_{0<u<v<1} \alpha_u(s_{i'},t_j)(dB_u(s_{i'},t_{j+1}) - dB_u(s_{i'},t_j))\alpha_v(s_{i'})(dB_v(s_{i'},t_{j+1}) - dB_v(s_{i'},t_j))$ where α_u and α_v are $B(s_{i'},t_j)$ measurable. By the same argument, we replace $\Delta t_{j'} g^{\cdot;2}(s_{i'},t_{j'})$ by a double stochastic integral $\int_{0<u<v<1} \alpha_u(s_{i'},t_{j'})(dB_u(s_{i'},t_{j'+1}) - dB_u(s_{i'},t_{j'}))$ $\alpha_v(s_{i'},t_{j'})(dB_v(s_{i'},t_{j'+1}) - dB_v(s_{i'},t_{j'})$ where $\alpha_u(s_{i'},t_{j'})$ and $\alpha_v(s_{i'},t_{j'})$ are $B_{.}(s_{i'},t_{j'})$ measurable. To study the behaviour when $N' \to \infty$, we can replace without difficulty in this last expression $\alpha_u(s_{i'},t_{j'})$ by $\alpha_u(s_{i'},t_j)$. We write:

$$dB_{.}(s_{i'},t_{j+1}) - dB_{.}(s_{i'},t_j) = \sum dB_{.}(s_{i'},t_{j'+1}) - dB_{.}(s_{i'},t_{j'}) \tag{125}$$

and we distribute in the first term of (124). The diagonal terms cancel, and we have to estimate when $N \to \infty$ the behaviour of

$$\begin{aligned} C_{i,j,2}^{11,N'} = \sum \sqrt{\Delta s_{i'}} < \omega(g(s_i,t_j)), g^{1,\cdot}(s_{i'},t_j), \sum_{t_k \neq t_{k'}} \int_{0<u<v<1} \\ < \alpha(u)(dB_u(s_{i'},t_{k+1}) - dB_u(s_{i'},t_k))\alpha(v)(dB_v(s_{i'},t_{k'+1}) - dB_v(s_{i'},t_{k'}) > \end{aligned} \tag{126}$$

where we sum over $[t_k,t_{k+1}] \subseteq [t_j,t_{j+1}]$ and $[t_{k'},t_{k'+1}] \subseteq [t_j,t_{j+1}]$ for the sharper dyadic subdivision associated to $2^{N'}$. Instead of taking the following expression in time 1, let us take it in time r. We get a process $\sum C_{i,j,2,r}^{11,N'}$ (We replace $g(s_i,t_j)$ by $g_r(s_i,t_j)$, $g^{1;\cdot}(s_{i'},t_j)$ by $g_r^{1;\cdot}(s_{i'},t_j)$ and the double integral between 0 and 1 by a double integral between 0 and r. Let us consider the finite variational part $V_r^{N'} = \sum V_{i,j,2,r}^{N'}$ and the martingale part $M_r^{N'} = \sum M_{i,j,2,r}^{N'}$ associated to this process.

Let us begin to study the finite variational part of this process $V_r^{N'}$. This can come from a contraction between $\omega(g(s_i,t_j))$ and $g^{1;\cdot}(s_i',t_j)$ which leads to a term in $\sqrt{\Delta s_i'}$, which is multiplied by a term in $\sqrt{\Delta s_i'}$. But the L^2 norm of the sum $\sum_{t_k \neq t_{k'}}$ can be estimated. We decompose first $\sum_{t_k \neq t_k}$ in a martingale term and a finite variational term. There is first a contraction between α_v and $dB_v(s_i',t_{k'+1}) - dB_v(s_i',t_{k'})$ which leads to a term in $t_{k'+1} - t_{k'}$ The stochastic integral in u can be estimated. We see the martingale term. By Itô formula $\| \sum_{t_k \neq t_{k'}} \int_0^v \alpha_u(\delta B_u(s_i',t_{k+1}) - \delta B_u(s_i',t_k)\|_{L^2}^2$ can be estimated in $\sum(t_{k'+1}-t_{k'})(t_{k"+1}-t_{k"}) + \sum(t_{k+1}-t_k) = (t_{j+1}-t_j)^2 + (t_{j+1}-t_j)$.

Therefore the L^2 norm of this term behaves in $\sqrt{t_{j+1} - t_j}$. But since there is $(t_{k'+1} - t_{k'})$ in time u, we have a behaviour of this contribution in $\Delta s_i(t_{j+1} - t_j)^{3/2}$ whose sum vanish when $N \to \infty$. The second term comes from a contraction between $dB_u(s'_i, t_{k+1}) - dB_u(s'_i, t_k)$ and $dB_v(s'_i, t_{k'+1}) - dB_v(s'_i, t_{k'})$ which leads to a term in $(t_{k+1} - t_k)(t_{k'+1} - t_{k'})$ and therefore to a contribution in $(t_{j+1} - t_j)^2$. Therefore the total contribution is in $\Delta s_i(t_{j+1} - t_j)^2$, whose sum vanish when $N \to \infty$, because $< g^{1;\cdot}(s'i', t_j), g^{1;\cdot}(s_{i"}, t_j) >= O\sqrt{\Delta s_{i'} \Delta s_{i"}}$

There is a contraction between $\omega(g(s_i, t_j))$ and $dB_v(s'_i, t_{k'+1}) - dB_v(s'_i, t_{k'})$ which is in $(t_{k'+1} - t_{k'})$. This term cancel, because when we take the square of the L^2 norm of the sum, it behaves in $\sum_{i',i"} \Delta_{s_{i'}} \Delta_{si"} I_{i',i"}$, where $I_{i',i"}$ where $I_{i',i"}$ is a sum of quadruple $t_{k'}, t_{k"}, t_{k^3}, t_{k^4}$ which behaves in $O(t_{j+1} - t_j)^3$ and a sum $\sum_{i'} \Delta s_i I_{i'}$ where $I_{i'}$ has a bound in $(t_{j+1} - t_j)^{3/2}$. The sum of these terms vanish, when $N \to \infty$ (See part III for analoguous considerations).

Let us estimate the martingale term $M^{N'}_{i,j,2,r}$. Let us estimate the L^2 norm of $M^{N'}_r$. We use Itô formula. It behaves as $\sum_{i',i"} \Delta s_{i"} \Delta s_{i'} I_{i,i'} + \sum_{i'} \Delta s_{i'} I_{i'}$ where $I_{i',i"}$ has a bound in $(t_{j+1} - t_j)^{3/2}$ and $I_{i'}$ the same. Therefore the L^2 norm of $M^{N'}_r$ vanish when $N \to \infty$.

Step I.1.3: study of the behaviour of the double classical integral C^N_5.

We writte

$$C^N_5 = \sum C^N_{i,j,5} = \sum \Delta s_i \Delta t_j < \omega(g(s_i, t_j)), g^{2;\cdot}(s_i, t_j), g^{\cdot;2}(s_i, t_j) > \tag{127}$$

We consider $N' > N$ and study:

$$D^{N'}_{i,j,5} = C^N_{i,j,5} - \sum_{T_{i',j'} \subseteq T_{i,j}} C^{N'}_{i',j',5} \tag{128}$$

We write

$$D^{N'}_{i,j,5} = C^{2,N}_{i,j,5} + C^{3,N'}_{i,j,5} \tag{129}$$

with

$$C^{2,N'}_{i,j,5} = \sum_{T_{i',j'} \subseteq T_{i,j}} \Delta s_{i'} \Delta t_{j'} < \omega(g(s_i, t_j)) - \omega(g(s_{i'}, t_{j'}), g^{2;\cdot}(s_i, t_j)., g^{\cdot;2}(s_i, t_j) > \tag{130}$$

and

$$\begin{aligned} C^{3,N'}_{i,j,5} = \sum_{T_{i',j'} \subseteq T_{i,j}} & \Delta s_{i'} \Delta t_{j'} \{< \omega(g(s_{i'}, t_{j'}), g^{2;\cdot}(s_i, t_j), g^{\cdot;2}(s_i, t_j) > \\ & - < \omega(g(s_{i'}, t_{j'}), g^{2;\cdot}(s'_i, t'_j), g^{\cdot;2}(s_{i'}, t_{j'}) > \end{aligned} \tag{131}$$

It is clear that $\sum C^{2,N'}_{i,j,5} \to 0$ in L^2 because $g^{2;2}(s_i, t_j)$ is bounded in L^2.

In order to estimate $C^{3,N'}_{i,j,5}$, we can replace $\omega(g(s_{i'}, t_{j'})$ by $\omega(g(s_i, t_j))$. We can replace $\Delta s_{i'} g^{2;\cdot}(s_{i'}, t_{j'})$ by a double stochastic integral in the dynamical time u $I^{2;\cdot}(s_{i'}, t_{j'})$ as it was done in (126) and do the same transformation for the other $g^{2;\cdot}$ and $g^{\cdot;2}$ which appear in $C^{3,N'}_{i,j,5}$ such that we have only to show that $\sum^{4,N'}_{i,j,5} \to 0$ in L^2 where

$$\begin{aligned} C^{4,N'}_{i,j,5} =& < \omega(g(s_i, t_j), I^{2;\cdot}(s_i,t_j), I^{\cdot;2}(s_i, t_j) > - \\ \sum_{T_{i',j'} \subseteq T_{i,j}} & < \omega(g(s_i, t_j)), I^{2;\cdot}(s_{i'}, t_{j'}), I^{\cdot;2}(s_{i'}, t_{j'}) > \end{aligned} \tag{132}$$

We write

$$d\Delta_{s_i} B_.(s_i, t_j) = \sum_{s_{i'}} d\Delta_{s_{i'}} B_.(s_{i'}, t_j) \tag{133}$$

and

$$d\Delta_{t_j} B_.(s_i, t_j) = \sum_{t'_j} d\Delta_{t_{j'}} B_.(s_i, t_{j'}) \tag{134}$$

and we distribute in $I^{2;\cdot}(s_i, t_j)$ and $I^{\cdot;2}(s_i, t_j)$. We get that the expression $I^{2;\cdot}(s_i, t_j)$ is equal to the expression $\sum_{s^1_{i'}, s^2_{i'} \in [s_i, s_{i+1}]} I^{2;\cdot}(s^1_{i'}, s^2_{i'}, t_j)$ and that $I^{\cdot;2} = \sum_{t^1_{j'}, t^2_{j'} \in [t_j, t_{j+1}]} I^{\cdot;2}(s_i, t^1_{j'}, t^2_{j'})$ after distributing in these stochastic integral. Only the contribution where $s^1_{i'} = s^2_{i'}$ and $t^1_{j'} = t^2_{j'}$ do not vanish when $N' \to \infty$, by the same considerations than in (54). These terms are nothing else, modulo some small error terms than $I^{2;\cdot}(s_{i'}, t_j)$ and $I^{\cdot;2}(s_i, t_{j'})$. We have only to show that $\sum_{i,j} C^{5,N'}_{i,j,5} \to 0$ in L^2 where

$$\begin{aligned} C^{5,N'}_{i,j,5} = \sum_{T_{i',j'} \subseteq T_{i,j}} & (< \omega(g(s_i, t_j)), I^{2;\cdot}(s_{i'}, t_{j'}), I^{\cdot;2}(s_{i'}, t_{j'}) > \\ & - < \omega(g(s_i, t_j)), , I^{2;\cdot}(s_{i'}, t_j), I^{\cdot;2}(s_i, t_{j'}) > \end{aligned} \tag{135}$$

But we can show that the L^2 norm of $I^{2;\cdot}(s_{i'}, t_j) - I^{2;\cdot}(s_{i'}, t'_j)$ is $O(4/N')\Delta s_{i'}$ because the right bracket of $B_.(s_{i'+1}, t_j) - B_.(s_{i'}, t_j) - B_.(s_{i'+1}, t_{j'}) + B(s_{i'}, t_{j'})$ is in $O((s_{i'+1} - s_{i'})(t_j - t_{j'}))$.

Step I.1.4: study of the vanishing term C^N_3.

We write $C^N_3 = \sum^N_{i,j,3}$ where the L^2 norm of $C^N_{i,j,3}$ is in $O(\Delta s_i \Delta t_j^{3/2})$. But we have if $s_i \neq s_{i'}$, by using the previous technics

$$\begin{aligned} E[< \omega(g(s_i, t_j)), g^{1;\cdot}(s_i, t_j), g^{\cdot;3}(s_i, t_j) >< \omega(g(s_{i'}, t_{j'})), g^{1,\cdot}(s_{i'}, t_{j'}), g^{\cdot,3}(s_i, t_{j'}) >] \\ = O(\sqrt{\Delta s_i}\sqrt{\Delta s_{i'}}) \end{aligned} \tag{136}$$

Therefore $E[(C^N_3)^2] \to 0$.

Step I.2: convergence of B^N_2.

We write in probability:

$$\begin{aligned} \omega(g^N(s, t)) - \omega(g(s_i, t_j)) = \nabla\omega(g(s_i, t_j))(g^N(s, t)) - g(s_i, t_j)) \\ + \nabla^2\omega(g(s_i, t_j))(g^N(s, t) - g(s_i, t_j))^2 + O(\Delta t_j^{3/2}(+O(\Delta s_i^{3/2}) \end{aligned} \tag{137}$$

The residual term converges to 0 by the previous arguments. It remains to treat the main term. We recall:

$$\begin{aligned} & g^N(s, t) - g(s_i, t_j) \\ = & \frac{s - s_i}{s_{i+1} - s_i}(g(s_{i+1}, t_j) - g(s_i, t_j)) + \frac{t - t_j}{t_{j+1} - t_j}(g(s_i, t_{j+1}) - g(s_i, t_j)) \\ + & \frac{t - t_j}{t_{j+1} - t_j}\frac{s - s_i}{s_{i+1} - s_i}(g(s_{i+1}, t_{j+1}) - g(s_i, t_{j+1}) - g(s_{i+1}, t_j) + g(s_i, t_j)) \end{aligned} \tag{138}$$

Moreover

$$\int_{s_i}^{s_{i+1}} \frac{s - s_i}{s_{i+1} - s_i} ds = s_{i+1} - s_i \tag{139}$$

The integral of the first term of (138) leads to the convergence of the sum of random quantities of a type analoguous to already considered quantities, which contains some "brackets" of the type $< \nabla\omega(g(s_i,t_j)).\Delta_{s_i}g(s_i,t_j), \Delta_{s_i}g(s_i,t_j), \Delta_{t_j}g(s_i,t_j) >$ which converges by the methods used before. We can treat by the same method the convergence of $< \nabla\omega(g(s_i,t_j))(g(s_i,t_{j+1}) - g(s_i,t_j)), \Delta_{s_i}g(s_i,t_j), \Delta_{t_j}g(s_j,t_j) >$ which converge by the same methods as before. The term in $\frac{(t-t_j)(s-s_i)}{(\Delta t_j \Delta s_i}$ lead to analoguous terms. If we consider the term where the square of $g^N(s,t) - g(s_i,t_j)$ appear, there is a term where the quantity $< \nabla^2\omega(g(s_i,t_j)); \Delta_{s_i}g(s_i,t_j)^2, \Delta_{s_i}g(s_i,t_j), \Delta_{t_j}g(s_i,t_j) >$ appears whose sum vanishes in L^2 by the same considerations as in Step I.1.4. The only problem comes when we take sum corresponding more and less to the double bracket of $(s,t) \to g_1(s,t)$ of the type $\sum_{i,j} < \nabla^2\omega(g(s_i,t_j)).\Delta_{s_i}g(s_i,t_j).\Delta_{t_i}g(s_i,t_j), \Delta_{s_i}g(s_i,t_j), \Delta_{t_i}g(s_i,t_j) >$ whose treatment is similar to step I.1.3 by expanding a product of integrals into iterated integrals of length 2.

Step II: convergence of A_2^N and A_3^N.

The treatment for A_2^N and A_3^N are similar. So we will treat only the case of A_2^N.

We write:

$$\begin{aligned} A_2^N &= \sum_{i,j} < \omega(g(s_i,t_j)), d_s\alpha_3^N(s,t), d_t\alpha_2^N(t) > \\ &= \sum_{i,j} \int_{T_{i,j}} < \omega(g^N(s,t)) - \omega(g(s_i,t_j)), d_s\alpha_3^N(s,t), d_t\alpha_2^N(t) >= B_1^N + B_2^N \end{aligned} \tag{140}$$

Step II.1: convergence of B_1^N.

$$\begin{gathered} \int_{T_{i,j}} < \omega(g(s_i,t_j)), df_s\alpha_3^N(s,t), d_t\alpha_2^N(t), >= \int_{T_{i,j}} \frac{ds}{s_{i+1}-s_i} \frac{(t-t_j)dt}{(t_{j+1}-t_j)^2} \\ < \omega(g(s_i,t_j)), g(s_{i+1},t_{j+1}) - g(s_i,t_{j+1}) - g(si+1,t_j) + g(s_i,t_j), g(s_i,t_{j+1}) - g(s_i,t_j) > \end{gathered} \tag{141}$$

The integral over $T_{i,j}$ is constant. We write:

$$\begin{gathered} g(s_{i+1},t_{j+1}) - g(s_i,t_{j+1}) - g(s_{i+1},t_j) + g(s_i,t_j) \\ = \{g(s_{i+1},t_{j+1}) - g(s_i,t_{j+1})\} - \{g(s_{i+1},t_j) - g(s_i,t_j)\} = \gamma_{i,j}^1 - \gamma_{i,j}^2 \end{gathered} \tag{142}$$

The term in $\gamma_{i,j}^2$ can be treated as in step I.1. The term in $\gamma_{i,j}^1$ can be treated as in step I.1, because the increments between $\Delta_{s_i}B(s_i,t_j)$ and $\Delta_{s_i}B(s_i,t_{j+1}) >$ satisfy to (121), and we can do as in the treatment of (121)

Step II.2: convergence of B_2^N.

We use (137) and we conclude as in step I.2.

Step III: convergence of A_4^N.

We write:

$$\begin{aligned} A_4^N &= \sum_{i,j} \int_{T_{i,j}} < \omega(g(s_i,t_j)), d_s\alpha_3^N(s,t), d_t\alpha_3^N(s,t) > \\ &+ \sum_{i,j} \int_{T_{i,j}} < \omega(g^N(s,t)) - \omega(g(s_i,t_j)), d_s\alpha_3^N(s,t), d_t\alpha_3^N(s,t) >= B_1^N + B_2^N \end{aligned} \tag{143}$$

Step III.1: convergence of B_1^N.

We write with the notations of (142):

$$\int_{T_{i,j}} < \omega(g(s_i,t_j)), d_s\alpha_s^N(s,t), d_t\alpha_3^N(s,t)) > \\ = 2\int_{T_{i,j}} \frac{(t-t_j)dt}{t_{j+1}-t_j}\frac{ds}{s_{i+1}-s_i} < \omega(g(s_i,t_j)), \gamma_{i,j}^1+\gamma_{i,j}^2, \gamma_{i,j}^1+\gamma_{i,j}^2 > \tag{144}$$

The integral over $T_{i,j}$ is constant. In order to treat the sum, we write the second $\gamma_{i,j}^1+\gamma_{i,j}^2$ as $\delta_{i,j}^1+\delta_{i,j}^2$ where

$$\delta_{i,j}^1 = g(s_{i+1},t_{j+1}) - g(s_{i+1},t_j) \tag{145}$$

and

$$\delta_{i,j}^2 = -g(s_i,t_{j+1}) + g(s_i,t_j) \tag{146}$$

and we perform the limit as in the previous considerations.

Step III.2: convergence of B_2^N.

We write

$$\int_{T_{i,j}} \alpha^N(s,t) < \omega(g^N(s,t)) - \omega(g(s,t)), \gamma_{i,j}^1+\gamma_{i,j}^2, \delta_{i,j}^1+\delta_{i,j}^2 > dsdt \tag{147}$$

and we use (137) for $\alpha^N(s,t)$ a suitable function of (s,t).

When the form depends on a finite dimensional parameter, we show that the approximation of the stochastic integrals converge for all the derivatives of ω and we conclude by using the Sobolev imbedding theorem as in [23]. That is we consider the integrals

$$\int_D < \nabla_u^\alpha \omega(g^N(s,t)), d_s g^N(s,t), d_t g^N(s,t) > \tag{148}$$

which converge in L^2 for all multiindices α. $\diamondsuit$

We would like to get the same theorem with a more intrinsic approximation $\tilde{g}^N(s,t)$ of the random field $g(s,t)$. As in the part III, the finite dimensional approximations of the integral $\int_{T^2} \tilde{g}^{N,*}\omega$ will converge in L^2, but we don't know if they will converge to the same limit integral of $\int_{T^2} g^{N,*}\omega$.

For that if $g(s,t_j)$ and $g(s,t_{j+1})$ are close, we use the functions:

$$F^N(t, g(s,t_j), g(s,t_{j+1}) = \exp[\frac{t-t_j}{t_{j+1}-t_j}\log(g(s,t_{j+1})g^{-1}(s,t_j)]g(s,t_j) \tag{149}$$

conveniently extended to the whole sets of matrices.

We approximate $g(s,t_{j+1})$, $g(s,t_j)$ as follows:

$$F_N(s, g(s_i,t_{j+1}), g(s_{i+1},t_{j+1})) = \exp[\frac{s-s_i}{s_{i+1}-s_i}\log(g(s_{i+1},t_{j+1})g^{-1}(s_i,t_{j+1}))]g(s_i,t_{j+1}) \tag{150}$$

conveniently extended over the whole matrix algebras as well as its inverse. Moreover,

$$F^N(s, g(s_i,t_j), g(s_{i+1},t_j)] = \exp[\frac{s-s_i}{s_{i+1}-s_i}\log(g(s_{i+1},t_j)g^{-1}(s_i,t_j))]g(s_i,t_j) \tag{151}$$

conveniently extended as well as its inverse to the set of all matrices.

We take as approximation:

$$\begin{aligned}\tilde{g}^N(s,t) = \exp[\frac{t-t_j}{t_{j+1}-t_j}\log(F^N(s,g(s_i,t_{j+1},g(s_{i+1},t_{j+1}))\\ (F^N)^{-1}(s,g(s_i,t_j),g(s_{i+1},t_j)))]F^N(s,g(s_i,t_j),g(s_{i+1},t_j))\end{aligned} \tag{152}$$

We have the asymptotic expansion:

$$\begin{aligned}F^N(t,g(s,t_j),g(s,t_{j+1}) = g(s,t_j)\\ +\frac{t-t_j}{t_{j+1}-t_j}(g(s,t_{j+1})-g(s,t_j))+O((\frac{t-t_j}{t_{j+1}-t_j})^2(g(s,t_{j+1}-g(s,t_j))^2)\end{aligned} \tag{153}$$

We imbed in this expression the approximation of $g(s,t_{j+1})$ and of $g(s,t_j)$. This shows that, in the expansion of $\tilde{g}^N(s,t)$, the more singular term is the same in (70), modulo some more regular terms which converge. The main Itô integral is the same, but we don't know if the correcting terms are the same.

We get the main result of this part:

Theorem 4.2: when $N\to\infty$, the traditional integral $\tilde{A}_v^N = \int_{T^2}(\tilde{g}^N)^*\omega_v$ converges in L^2 to the stochastic Stratonovitch integral:

$$\int_D g^*\omega_v = \int_{S^1\times[0,1]} < \omega(g(s,t)), d_s g(s,t), d_t g(s,t) > \tag{154}$$

Moreover, $\int_D g^*\omega_v$ has a smooth version in v.

Remark: we ignore if the stochastic integral of Theorem IV.2 is equal to the stochastic integral of Proposition IV.1. In the rest of this paper, we will use the version of Theorem IV.2.

Remark: we can consider in the previous theorem a 2-tensor which is not necessarily a 2-form.

5 Stochastic W.Z.N.W. model on the punctured sphere

Let us consider the 3-form closed Z-valued ω over G which is supposed simple simply connected, which at the level of the Lie algebra of G is equal to

$$\omega(X,Y,Z) = K < [X,Y], Z > \tag{155}$$

We extend ω in a 3-form over the whole matrix algebra bounded with bounded derivatives of all orders. We can suppose that ω is Z-valued on G.

Let $\Sigma(1,n)$ be a $(1+n)$ punctured sphere. We deduce a family of loops $s\to g(s,t)$. Let $s\to g(s,t)$ such a loop. We repeat the considerations of [28] and [31] in order to define over such loop group $L_t^i(G)$ the stochastic 2-form:

$$\tau_{st}(\omega) = \int_{S^1} < \omega(g(s,t)), d_s g(s,t), . > \tag{156}$$

We can define for that the following poor stochastic diffeology (see [10], [46] for the introduction of this notion in the deterministic case). Let Ω be the probability space where the random $(1+n)$ punctured sphere is defined:

Definition 5.1: A stochastic plot of dimension m of $L(G)$ is given by a countable family (O, ϕ_i, Ω_i) where O is an open subset of R^m such that:

i)The Ω_i constitute a measurable partition of Ω.

ii) $\phi_i(u)(.) = \{s \to F_i(u, s, g(s,t))\}$ where F_i is a smooth function over $O \times S^1 \times R^N$ with bounded derivatives of all orders (R^N is the matrix algebra where we have imbedded G).

iii)Over Ω_i, for all $u \in U$, $\phi_i(u)(.)$ belongs to the loop group $L(G)$.

We identify two stochastic plots $(O, \phi_i^1, \Omega_i^1)$ and $(O, \phi_j^2, \Omega_j^2)$ if $\phi_i^1 = \phi_j^2$ almost surely over $\Omega_i^1 \cap \Omega_j^2$.

If $\phi_i(u)$ is a stochastic plot,

$$\begin{aligned} \phi_i^* \tau_{st}(\omega)(X,Y) = \int_{S^1} &< \omega(F_i(u,s,g(s,t)) \\ &, d_s F_i(u,s,g(s,t)), \partial_X F_i(u,s,g(s,t)), \partial_Y F_i(u,s,g(s,t)) > \end{aligned} \tag{157}$$

which defines a random smooth form over O by the rules of the Part III.

We can look at the apparatus of [28], [30], [31] to define a stochastic line bundle ξ_t^i over $L_t^i(G)$, with curvature $2\pi\sqrt{-1}k\tau_{st}(\omega)$ for k an integer. Let us recall how to do (See [28], p 463-464): let g_i be a countable system of finite energy loops in the group such that the ball of radius δ and center g_i for the uniform norm O_i determine an open cover of $L(G)$. We can suppose that δ is small. The loop g_i constitutes a distinguished point in O_i. We construct if g belongs to O_i a distinguished curve joining g to g_i, called $l(g_i, g)$: since δ is small, $g_i(s)$ and $g(s)$ are joined by a unique geodesic for the group structure. $l_u(g_i, g)$ is the loop $s \to \exp_{g_i(s)}[u(g(s) - g_i(s))]$ where $g(s) - g_i(s)$ is the vector over the unique geodesic joining $g_i(s)$ to $g(s)$ and exp the exponential of the Lie group associated to the canonical Riemannian structure over the Lie group. This allows to define over O_i a distinguished path joining $g(.)$ to $g_i(.)$. We choose a deterministic path joining the unit loop $e(.)$ to $g_i(.)$ $l_i(e(.), g_i(.))$, and by concatenation of the two paths, we get a distinguished path joining $g(.)$ to $e(.)$ $l_i(g(.), g_i(.))$ over O_i.

The second step is to specify a distinguished surface bounded by $l_i(e(.), g(.))$ and $l_j(e(.), g(.))$, where $g(.)$ belongs to $O_i \cap O_j$. Since δ is small, there is a path $u \to \exp_{g_i(.)}[u(g_j(.) - g_i(.))]$ joining $g_i(.)$ to $g_j(.)$. Because $L(G)$ is simply connected, because G is two-connected, the loop constituted of the path joining $e(.)$ to $g_i(.)$, the path joining $g_i(.)$ to $g_j(.)$ and the path joining $g_j(.)$ to $e(.)$ can be filled by a deterministic surface in the smooth loop group. We can moreover fill the small stochastic triangle constituted of $l_.(g_i(.), g(.))$, $l_.(g_j(.), g(.))$ and the the exponential curve joining $g_i(.)$ to $g_j(.)$ by a small stochastic surface (See [28] for analoguous statements). We get a surface $B_{i,j}^t(g(.))$ which satisfies to our request and which is a stochastic plot. By pulling back (See [28], [30], [31]), we can consider the stochastic Z-valued form $\tau_{st}(\omega)$ and integrate it over the surface $B_{i,j}^t(g(.))$. We put

$$\rho_{i,j}^t(g(.)) = \exp[-\sqrt{-1}2\pi k \int_{B_{i,j}^t(g(.))} \tau_{st}(\omega)] \tag{158}$$

(See [30]).

Definition 5.2: a measurable setion ϕ^t of the line bundle ξ_i^t associated to the stochastic transgression $\tau_{st}(2\pi\omega)$ over $L_t^i(G)$ is a collection of random variable $\alpha_j^{t,i}$ $L_t^i(G)$ measurables over O_j submitted to the rules

$$\alpha_{j'}^{t,i} = \alpha_j^{t,i}\rho_{j,j'}^t \tag{159}$$

almost surely over $O_i \cap O_j$. The Hilbert space of section Ξ_i^t of the line bundle ξ_t^i is the space of measurable sections of ξ_t^i such that

$$E[\|\phi^t\|^2] < \infty \tag{160}$$

where $\|\phi^t\| = |\alpha_j^{t,i}|$ over O_j, definition which is consistent, because $\rho_{j,j'}^t$ is almost surely of modulus 1 in (159).

Let us work in a loop space where the loop splits in two loops. We get a splitting map $g_t^{tot} \to (g_t^1, g_t^2)$. Moreover,

$$\tau_{st}^{tot} = \tau_{st}^1(\omega) + \tau_{st}^2(\omega) \tag{161}$$

If we consider a couple of stochastic sections ($\phi^{1,t}$) and $\phi^{2,t}$ over the two small loop groups, this gives therefore a stochastic section $\phi^{tot,t}$ over the big loop group (See [30] for analoguous considerations), and the different operations are consistent with the glueing property of two loops, especially the notion of stochastic connection, we will define now [28]).

Over O_i, the stochastic 1-form associated to the bundle ξ (we omitt to writte we work over $L_t^i(G)$ by writting only $L(G)$), is given by:

$$A_i(g(.)) = 2\pi k \int_0^1 \tau_{st}(l_{i,t}(e(.), g(.)))(\omega)(d/dtl_{i,t}(e(.), g(.)), \partial l_{i,t}(e(.), g(.)) \tag{162}$$

This gives the double integral:

$$\begin{gathered} 2\pi k \int_0^1 \int_0^1 < \omega(l_{i,u}(e(.), g(.))(s)), \\ d_s l_{i,u}(e(.), g(.))(s), d_u l_{i,u}(e(.),g(.))(s), \partial l_{i,u}(e(.), g(.))(s) > \end{gathered} \tag{163}$$

Let us consider a stochastic plot (O, ϕ_j, Ω_j) of dimension m. $\phi_j^* A_i$ is a random one form over O given if $u \in O$ by:

$$\begin{gathered} 2\pi k \int_0^1 \int_0^1 \omega(l_{i,t}(e(.), F_j(u, ., g(.))(s)), d_s l_{i,t}(e(.), F_j(u, ., g(.))(s), \\ d_t l_{l_{i,t}(e(.), F_j(u,.,g(.))(s)}, \partial_X l_{i,t}(e(.),F_j(u, ., g(.))(s) >= \phi_j^* A_i(X) \end{gathered} \tag{164}$$

where X is a vector field over the parameter space O whose generic element is u. By the results of part II, this give a random smooth one form on O. This connection form are compatible with the application $g^{tot} \to (g^1, g^2)$ when the big loop splits in two small loops.

Let be an elementary cylinder in the $(1+n)$ punctured sphere. Let $\Omega_i, [t_i, t_{i+1}]$ where $\Omega_i \subseteq \Omega$ is a set of probability strictly positive and such over Ω_i $t \to \{s \to g(s,t)\}$ belongs to O_i. We suppose $t_{i+1} > t_i$ with the natural order which is inherited from the fact we consider over the $(1 + n)$

punctured sphere n exit loop groups and one input loop group. We can define the stochastic parallel transport from ξ^{t_i} to $\xi^{t_{i+1}}$ over Ω_i along the path $t \to \{s \to g(s,t)\}$ by the formula

$$\exp[-2\pi ik \int_{t_i}^{t_{i+1}} \int_0^1 \int_0^1 \omega(l_{i,u}(e(.), g(.,t))(s)), d_s l_{i,u}(e(.), g(.,t))(s), d_u l_{i,u}(e(.) \\ , g(.,t)(s), d_t l_{i,u}(e(.), g(.,t))(s) >= \tilde{\tau}^{t_i,t_{i+1}} \tag{165}$$

(See Part IV for the definition of the double stochastic integral). Let $\Sigma(1,n)$ be a $(1+n)$ punctured sphere. Let $L^i_{out}(G)$ the n output loop groups and $L^1_{in}(G)$ the input loop group. We can define, by iterating, a generalization of the stochastic parallel transport, which applies a tensor product of sections ϕ^i_{out} over the output loop spaces to an element over the input loop space, because the different operations are compatible with the notion of glueing loops. We call this generalized parallel transport $\tilde{\tau}^{1,n}$. It is not measurable with respect of the σ-algebras given by the restriction to the random $1+n$ punctured to its boundary. Moreover, over each boundary, the laws of the loops are identical, and the Hilbert space of section of the bundle ξ^i_{out} and ξ_{in} are identical. We denote it by Ξ. We consider the map $\tau^{1,n}$ which associates to an element ξ_{tot} of the the tensor product of the Hilbert spaces of section at the exit boudary the section conditional expectation of $\tilde{\tau}^{1,n}\xi_{tot}$ with respect to the σ-algebra spanned by the input boudary. We get. :

Theorem 5.3: $\tau^{1,n}$ associated to the $1+n$ punctured sphere defines an element of $Hom(\Xi^{\otimes n}, \Xi)$.

Moreover, when we give n random punctured spheres $\Sigma(1,n_i)$, and a punctured sphere $\Sigma(1,n)$, we can glue then in order to get a sphere $\Sigma(1+\sum n_i)$ according the rules of Part II. We get $\tau^{1,\sum n_i}$ which is got by Markov property of part II by composing over the input boundary of $\Sigma(,n_i)$ τ^{1,n_i} and $\tau^{1,n}$ along the output boundary of $\Sigma(1,n)$.

Let σ_i be elements of $Hom(\Xi^{\otimes n_i}, \Xi)$. We deduce by composition an element of $Hom(\Xi^{\otimes \sum n_i}, \Xi)$. Moreover, it is naturally equivariant under the action of the symmetric groups over the n elements σ_i. We say that the collection of vector spaces $Hom(\Xi^{\otimes n}, \Xi)$ constitutes an operad (See [40], [38], [39])

We deduce form the Markov property of the random field parametrized by $\Sigma(1, \sum n_i)$ along the sewing boundary that:

Theorem 5.4: $\tau^{1,n}$ realizes a morphism from the topological operad $\Sigma(1,n)$ got by sewing $1+n$ punctured spheres along their boundary into the operad $Hom(\Xi^{\otimes n}, \Xi)$.

We refer to [21] and [22] for the motivation of this part.

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